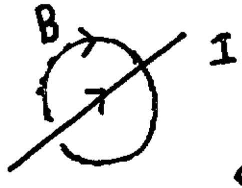
Lect: 2: Magnetic fields from stedy currents

§ Ampere's law



$$\oint \vec{B}\cdot d\vec{l} = \frac{4\pi}{c} I$$ ← we start from here to derive Biot - Savart's law

$$\oint \vec{B}\cdot d\vec{l} = \iint (\vec{\nabla}\times\vec{B})\, d\vec{S} = \frac{4\pi}{c}\iint \vec{j}\cdot d\vec{S}$$

$$\Rightarrow \boxed{\nabla\times\vec{B} = \frac{4\pi}{c}\vec{j}}$$ clearly Ampere's law only applies to stedy - current

$$\nabla\cdot(\nabla\times\vec{B}) = \frac{4\pi}{c}\nabla\cdot\vec{j} = 0 \Rightarrow \frac{\partial\rho}{\partial t} = 0.$$ we will consider time-dependent case later.

Suppose that we know the distribution of $\vec{j}(x,y,z)$, but it's not enough. We need to the divergence of $\vec{B}$. The fact is that so far no magnetic monopole is discovered. Any closed surface, the magntic flux is zero $\oiint \vec{B}\cdot d\vec{S} = 0 \Rightarrow \boxed{\nabla\cdot\vec{B} = 0}$.

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§ Vector potential

$$\nabla\cdot\vec{B} = 0 \quad\Rightarrow\quad \vec{B} = \nabla\times\vec{A},$$ $\vec{A}$ is well-define up to $\vec{A} + \nabla f$

$$\nabla\times\vec{B} = \nabla\times(\nabla\times\vec{A}) = \frac{4\pi}{c}\vec{j}$$

$$\nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A} = \frac{4\pi}{c}\vec{j}$$

we further impose the condition $\nabla\cdot\vec{A} = 0 \Rightarrow$

$$\boxed{\begin{aligned}\nabla^2\vec{A} &= -\frac{4\pi}{c}\vec{j}\\ \nabla\cdot\vec{A} &= 0\end{aligned}}$$

Assuming $\vec{J}$ goes to zero at infinity, we can read off from the solution of Poisson equation:

$$\vec{A}(\vec{r}) = \frac{1}{c}\iiint \frac{\vec{j}(\vec{r}')\, d^3r'}{|\vec{r}-\vec{r}'|}$$

check $\nabla\cdot\vec{A}(\vec{r}) = 0$. $\Rightarrow$ $\nabla\cdot\vec{A}(\vec{r}) = \frac{1}{c}\iiint \vec{j}(\vec{r}')\, \nabla_r \frac{1}{|\vec{r}-\vec{r}'|}\, d^3\vec{r}'$

$$= -\frac{1}{c}\iiint \vec{j}(\vec{r}')\, \nabla_{r'} \frac{1}{|\vec{r}-\vec{r}'|}\, d^3\vec{r}'$$

$$= -\frac{1}{c}\iiint \left[\nabla_{r'}\left[\frac{\vec{j}(r')}{|\vec{r}-\vec{r}'|}\right] - \left(\nabla'\vec{j}(r')\right)\frac{1}{|\vec{r}-\vec{r}'|}\right] d^3\vec{r}'$$

$$\nabla\cdot A(\vec{r}) = -\frac{1}{c}\oiint \frac{\vec{j}(r')}{|\vec{r}-\vec{r}'|}\cdot dS' = 0 \leftarrow \boxed{\text{assuming } j(r)\to 0 \text{ at } r\to+\infty\text{, or boundry}}$$

Example: <u>2D, E and B duality:</u>

Assume an 2D distribution $j_z(x,y)$, which does not depend on z. we cannot use the above formula becasuse, j_z extends to $z=\pm\infty$.

B only has in-plane component. define $\vec{E} = \vec{B}\times\hat{z}$, and $\vec{A} = \varphi(x,y)\,\hat{z}$

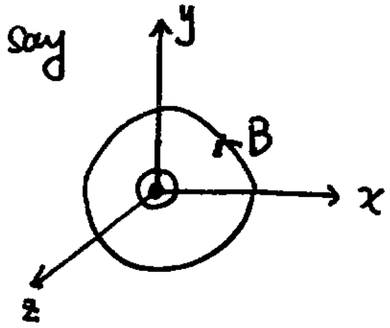
$$\Rightarrow \nabla\times\vec{A} = \nabla\times(\varphi(x,y)\hat{z}) = (\nabla\cdot\varphi)\times\hat{z} = \vec{B}$$

$$[(\nabla\cdot\varphi)\times\hat{z}]\times\hat{z} = \vec{B}\times\hat{z}$$

$$\boxed{-\nabla\varphi = \vec{E} = \boxed{\vec{B}\times\hat{z}}} \Rightarrow -\nabla^2\varphi = \nabla(\vec{B}\times\hat{z}) = \hat{z}\cdot\nabla\times\vec{B} = j_z(x,y)$$

so we ~~trans~~form it to an electro-static problem.

say



$$\vec{B} = \frac{2I}{cr} e_\varphi = \frac{2I}{c}\left(\frac{-y\hat{x} + x\hat{y}}{x^2+y^2}\right)$$

$$j_z = \delta(r)$$

$$\Rightarrow \vec{E} = \vec{B} \times \vec{z} = \frac{2I}{c}\frac{\hat{r}}{r} \Rightarrow -\nabla\varphi = \frac{2I}{c}\frac{\hat{r}}{r}$$

$$\varphi(r) - \varphi_{(a)} = \frac{-2I}{c} \ln r \Big|_a^r \Rightarrow \varphi(r) = -\frac{2I}{c}\ln r$$

$$= -\frac{I}{c}\ln\left(\frac{x^2+y^2}{a^2}\right)$$

short range cut off

§ Magnetic fields from a general shape of wire

$J = \frac{I}{a}$ $\quad$ a : cross section of wire

$d^3r' = a\,d\ell$

$$\Rightarrow \vec{J} d^3 r' = I\, d\vec{\ell}$$

$$\Rightarrow A(\vec{r}) = \frac{I}{c}\int \frac{d\vec{\ell}}{|\vec{r}-\vec{r}'|},$$ we will from here $\Rightarrow$ Biot-Savart law.

$$dA(\vec{r}) = \frac{I}{c}\frac{d\vec{\ell}}{|\vec{r}-\vec{r}'|}$$ thus the contribution to $\vec{B}$ from this line segement is

$$dB(r) = \nabla \times dA(\vec{r}) = \frac{I}{c}\nabla_r \times \left(\frac{d\vec{\ell}}{|\vec{r}-\vec{r}'|}\right) = \frac{I}{c}\left\{ -d\vec{\ell} \times \nabla_r \frac{1}{|\vec{r}-\vec{r}'|}\right)$$

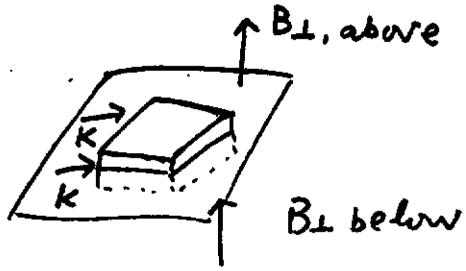
$d\vec{\ell}$ is a const vector respect to ∇_r

$$\Rightarrow \boxed{d\vec{B}(\vec{r}) = -\frac{I}{c} d\vec{\ell} \times \nabla_r \frac{1}{|\vec{r}-\vec{r}'|} = -\frac{I}{c} d\vec{\ell} \times \left(-\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\right) = \frac{I}{cr^2} d\vec{\ell} \times \hat{r}, \quad \text{where } \hat{r} = \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|}.}$$

$$\vec{B}(\vec{r}) = \frac{I}{c}\oint \frac{d\vec{\ell} \times \hat{r}}{r^2}$$

§ magneto-static boundary condition

When there's surface current K, magnetic fields can be discontinuous.



But the norm component remains continuous.

$$\oiint \vec{B}\cdot d\vec{S} = 0 \Rightarrow \left(B_{\perp,above} - B_{\perp,below}\right)\cdot S = 0 \Rightarrow \boxed{B_{\perp,above} = B_{\perp,below}}$$

$$\oint \vec{B}\cdot d\vec{l} = \frac{4\pi}{c}\vec{K}\, d\vec{l}$$

$$\Rightarrow \left(\vec{B}''_{above} - \vec{B}''_{below}\right)\cdot\Delta\vec{l} = \frac{4\pi}{c}\vec{K}\cdot\left(\hat{n}\times\Delta\vec{l}\right) = \frac{4\pi}{c}\Delta\vec{l}\cdot\left(\vec{K}\times\vec{n}\right)$$

$$\Rightarrow \boxed{\vec{B}''_{above} - \vec{B}''_{below} = \frac{4\pi}{c}\vec{K}\times\hat{n}}$$

We may also need boundary conditions for vector potential A. Since A satisfies 2nd differential Eq, its discontinuity appears at its derivative. $\vec{A}$ itself is continuous.

$\nabla\cdot\vec{A} = 0 \Rightarrow$ the normal component of A is continuous

$\oint \vec{A}\cdot d\vec{l} = B\cdot(\Delta l\cdot\Delta h) \longrightarrow 0$ as the thickness of $\Delta h \to 0$
B is regular.

set the triod frame $e_{\hat{n}}$, $e_{\hat{K}}$, and $e_{\hat{K}\times\hat{n}}$.

$\vec{B}\cdot e_{\hat{K}\times\hat{n}} = -\partial_n A_{\hat{K}} + \partial_{\hat{K}} A_{\hat{n}}$, only this component of B is discontinuous

we don't expect discontinuity of $\partial_{\hat{K}} A_{\hat{n}}$, because $A_{\hat{n}}$ is continuous and $\hat{K}$ is parallel to the boundary. The discontinuity comes from $\partial_n A_{\hat{K}}$.

$$\Rightarrow \quad (\partial_n A_{\hat{k}})_{above} - (\partial_n A_{\hat{k}})_{below} = -\frac{4\pi}{c} K$$

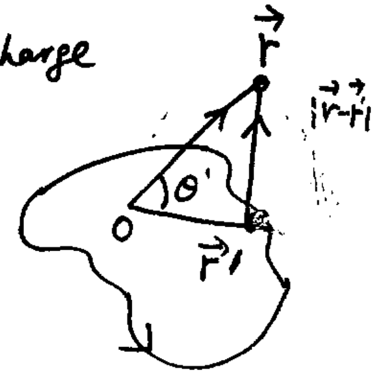
$$\text{or} \quad \boxed{(\partial_n \vec{A})_{above} - (\partial_n \vec{A})_{below} = -\frac{4\pi}{c} \vec{K}}$$

in comparison: boundary condition for electric surface charge

$$(\vec{E}_{\parallel})_{above} = (\vec{E}_{\parallel})_{below}$$

$$(\vec{E}\cdot\hat{n})_{above} - (\vec{E}\cdot\hat{n})_{below} = 4\pi\sigma$$

$$\text{or} \quad -\left(\frac{\partial\varphi}{\partial n}\right)_{above} + \left(\frac{\partial\varphi}{\partial n}\right)_{below} = 4\pi\sigma$$



§ Multiple expansion of the vecter potential

we need to use

$$\frac{1}{|r-r'|} = \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\theta'}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta') .$$

$$\vec{A}(\vec{r}) = \frac{I}{c}\oint\frac{d\vec{l}'}{|\vec{r}-\vec{r}'|} = \frac{I}{c}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint(r')^n P_n(\cos\theta')\, d\vec{l}'$$

$$= \frac{I}{c}\Big[\underbrace{\frac{1}{r}\oint d\vec{l}'}_{\text{monopole}=0} + \underbrace{\frac{1}{r^2}\oint r'\cos\theta' d\vec{l}'}_{\text{dipole}} + \underbrace{\frac{1}{r^3}\oint r'^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right) d\vec{l}'}_{\text{gudrapole}} + \cdots\Big]$$

The magnetic dipole component

$$\vec{A}_{dip}(r) = \frac{I}{cr^2}\oint r'\cos\theta' d\vec{l}' = \frac{I}{cr^2}\oint(\hat{r}\cdot\vec{r}')\, d\vec{l}'$$

using the identity $\oint(\vec{c}\cdot\vec{r})\,d\vec{l} = \vec{a}\times\vec{c}$, where $\vec{c}$ is a const vecter, $\vec{a} = \frac{1}{2}\oint\vec{r}\times d\vec{l}$

$$\oint (\hat{r}\cdot\vec{r}\,')d\vec{l}\,' = \vec{a}\times\hat{r} = \left[\frac{1}{2}\oint \vec{r}\,'\times d\vec{l}\,'\right]\times\hat{r}$$

$$\Rightarrow \vec{A}_{dip}(\vec{r}) = \frac{1}{c}\frac{\vec{m}\times\hat{r}}{r^2}, \text{ where } m = \frac{I}{2}\oint \vec{r}\,'\times d\vec{l}\,' .$$

where $\vec{a} = \oint \vec{r}\,'\times d\vec{l}\,'$ is the vecter area of a surface, which is determined by the boundary. (c.f. Prob 1.61).

§ field from a dipole.

$$\vec{B}_{dip}(\vec{r}) = \nabla\times\vec{A}_{dip}(\vec{r}) = \frac{1}{c}\nabla\times\left(\frac{\vec{m}\times\hat{r}}{r^2}\right)$$

using $\nabla\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{A} - (\vec{A}\cdot\nabla)\vec{B} + \vec{A}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{A})$

$$\nabla\times(\vec{m}\times\frac{\hat{r}}{r^2}) = -(\vec{m}\cdot\nabla)\left(\frac{\hat{r}}{r^2}\right) + \vec{m}\left(\nabla\cdot\frac{\hat{r}}{r^2}\right) \leftarrow \boxed{\vec{m}\cdot 4\pi\,\delta(\vec{r})}$$

singluar point at origin, neglected!

$$\nabla(\vec{A}\cdot\vec{B}) = \vec{A}\times(\nabla\times\vec{B}) + \vec{B}\times(\nabla\times\vec{A}) + (\vec{A}\cdot\nabla)\vec{B} + (\vec{B}\cdot\nabla)\cdot\vec{A} \Rightarrow$$

$$\Rightarrow \nabla\left(\frac{\vec{m}\cdot\hat{r}}{r^2}\right) = \vec{m}\cdot\nabla\frac{\hat{r}}{r^2}$$

$$\Rightarrow \nabla\times(\vec{m}\times\frac{\hat{r}}{r^2}) = -\nabla\left(\frac{\vec{m}\cdot\hat{r}}{r^2}\right) = -(\vec{m}\cdot\vec{r})\nabla\frac{1}{r^3} - \nabla(\vec{m}\cdot\vec{r})\frac{1}{r^3}$$

$$\nabla\frac{1}{r^3} = \frac{-3\hat{r}}{r^4} \qquad \nabla(\vec{m}\cdot\vec{r}) = \vec{m}$$

$$\Rightarrow \nabla\times(\vec{m}\times\hat{r}/r^2) = \frac{3(\vec{m}\cdot\hat{r})\hat{r}}{r^3} - \frac{\vec{m}}{r^3}$$

$$\Rightarrow \vec{B}_{dip}(\vec{r}) = \frac{1}{c}\left(\frac{3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}}{r^3}\right)$$

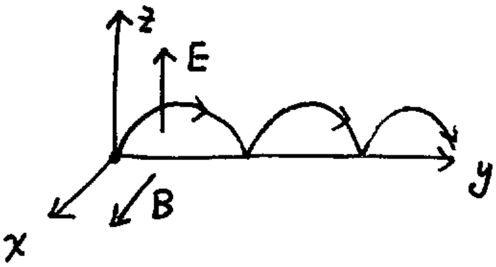
The interaction energy between two magnetic dipoles

$$V = -\vec{B}_{12}\cdot\vec{m}_2 = \frac{1}{c}\frac{1}{r^3}\left[\vec{m}_1\cdot\vec{m}_2 - 3(\vec{m}_1\cdot\hat{r})(\vec{m}_2\cdot\hat{r})\right]$$

Lect 3: Examples of magnetic fields

Griffiths Ex 5.2:

related to the classic picture of quantum Hall edge states



Solution: no force in the x-direction. The motion is in the yz-plane.

$$\vec{r}(t) = (0,\ y(t),\ z(t))$$

$$\vec{F}_L = q\frac{\vec{v}}{c}\times\vec{B} = \frac{qB}{c}(\dot{z}\hat{y} - \dot{y}\hat{z})\ , \quad \vec{F}_E = qE_0\hat{z}$$

$$\Rightarrow\ q(E - \frac{B}{c}\dot{y})\hat{z} + \frac{qB}{c}\dot{z}\hat{y} = m\ddot{y}\hat{y} + \ddot{z}\hat{z}$$

$$\Rightarrow \begin{cases} \frac{qB\dot{z}}{c} = m\ddot{y} \\ qE - \frac{qB}{c}\dot{y} = m\ddot{z} \end{cases}$$

define $\omega = \frac{qB}{mc}$. $\leftarrow$ cyclotron frequency

$$\Rightarrow \begin{cases} \ddot{y} = \omega\dot{z} \\ \ddot{z} = \omega(\frac{Ec}{B} - \dot{y}) \end{cases}$$

$$\Rightarrow\ \dddot{z} = -\omega\ddot{y} = -\omega^2\dot{z}$$

$$\Rightarrow\ \dot{z} = A\cos\omega t + B\sin\omega t$$

$$\Rightarrow\ \underline{z = C_1\cos\omega t + C_2\sin\omega t + C_3}$$

$$\ddot{z} = -\omega^2[C_1\cos\omega t + C_2\sin\omega t]$$

$$\Rightarrow\ \dot{y} = -\frac{\ddot{z}}{\omega} + \frac{Ec}{B} = \omega[C_1\cos\omega t + C_2\sin\omega t] + \frac{Ec}{B}$$

$$\Rightarrow\ \underline{y = +C_1\sin\omega t - C_2\cos\omega t + \frac{Ec}{B}t + C_4}$$

plug into the initial condition $y(0) = z(0) = \dot{y}(0) = \dot{z}(0) = 0$

$$\Rightarrow \begin{cases} y(t) = \frac{Ec}{\omega B}(\omega t - \sin\omega t) \\ z(t) = \frac{Ec}{\omega B}(1 - \cos\omega t) \end{cases}$$

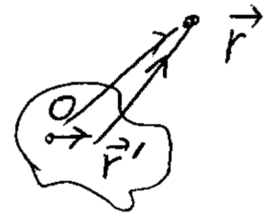
define $R = \frac{Ec}{B\omega}$ radius

$$\Rightarrow\ (y - \underbrace{R\omega t}_{vt})^2 + (z - R)^2 = R^2$$

Cycloid $\boxed{v = \frac{Ec}{B}}$

* read Page 209 – 211.

2: Start from Biot - Savart law to prove $\nabla \cdot \vec{B} = 0$.



$$\vec{B}(\vec{r}) = \frac{1}{c} \int \frac{\vec{J}(\vec{r}\,') \times \hat{r}_{12}}{r_{12}^2} \, d^3r'$$

$$\nabla \cdot \vec{B}(r) = \frac{1}{c} \int \nabla_r \cdot \left[\frac{\vec{J}(\vec{r}\,') \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} \right] d^3r'$$

$$= \frac{1}{c} \int \vec{J}(\vec{r}\,') \cdot \left[\nabla_r \times \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3} \right] d^3r' = 0$$

$$\nabla \times \vec{B}(r) = \frac{1}{c} \int \nabla_r \times \left[\frac{\vec{J}(\vec{r}\,') \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} \right] d^3r'$$

$$\nabla_r \times \left(\vec{J}(r') \times \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3} \right) = \vec{J}(r') \left(\nabla_r \cdot \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3} \right) - \left(\vec{J}(r') \cdot \nabla_r \right) \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3}$$

The second term

$$-\vec{J}(r') \cdot \nabla_r \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3} = \left(\vec{J}(r') \cdot \vec{\nabla}_{r'} \right) \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3} = \overbrace{\vec{\nabla}_{r'} \left(\frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3} \vec{J}(r') \right)}^{\text{total derivative}}$$

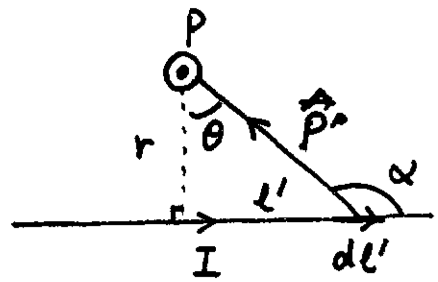
~~using $\nabla(A \cdot B)$ $\nabla \times (A \times B)$~~ $= \quad - \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3} \underbrace{\vec{\nabla}_{r'} \vec{J}(r')}_{0}$

$\Rightarrow$ the second term vanishes after volume integral

$$\Rightarrow \nabla \times \vec{B}(\vec{r}) = \frac{1}{c} \int \vec{J}(r') \, \nabla_r \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3} \, d^3r' = \frac{4\pi}{c} \int \vec{J}(r') \, \delta(\vec{r} - \vec{r}\,') \, d^3r'$$

$$= \frac{4\pi}{c} \vec{J}(r)$$

3: Application of Biot-Sarvart law



§ B-field from a long straight line:

$d\vec{\ell}' \times \hat{\rho}$ points out of page, with the magnitude $d\ell' \sin\alpha = d\ell' \cos\theta$. $\ell' = r\,tg\,\theta \Rightarrow d\ell' = r\sec^2\theta\, d\theta$

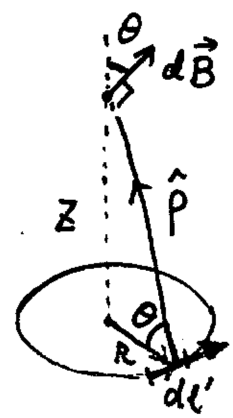
$$\rho^2 = \frac{r^2}{\cos^2\theta} \quad \Rightarrow B = \frac{I}{c}\int \frac{d\vec{\ell}' \times \hat{\rho}}{\rho^2}$$

$$B = \frac{I}{c}\int_{-\frac{\pi}{2}}^{\pi/2} \frac{r\sec^2\theta\cos\theta\, d\theta}{r^2/\cos^2\theta} = \frac{I}{cr}\int_{-\frac{\pi}{2}}^{\pi/2} \cos\theta\, d\theta = \frac{I}{cr}\sin\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{2I}{cr}$$

The force between two parallel wire $\vec{F} = \int(\frac{\vec{v}}{c}\times\vec{B})\,dq = \int(\frac{\vec{v}}{c}\times\vec{B})\lambda\, d\ell$

$$\Rightarrow \vec{F} = \int(\vec{I}\times\vec{B})\frac{d\ell}{c} \quad \Rightarrow \boxed{\frac{F}{\ell} = \frac{2I_1I_2}{c^2d}}$$

§: B-field at distance z above the center of a circular loop of radius R with a steady current I.



$d\vec{\ell}' \times \hat{\rho}$ has the magnetude $d\ell'$, it's direction as plotted forming a polar angle θ with the z-axis.

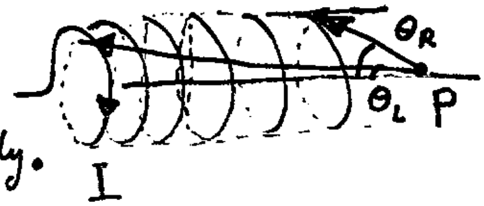
$$\Rightarrow B_z = \frac{I}{c}\int\frac{d\ell'\cos\theta}{\rho^2} = \frac{I\cos\theta}{\rho^2 c}\cdot 2\pi R = \frac{2\pi I}{c}\frac{R^2}{\rho^3}$$

$$= \frac{2\pi I}{c}\frac{R^2}{(R^2+z^2)^{3/2}} \quad \text{or} \quad \boxed{B_z = \frac{2\pi I\cos^3\theta}{cR}}$$

Other components average to zero.

§ B-field at axis of a wound solenoid

Suppose that the left and right ends span the polar angles of θ_L and θ_R, respectively.



$$dB = \frac{2\pi\lambda\, dl}{CR}\sin^3\theta = \frac{+2\pi\lambda}{CR}\sin\theta\, d\theta$$

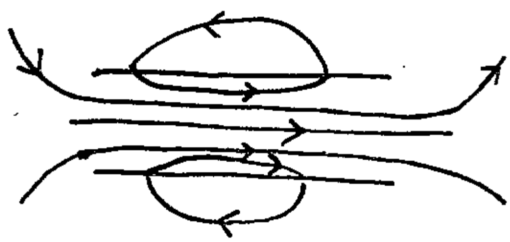
$$B = \int_{\theta_L}^{\theta_R} dB = \frac{-2\pi\lambda}{CR} R\cos\theta \Big|_{\theta_L}^{\theta_R} = \frac{2\pi\lambda}{c}\left[\cos\theta_L - \cos\theta_R\right]$$

$$l = R\cot\theta$$

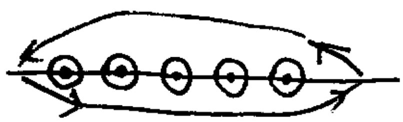
$$dl = +R\csc^2\theta\, d\theta$$

for an infinitely long solenoid,

$\theta_L = 0$, $\theta_R = \pi$ $\Rightarrow$ $\boxed{B = \frac{4\pi\lambda}{c}}$, where $\lambda = I\cdot n$, and n is the num of turns per length.

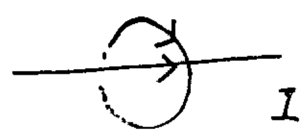


B-field of a solenoid.



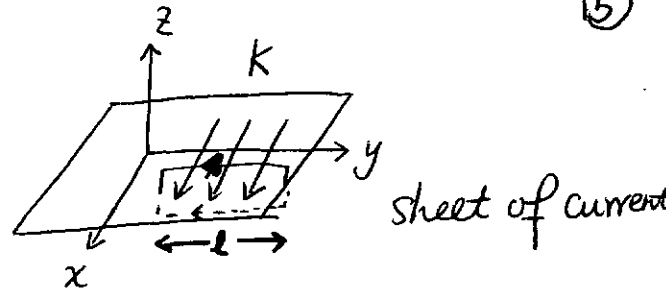
4: Application of Ampere's law + symmetric analysis

① B can only along "circumferential" direction.



$$\oint B\cdot dl = \frac{4\pi I}{c} \Rightarrow B\cdot 2\pi r = \frac{4\pi I}{c} \Rightarrow B = \frac{2I}{cr}$$

★ B- field from a sheet - curret.



B should have translational symmetry, i.e. B is uniform along xy-direction.

① Can B has a z-component?

no. The system has the symmetry of combined tim-reversal and rotation along z-axis 180°.

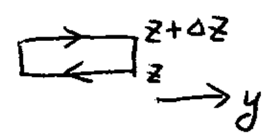
This operations flip the direction of B_z. So B can only be in the plane.

② The system has the symetry of time-reversal and reflectional repect to zy plane.

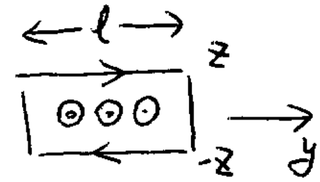
B is an axial vector $\Rightarrow$ com $B_x \xrightarrow{TR} -B_x \xrightarrow{Ref} -B_x$. Thus $B_x = 0$.

③ B can only along y-direction. B_y should not depend on z. for $z>0$

Let us choose a loop at $z>0 \Rightarrow B_y(z) = B_y(z+\Delta z)$.



for a loop crossing the current sheet.



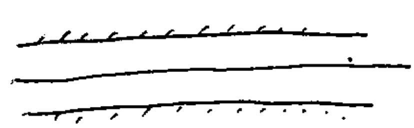
~~$B_y \cdot 2l$~~

The system has rotation symmetry arond x-axis at 180°.

$\Rightarrow B_y(z) = -B_y(-z) \quad \Rightarrow \quad B_y(z) \cdot 2 \cdot l = -\frac{4\pi}{c} K \cdot l$

$$\Rightarrow \boxed{B_y(z) = \begin{cases} -\frac{2\pi}{c} K & \text{for } z>0 \\ \frac{2\pi}{c} K & \text{for } z<0 . \end{cases}}$$

★ B- field from an infinitely long solenoid

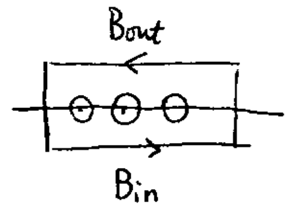


The system has rotational symmetry around the axis.

① B cannot have radial component, otherwise $\oint B \cdot ds \neq 0$.

2 B cannot have "circumferential" component, otherwise $\oint \vec{B}\cdot d\vec{l} \neq 0$.

3 B can only be along axial. It can also be proved that B is uniform inside the solenoid, and outside.



$$(B_{in} - B_{out})\cdot l = \frac{4\pi}{c} I N$$

$B_{out} = 0$ if we set $r \to \infty$

$$\Rightarrow \boxed{B_{in} = \frac{4\pi}{c} I n \text{ along axis.}}$$

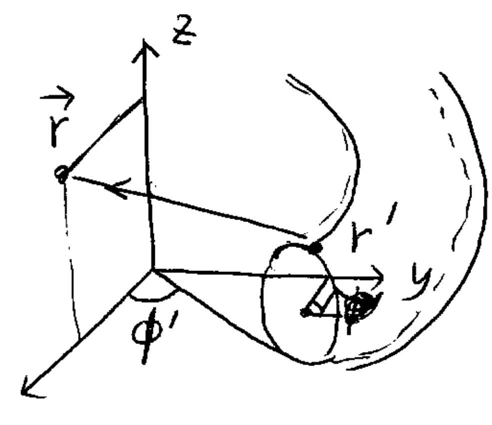
☆: a toroidal coil of a circular ring (actually can be any shape). The winding is uniform. What's the distribution of B-field?

we first use Biot-Savart law to prove that B is only circumferential. This can also be proved simply by symmetry.

Our system has rotational symmetry around "z"-axis, without loss of generality, let us consider a point r in the xz-plane. with $\vec{r} = (x, 0, z)$

the coordinate of r' on the toroidal coil

$$\vec{r}' = (s'\cos\phi', s'\sin\phi', z')$$ azimuthal angel.



the current I has no ϕ dependence

$$\vec{I} = (I_s\cos\phi', I_s\sin\phi', I_z)$$

$$\Rightarrow d\vec{B} = \frac{1}{c}\,\frac{\vec{I}\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\,dl'$$

$$\vec{I}\times(\vec{r}-\vec{r}') = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ I_s\cos\phi' & I_s\sin\phi' & I_z \\ x-s'\cos\phi' & -s'\sin\phi' & z-z' \end{vmatrix} = \sin\phi'\left(I_s(z-z')+s'I_z\right)\hat{x}$$
$$+\left[I_z(x-s'\cos\phi')-I_s\cos\phi'(z-z')\right]\hat{y}$$
$$+\left[-I_s x\sin\phi'\right]\hat{z}$$

the contribution to $\hat{x}$ and $\hat{z}$ are odd functions of ϕ' $\Rightarrow$ vanishes after integration

$\Rightarrow$ $d\vec{B}$ only along the $\hat{y}$-direction. or $\vec{B}$ is along "circumferential".

Or we can simply get it from symmetry analysis.

$\vec{B}$ is axial-vector. It has different properties under reflection operation.

Can you explain why?

~~① our system has rotation symmetry~~

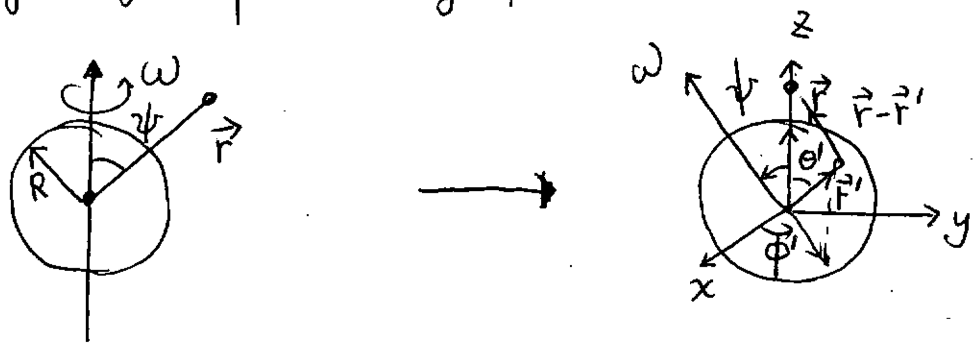
① our system has reflection symmetry respect to any vertical radial plane $\Rightarrow$ B cannot parallel to that plane. B can only perpendicular to the vertical-radial plane.
$\Rightarrow$ B is circumferential.

Then the results are straight-forward. For point $\vec{r}$ inside the torus with radius ρ to the z-axis

$$\cancel{B(\rho)\cdot 2\pi\rho = \frac{4\pi}{c}I\cdot N \to B(\rho) = \frac{2IN}{\rho c} \text{ for } \rho \text{ in}}$$

$$B(r) = \frac{2I}{c}\frac{N}{\rho}, \quad \text{otherwise } B(r) = 0.$$

§ Magnetic field of a rotating spherical shell



we rotate $\vec{r}$ to the z-axis, and $\vec{\omega}$ in the x-z plane. $\vec{r}'$ is on the sphere with (θ', ϕ')

$$A(\vec{r}) = \frac{1}{c}\int \frac{\vec{K}(r')}{|\vec{r}-\vec{r}'|}\, da' \qquad \text{where } \vec{K}(r') = \sigma \vec{v}$$

$$|\vec{r}-\vec{r}'| = \sqrt{R^2 + r^2 - 2Rr\cos\theta'}$$

$$da' = R^2 \sin\theta'\, d\theta'\, d\phi'$$

$$\vec{v} = \vec{\omega}\times\vec{r}' = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \omega\sin\psi & 0 & \omega\cos\psi \\ R\sin\theta'\cos\phi' & R\sin\theta'\sin\phi' & R\cos\theta' \end{vmatrix}$$

$$= R\omega\left[-\cos\psi\sin\theta'\sin\phi'\,\hat{x} + (\cos\psi\sin\theta'\cos\phi' - \sin\psi\cos\theta')\,\hat{y} + \sin\psi\sin\theta'\sin\phi'\,\hat{z}\right]$$

those terms contains $\cos\phi'$ & $\sin\phi'$ go to zero after average over ϕ'

$\Rightarrow$ only $A_y(\vec{r}) \neq 0$. $-\hat{y}$ is the direction of $\vec{\omega}\times\vec{r}$.

$$\Rightarrow \vec{A}(\vec{r}) = \underbrace{-\frac{R^3\sigma\omega}{c}}_{2\pi}\int_0^{\pi} \frac{\sin\psi\cos\theta'\sin\theta'}{\sqrt{R^2+r^2-2Rr\cos\theta'}}\, d\theta' = \frac{2\pi R^3\sigma\omega}{c}\overset{\sin\psi}{\vee}\int_0^{\pi}\frac{\cos\theta'\, d\cos\theta'}{\sqrt{R^2+r^2-2Rr\cos\theta}}$$

$$\int_{-1}^{1}\frac{u\,du}{\sqrt{R^2+r^2-2Rru}} = -\frac{R^2+r^2+Rru}{3R^2r^2}\sqrt{R^2+r^2-2Rru}\,\Bigg|_{-1}^{+1} =$$

$$= -\frac{1}{3R^2r^2}\left[(R^2+r^2+Rr)|R-r| - (R^2+r^2-Rr)(R+r)\right]$$

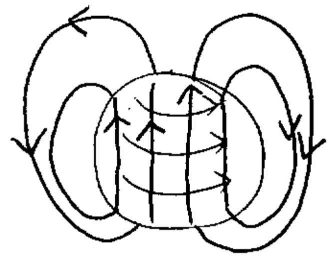
$$= \begin{cases} \dfrac{2r}{3R^2} & (R > r) \\ \dfrac{2R}{3r^2} & (R < r) \end{cases}$$

remember $\vec{\omega}\times\vec{r} = -\omega r\sin\psi\,\hat{y}$

$$\Rightarrow \vec{A}(r) = \begin{cases} \frac{4\pi}{3c} R\sigma (\vec{\omega} \times \vec{r}) & \text{if } r \text{ is inside sphere} \\ \frac{4\pi}{3c} \frac{R^4}{r^3} \sigma (\vec{\omega} \times \vec{r}) & \text{if } r \text{ is outside sphere} \end{cases}$$

if set back $\vec{\omega}$ along z-axis $\Rightarrow$

$$A(r, \theta, \varphi) = \begin{cases} \frac{4\pi}{3c} R \omega \sigma \, r \sin\theta \, \hat{e}_\varphi & (r \leq R) \\ \frac{4\pi}{3c} \frac{R^4}{r^2} \omega \sigma \sin\theta \, \hat{e}_\varphi & (r \geq R) \end{cases}$$



$$\vec{B} = \nabla \times \vec{A}(r)$$

$$\nabla \times (\vec{\omega} \times \vec{r}) = -\underbrace{(\vec{\omega} \cdot \vec{\nabla}) \vec{r}}_{\vec{\omega}} + \vec{\omega} \overbrace{(\nabla \cdot \vec{r})}^{3} = 2\vec{\omega}$$

$$\Rightarrow \vec{B}_{inside} = \frac{8\pi}{3c} R\sigma \vec{\omega}$$

$$\nabla \times \left(\vec{\omega} \times \frac{\hat{r}}{r^2}\right) = -(\vec{\omega} \cdot \vec{\nabla})\left(\frac{\hat{r}}{r^2}\right) + \vec{\omega}\left(\nabla \cdot \frac{\hat{r}}{r^2}\right)$$

$$= -\nabla\left(\frac{\vec{\omega} \cdot \hat{r}}{r^2}\right) + \vec{\omega} 4\pi \delta(\vec{r})$$ go to zero, because $|\vec{r}| > R$

$$\nabla\left(\frac{\vec{\omega} \cdot \vec{r}}{r^3}\right) = -(\vec{\omega} \cdot \vec{r}) \nabla \frac{1}{r^3} - \nabla(\vec{\omega} \cdot \vec{r}) \frac{1}{r^3}$$

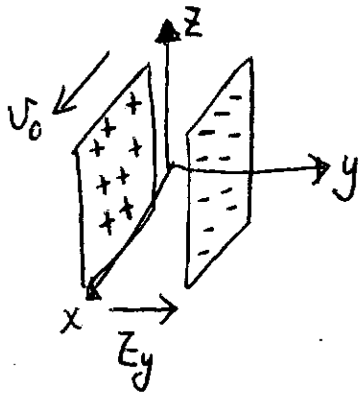
$$\nabla \times \left(\vec{\omega} \times \frac{\hat{r}}{r^2}\right) = \frac{3(\vec{\omega} \cdot \hat{r})\hat{r}}{r^3} - \frac{\vec{\omega}}{r^3}$$

$$\vec{B}_{outside} = \frac{4\pi}{3c} R^4 \sigma \left[\frac{3(\vec{\omega} \cdot \hat{r})\hat{r} - \vec{\omega}}{r^3}\right]$$ its a dipolar field

with $\boxed{\vec{m} = \frac{4\pi}{3c} R^4 \sigma \vec{\omega}}$

Lect 4: Transformation of E-M fields

Let us consider two plates (in Frame F) in the xz-plane. The surface charge density $\pm\sigma$

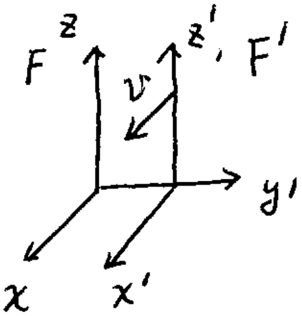


$$E_y = 4\pi\sigma .$$

The two plates move along x-direction at the speed v_0

$$\Rightarrow J_x = \sigma v_0 , \text{ thus } B_z = \frac{4\pi J_x}{c} = \frac{4\pi\sigma v_0}{c} .$$

we consider another frame F', which moves an the speed v along x-axis repect with F, what the fields observed in F'?



In F', the velocity of the two plates

$$v_0' = \frac{v_0 - v}{1 - \frac{v_0 v}{c^2}} = c\,\frac{\beta_0 - \beta}{1-\beta_0\beta} \quad \leftarrow \beta_0 = \frac{v_0}{c}, \quad \beta = \frac{v}{c}$$

in fram F', the charge desity $\sigma' = \frac{\sigma}{\gamma_0}\gamma_0'$, where $\gamma_0 = \frac{1}{\sqrt{1-\beta_0^2}}$

$$\gamma_0' = \frac{1}{\sqrt{1-\left(\frac{\beta_0-\beta}{1-\beta_0\beta}\right)^2}} = \frac{1-\beta_0\beta}{\sqrt{(1-\beta_0^2)(1-\beta^2)}} \Rightarrow \sigma' = \sigma\,\frac{1-\beta_0\beta}{\sqrt{1-\beta^2}}$$

$$= \gamma\sigma\,(1-\beta_0\beta)$$

thus $J_x' = \sigma' v_0' = \gamma\sigma(1-\beta_0\beta)\,v_0' = \gamma\sigma(\beta_0-\beta)\,c$

$$\Rightarrow E_y' = 4\pi\sigma' = 4\pi\sigma\,\gamma(1-\beta_0\beta) = \gamma\left[4\pi\sigma - \frac{4\pi\sigma v_0}{c}\left(\frac{v}{c}\right)\right]$$

$$B_z' = \frac{4\pi}{c} J_x' = \gamma\left[\frac{4\pi\sigma v_0}{c} - 4\pi\sigma\left(\frac{v}{c}\right)\right]$$

or $$\boxed{\begin{aligned} E_y' &= \gamma(E_y - \beta B_z) \\ B_z' &= \gamma(-\beta E_y + B_z) . \end{aligned}}$$

We can derive the rules for other components: F' is moving at speed of v along the x-direction, respect to F, then.

$$\boxed{\begin{aligned} E_x' &= E_x, \quad E_y' = \gamma(E_y - \beta B_z), \quad E_z' = \gamma[E_z + \beta B_y] \\ B_x' &= B_x, \quad B_y' = \gamma(B_y + \beta E_z) \quad B_z' = \gamma[B_z - \beta E_y] \end{aligned}} \xrightarrow{\text{firstorder}} \begin{aligned} \vec{E}' &= \vec{E} + \frac{\vec{v}}{c} \times \vec{B} \\ \vec{B}' &= \vec{B} - \frac{\vec{v}}{c} \times \vec{E} \end{aligned}$$

Suppose in the Frame F, that $B = 0$, $\Rightarrow$

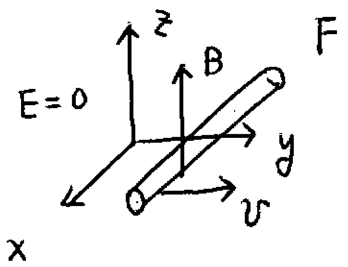
$$E_x' = E_x, \quad E_y' = \gamma E_y \quad E_z' = \gamma E_z$$
$$B_x' = 0, \quad B_y' = \beta\gamma E_z \quad B_z' = -\gamma\beta E_y$$

then $\boxed{\vec{B}' = -\left(\frac{\vec{v}}{c}\right) \times \vec{E}'}$ $\vec{v}$ is the velocity of F' respect to F.

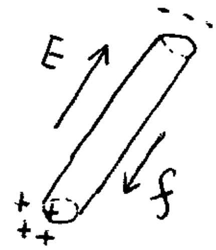
Similarly, if in the frame F in which $E = 0$, then in the frame F'

we have $\boxed{\vec{E}' = +\left(\frac{\vec{v}}{c} \times \vec{B}'\right).}$

§ A conducting rod moving in B-field



in the F-frame, $E=0$, $B = B\hat{z}$, the rod is moving along $\hat{e}_y$.



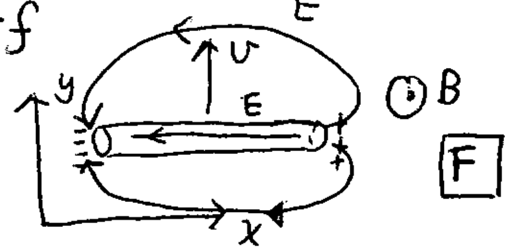
Lorentz force $\vec{f} = \frac{q}{c}\, \vec{v} \times \vec{B}$

f drives charge accumulate at ends that there's internal electric field

$\Rightarrow \quad q\vec{E} = -\vec{f}$

$$\boxed{\vec{E} = \frac{-1}{c}\, \vec{v} \times \vec{B}}$$

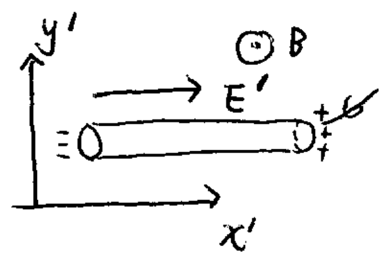
in frame F



Now let us sit the co-moving frame F′ with F. For the moment, we neglect the rod, the we will see in the frame F′, there exist B′ and E′.

$$\vec{B}' \approx \vec{B} - \frac{\vec{v} \times \vec{E}}{c} \approx \vec{B} \quad \text{up to } \beta^2 .$$

$$\vec{E}' = \frac{\vec{v}}{c} \times \vec{B}' .$$



in F′, the rod is at rest. $\vec{E}'$ field induces charge distributions on the rod. There's no eletric field inside the rod! Thus no motion of electric charge!