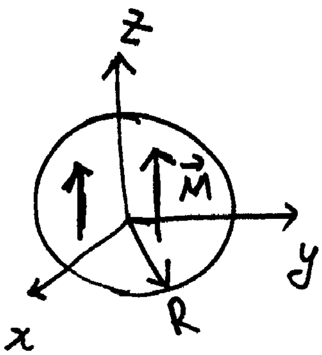
Lecture 9: Examples on magnetic fields

example 1: find the magnetic field of a uniformly magnetized sphere.



Set $\vec{M}$ along the $\vec{z}$-axis. $\Rightarrow$ $\vec{j}_b = c\,\nabla\times\vec{M} = 0$

$$\vec{K}_b = c\vec{M}\times\hat{n} = cM\sin\theta\,\hat{e}_\phi$$

This current distribution is the same as the uniformly charged sphere under rotation $\vec{K} = \sigma\vec{v} = \sigma\omega R\sin\theta\,\hat{e}_\phi$

we identify $cM = \sigma\omega R$, have inside the sphere

$$\vec{B}_{inside} = \frac{8\pi}{3c}R\sigma\omega\,\hat{z} = \frac{8\pi}{3}\,\frac{R\sigma\omega}{c}\,\hat{z} = \frac{8\pi}{3}\vec{M}$$

outside $\vec{B}_{outside} = \dfrac{3(\vec{M}_{tot}\cdot\hat{r})\hat{r} - \vec{M}_{tot}}{r^3}$ with

$$\vec{M}_{tot} = \frac{4\pi}{3c}R^4\sigma\omega\hat{z} = \frac{4\pi}{3}R^3\,\frac{\sigma\omega R}{c}\,\hat{z} = \frac{4\pi}{3}R^3 M\,\hat{z}$$

we can also solve this problem by using magnetic scaler potentials W.

$\nabla\times\vec{H} = 0$ in our problem. We can write

$$\vec{H} = -\nabla W. \qquad \nabla\cdot\vec{H} = \nabla\cdot(\vec{B} - 4\pi\vec{M}) = -4\pi\,\nabla\cdot\vec{M}$$

$\Rightarrow -\nabla^2 W = -4\pi\,\nabla\cdot\vec{M} \Rightarrow$ Possion equation.

$$\Rightarrow \quad W_{in}(r, \theta, \varphi) = \frac{4\pi M}{3} r\cos\theta = \frac{4\pi}{3} M z$$

$$\vec{H} = -\nabla W_{in} = -\frac{4\pi}{3} M \hat{z}$$

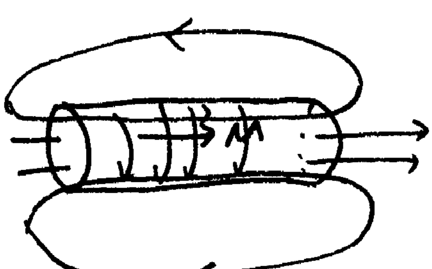
$$\vec{B} = \vec{H} + 4\pi\vec{M} = \left(-\frac{4\pi}{3} + 4\pi\right) M = \frac{8\pi}{3}\vec{M}$$

$$W_{out}(r, \theta, \varphi) = \frac{B_1}{r^2} P_1(\cos\theta) = \frac{\cos\theta}{r^2}\frac{4\pi M R^3}{3} = \frac{z}{r^3}\cdot M_{tot}$$

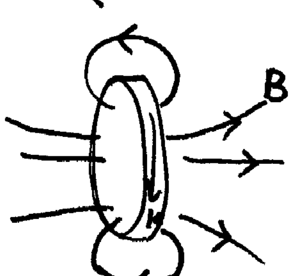
$$\Rightarrow \quad \vec{H} = \vec{B} = -\nabla W_{out} = \frac{3(\vec{M}_{tot}\cdot\hat{r})\hat{r} - \vec{M}_{tot}}{r^3}$$

example 2: Problem 6.9 a short circular of radius a and length L carries a "frozen-in" uniform magnetization $\vec{M}$ parallel to its axis. Find the bond currents, and sketch the magnetic field of the cylinder (for $L \gg a$, $L \ll a$ and $L \simeq a$).

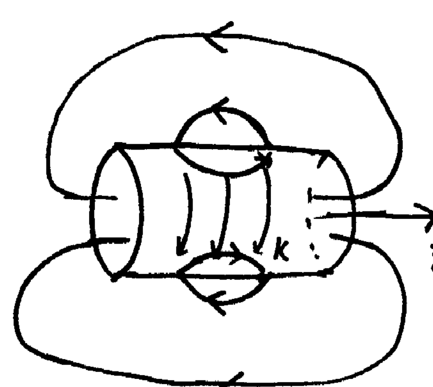
$$\vec{j}_b = c\,\nabla\times\vec{M} = 0, \quad \vec{K}_b = c\vec{M}\times\hat{n} = cM\,\hat{e}_\varphi$$



(a long solenoid $L \gg a$)



a dipole



$$B''_{above} - B''_{below} = \frac{4\pi}{c}\vec{K}\times\hat{n} = -4\pi M\hat{z}$$

$\nabla\cdot\vec{M} = 0$ everywhere, except on the boundary. We need the following boundary condititions

① $W_{in}(R,\theta) = W_{out}(R,\theta)$ ← $W(\vec{b}) - W(\vec{a}) = \int_a^b \nabla W\, dl = -\int_a^b H\, dl$

② $\rightarrow 0$ as $b \to a$

$$\Rightarrow -\left.\frac{\partial W_{out}}{\partial r}\right|_R + \left.\frac{\partial W_{in}}{\partial r}\right|_R = 4\pi\sigma_M = 4\pi\,\vec{M}\cdot\hat{e}_r = 4\pi M\cos\theta$$

in and outside the sphere, we solve Laplace equation $\nabla^2 W = 0$ using the method of variable sparation.

$$W(r,\theta,\varphi) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta).$$

at $r < R$, we take the branch of r^l, $W(r,\theta,\varphi) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$

$r > R$ we take the r^{l+1}, $W(r,\theta,\varphi) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$

at $r = R \Rightarrow$ $A_l R^l = \frac{B_l}{R^{l+1}} \Rightarrow B_l = R^{2l+1} A_l$

$$\sum (l+1)\frac{B_l}{R^{l+2}} P_l(\cos\theta) + \sum l A_l R^{l-1} P_l(\cos\theta) = 4\pi M\cos\theta$$

$$\Rightarrow \begin{cases} \frac{2B_1}{R^3} + A_1 = 4\pi M, & \text{and all other } A_l = B_l = 0 \\ B_1 = A_1 R^3 \end{cases} \Rightarrow A_1 = \frac{M}{3}4\pi$$

for $L \simeq 2a$.

$$B = H + 4\pi M$$

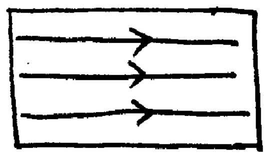
$$\nabla \cdot \vec{H} = \underbrace{-4\pi\, \nabla \cdot M}_{\rho_m}$$

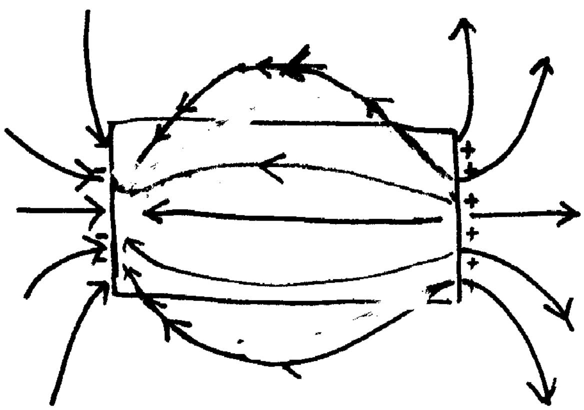
$\Rightarrow \vec{H}$ is field lines of uniform polarization charge

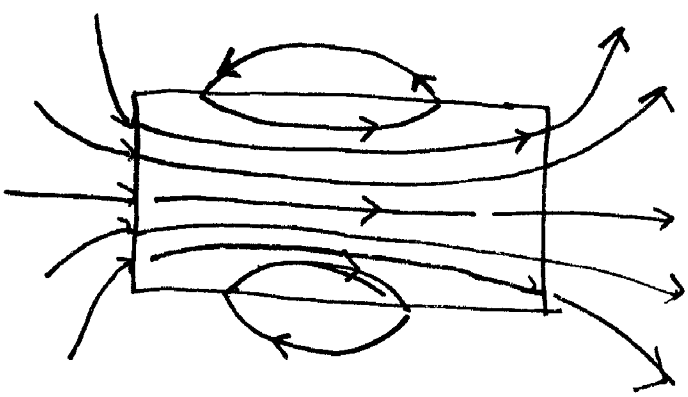
$$H^{\parallel}_{above} - H^{\parallel}_{below} = 0$$

$$H^{\perp}_{above} - H^{\perp}_{below} = -\left(M^{\perp}_{above} - M^{\perp}_{below}\right)$$

M

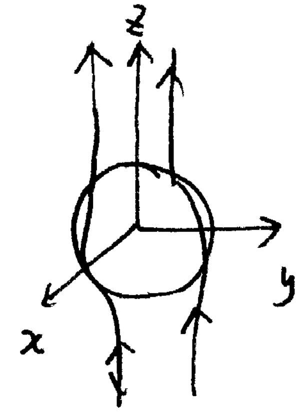






$$B = H + 4\pi M$$

Prob 6.18 A spherical magnetic material with χ_m is put in the external magnetic field $\vec{B} = B^e \hat{z}$ (as $r \to \infty$). What's the field inside?



Sol: Outside $\vec{B} = \vec{H} = B_0 \hat{z} \Rightarrow W = -B^e r\cos\theta$ (as $r \to +\infty$)

inside the material $\vec{B} = \vec{H} + 4\pi\vec{M} = (1+4\pi\chi_m)\vec{H} \Rightarrow \nabla\cdot\vec{H} = 0$

outside $\Rightarrow \nabla\cdot\vec{H} = 0$

$\Rightarrow$ Laplace equation for $r<R$ and $r>R$ $\Rightarrow$

magnetic potential

$$W_{in}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$$
$$W_{out}(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta) - B^e r\cos\theta$$

boundary condition ① $W_{in}(R,\theta) = W_{out}(R,\theta)$

② $\left.\frac{\partial W_{out}}{\partial r}\right|_{r=R} = \mu \left.\frac{\partial W_{in}}{\partial r}\right|_{r=R}$ ← B-continuous

$$\Rightarrow -B_0\cos\theta - \sum(\ell+1)\frac{B_\ell}{R^{\ell+2}}P_\ell(\cos\theta) = \mu\sum \ell A_\ell R^{\ell-1}P_\ell(\cos\theta)$$

for $\ell \neq 1$, combine with $B_\ell = R^{2\ell+1}A_\ell \Rightarrow A_\ell = B_\ell = 0$

for $\ell=1$. $\Rightarrow \begin{cases} A_1 R = -B^e R + \frac{B_1}{R^2} \\ -B^e - \frac{2B_1}{R^3} = \mu A_1 \end{cases} \Rightarrow A_1 = -\frac{3B^e}{2+\mu}$

$$B_1 = \frac{\mu-1}{\mu+2} B^e R^3$$

$$\Rightarrow \quad W_{in}(r,\theta) = -\frac{3B^e}{\mu+2} \, r\cos\theta = -\frac{3B_0^e z}{\mu+2}$$

$$H = -\nabla W = \frac{3B^e}{\mu+2}$$

$$B = \mu H = \frac{3\mu B^e}{\mu+2} = \frac{3(1+4\pi\chi_m)}{3+4\pi\chi_m} B^e = \frac{1+4\pi\chi_m}{1+\frac{4\pi}{3}\chi_m} B^e$$