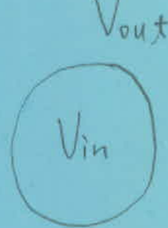
Problem 3.19

The only charge is on the surface. Laplacian equation are both valid inside and outside the sphere but not on the surface. Thus, we can express V_{in} and V_{out} in the form of eq. 3.65

$$V_{in} = \sum_{l=0}^{\infty} (A_l r^l + B_l r^{-l-1}) P_l(\cos\theta)$$

$$V_{out} = \sum_{l=0}^{\infty} (C_l r^l + D_l r^{-l-1}) P_l(\cos\theta)$$



For V_{in}, since it covers the origin, B_l must be zero, or $B_l r^{-l-1}\big|_{r\to 0}$ is singular.

$$V_{in} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

For V_{out}, we know $V_{out}(r=\infty) = 0$, so $C_l = 0$

$$V_{out} = \sum_{l=0}^{\infty} D_l r^{-l-1} P_l(\cos\theta)$$

Moreover, V_{out} and V_{in} should be equal at $r = R$

$$\Rightarrow \quad \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = \sum_{l=0}^{\infty} D_l R^{-l-1} P_l(\cos\theta)$$

$$\Rightarrow \quad D_l = R^{2l+1} A_l$$

$V(r=R) = V_0(\theta)$ is given by the problem.

$$\Rightarrow V_{out}(R) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = V_0(\theta) \quad \text{—— ①}$$

To determine A_l, we use the orthogonality of $P_l(\cos\theta)$

$$\int_0^{\pi} P_l(\cos\theta) P_{l'}(\cos\theta) \sin\theta \, d\theta = \frac{2\delta_{ll'}}{2l+1}$$

Multiplying $\int_0^\pi P_{\ell'}(\cos\theta)\sin\theta\, d\theta$ on the both sides of ①,

$$\int_0^\pi \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta) P_{\ell'}(\cos\theta)\sin\theta\, d\theta = \int_0^\pi V_0 P_{\ell'}(\cos\theta)\sin\theta\, d\theta$$

$$\Rightarrow \sum_{\ell=0}^{\infty} A_\ell R^\ell \frac{2\delta_{\ell\ell'}}{2\ell'+1} = \int_0^\pi V_0 P_{\ell'}(\cos\theta)\sin\theta\, d\theta$$

$$\Rightarrow \quad A_{\ell'} R^{\ell'} \frac{2}{2\ell'+1} = \int_0^\pi V_0 P_{\ell'}(\cos\theta)\sin\theta\, d\theta$$

$$\Rightarrow A_{\ell'} = \frac{2\ell'+1}{2} R^{-\ell'} C_{\ell'}, \text{ where } C_{\ell'} = \int_0^\pi V_0 P_{\ell'}(\cos\theta)\sin\theta\, d\theta$$

Now, we have

$$V_{out} = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2}\left(\frac{r}{R}\right)^{\ell} P_\ell(\cos\theta)$$

$$V_{in} = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2}\left(\frac{r}{R}\right)^{-\ell-1} P_\ell(\cos\theta)$$

Using $\left.\frac{\partial V_{out}}{\partial r}\right|_{r=R} - \left.\frac{\partial V_{in}}{\partial r}\right|_{r=R} = \frac{\sigma(\theta)}{\epsilon_0}$, we obtain

$$\sigma(\theta) = \frac{\epsilon_0}{2R}\sum_{\ell=0}^{\infty}(2\ell+1)^2 C_\ell P_\ell(\cos\theta).$$

Problem 3.22

The geometry of this problem is the same as that of 3.19.
Lets us use the result of 3.19 (see the solution of 3.19)

$$A_\ell = \frac{2\ell+1}{2} R^{-\ell} \int_0^\pi V_0 P_\ell(\cos\theta)\sin\theta\, d\theta \equiv \frac{2\ell+1}{2} R^{-\ell} C_\ell \quad \text{— ①}$$

$$D_\ell = R^{2\ell+1} A_\ell \quad \text{— ②}$$

(where I use the different symbol, D_ℓ, in 3.19, and my D_ℓ is equavilent to B_ℓ in 3.22 and in the textbook. They are the coefficients of the inside potential.)

From 3.19, we know

$$\sigma(\theta) = \frac{\epsilon_0}{2R}\sum_{\ell=0}^{\infty}(2\ell+1)^2 C_\ell P_\ell(\cos\theta) \quad \text{— ③}$$

Substituting C_ℓ for A_ℓ, we obtain

$$\sigma(\theta) = \epsilon_0 \sum_{\ell=0}^{\infty} R^{\ell-1}(2\ell+1) A_\ell P_\ell(\cos\theta) \quad \text{— ④}$$

We exploit the orthogonality by multiplying $\int_0^\pi P_{\ell'}(\cos\theta)\sin\theta\, d\theta$ on the both side of ④

$$\int_0^\pi \sigma(\theta) P_{\ell'}(\cos\theta)\sin\theta\, d\theta = \epsilon_0 \sum_{\ell=0}^{\infty} R^{\ell-1}(2\ell+1)A_\ell \underbrace{\int_0^\pi P_\ell(\cos\theta)P_{\ell'}(\cos\theta)\sin\theta\, d\theta}_{\frac{2\delta_{\ell\ell'}}{2\ell'+1}}$$

$$\Rightarrow \int_0^{\pi} \sigma(\theta) P_{l'}(\cos\theta) \sin\theta \, d\theta = \epsilon_0 \sum_{l=0}^{\infty} R^{l-1} (2l+1) A_l \frac{2\delta_{ll'}}{2l'+1}$$

$$\Rightarrow \int_0^{\pi} \sigma(\theta) P_{l'}(\cos\theta) \sin\theta \, d\theta = 2\epsilon_0 R^{l'-1} A_{l'}$$

$$\Rightarrow A_{l'} = \frac{R^{1-l'}}{2\epsilon_0} \int_0^{\pi} \sigma(\theta) P_{l'}(\cos\theta) \sin\theta \, d\theta \quad \text{— ⑤}$$

To evaluate A_0 to A_6, let's recall

$$\sigma(\theta) = \begin{cases} \sigma_0, & \theta < \frac{\pi}{2} \\ -\sigma_0, & \theta > \frac{\pi}{2} \end{cases}$$

and

$$P_0 = 1, \quad P_1 = x, \quad P_2 = \frac{3x^2-1}{2}, \quad P_3 = \frac{5x^3-3x}{2}, \quad P_4 = \frac{35x^4-30x^2+3}{8}$$

$$P_5 = \frac{63x^5-70x^3+15x}{8}, \quad P_6 = \frac{1}{16}(231x^6-315x^4+105x^2-5).$$

Putting these in ⑤, you can evaluate A_l

$$A_0 = 0, \quad A_1 = \frac{\sigma_0}{\epsilon_0}\left(\frac{1}{2}\right), \quad A_2 = 0, \quad A_3 = \frac{\sigma}{\epsilon_0 R^2}\left(-\frac{1}{8}\right), \quad A_4 = 0, \quad A_5 = \frac{\sigma_0}{\epsilon_0 R^4}\left(\frac{1}{16}\right), \quad A_6 = 0$$

To get D_l (or B_l), we use ②

$$B_0 = 0, \quad B_1 = \frac{\sigma_0 R^3}{\epsilon_0}\left(\frac{1}{2}\right), \quad B_2 = 0, \quad B_3 = \frac{\sigma_0 R^5}{\epsilon_0}\left(-\frac{1}{8}\right), \quad B_4 = 0, \quad B_5 = \frac{\sigma_0 R^7}{\epsilon_0}\left(\frac{1}{16}\right), \quad B_6 = 0$$

You may notice all the even terms are zeros. That's because.

$P_l(\cos\theta) = P_l(\cos(\pi-\theta))$, for l is even.

$\left(P_l(-x) = (-1)^l P_l(x) \right)$

Problem 3.37.

The potentials satisfy Laplacian equation in each region except the two surface. Thus, we can use eq 3.65 to describe potential in each region.

Let

$$V_1(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + B_l r^{-l-1}\right) P_l(\cos\theta) \quad \text{—— ①}$$

be the potential for $a \le r \le b$

$$V_2(r,\theta) = \sum_{l=0}^{\infty}\left(C_l r^l + D_l r^{-l-1}\right) P_l(\cos\theta)$$

be the potential for $r \geq b$

We assume $V_2(\infty, \theta) = 0$, so $C_l = 0$ and

$$V_2(r,\theta) = \sum_{l=0}^{\infty} D_l r^{-l-1} P_l(\cos\theta) \quad \text{—— ②}$$

It's time to apply the boundary conditions to find A_l, B_l, and D_l.

We need three equations to make A_l, B_l, and D_l unique.

We know according to the problem.

$$V_1(r=a, \theta) = V_0 \quad \text{—— ③}$$

$$\left.\frac{\partial V_2}{\partial r}\right|_{r=b} - \left.\frac{\partial V_1}{\partial r}\right|_{r=b} = \frac{\sigma(\theta)}{\epsilon_0} = \frac{k\cos\theta}{\epsilon_0} = \frac{k}{\epsilon_0} P_1(\cos\theta) \quad \text{—— ④}$$

⑤

The last equation is that at $r=b$, $V_1(b,\theta) = V_2(b,\theta)$. (Why? the discontinuity of the potential at space means infinite electric field, which is impossible for finite surface density)

Applying ③ to V_1, we obtain

$$V_0 = \sum_{l=0}^{\infty} \left(A_l a^{l} + B_l a^{-l-1}\right) P_l(\cos\theta)$$

Multiplying $\int_0^{\pi} P_{l'}(\cos\theta)\sin\theta \, d\theta$ on the both sides, we exploit the orthogonality.

(eq. 3.68)

$$V_0 \int_0^{\pi} P_{l'}(\cos\theta)\sin\theta \, d\theta = \left(A_{l'} a^{l'} + B_{l'} a^{-l'-1}\right) \times \frac{2}{2l'+1}$$

$$\Rightarrow A_l a^{l} + B_l a^{-l-1} = \left(\frac{2l+1}{2}\right) V_0 \int_0^{\pi} P_l(\cos\theta)\sin\theta \, d\theta$$

$$\Rightarrow A_l a^{l} + B_l a^{-l-1} = \left(\frac{2l+1}{2}\right) V_0 \int_0^{\pi} P_l(\cos\theta)\cdot P_0(\cos\theta)\sin\theta \, d\theta$$

$P_0(x) = 1$

$$\Rightarrow A_l a^{l} + B_l a^{-l-1} = \left(\frac{2l+1}{2}\right) V_0 \frac{2\delta_{l0}}{2l+1}$$

the tricky step

$$\Rightarrow \begin{cases} A_0 + B_0 a^{-1} = V_0 \\ A_l a^{l} + B_l a^{-l-1} = 0, \quad \text{for } l \neq 0 \end{cases} \qquad \text{———— ⑥}$$

Applying ⑤ to V_1 and V_2

$$\sum_{l=0}^{\infty} \left(A_l b^{l} + B_l b^{-l-1}\right) P_l(\cos\theta) = \sum_{l=0}^{\infty} D_l b^{-l-1} P_l(\cos\theta)$$

$$\Rightarrow A_l b^{l} + B_l b^{-l-1} = D_l b^{-l-1} \qquad \text{———— ⑦}$$

Applying ④ to V_1 and V_2,

$$\sum_{l=0}^{\infty}\left(l A_l b^{l-1}+(-l-1)B_l b^{-l-2}\right)P_l(\cos\theta)-\sum_{l=0}^{\infty}(-l-1)D_l b^{-l-2}P_l(\cos\theta)=\frac{k}{\epsilon_0}P_1(\cos\theta)$$

$$\Rightarrow \begin{cases} A_1-2B_1 b^{-3}+2D_1 b^{-3}=\frac{k}{\epsilon_0} \\ l A_l b^{l-1}+(-l-1)B_l b^{-l-2}+(l+1)D_l b^{-l-2}=0,\ \text{for } l\neq 1 \end{cases} \quad — ⑧$$

Using ⑥, ⑦ and ⑧ for $l\neq 0$, $l\neq 1$.

$$\begin{cases} A_l a^l+B_l a^{-l-1}=0 \\ A_l b^l+B_l b^{-l-1}=D_l b^{-l-1} \\ A_l b^l+\frac{(-l-1)}{l}B_l b^{-l-1}=\frac{(-l-1)}{l}D_l b^{-l-1} \end{cases}$$

You can show $A_l=B_l=D_l=0$ is the only solution.

For $l=0$, $\begin{cases} A_0+\frac{B_0}{a}=V_0 \\ A_0+\frac{B_0}{b}=\frac{D_0}{b} \\ -B_0+D_0=0 \end{cases} \Rightarrow \begin{cases} A_0=0 \\ B_0=D_0=aV_0 \end{cases}$

For $l=1$, $\begin{cases} A_1 a+B_1 a^{-2}=0 \\ A_1 b+B_1 b^{-2}=D_1 b^{-2} \\ A_1 b-2B_1 b^{-2}=-2D_1 b^{-2}+\frac{k}{\epsilon_0}b \end{cases} \Rightarrow \begin{cases} A_1=\frac{k}{3\epsilon_0} \\ B_1=-\frac{a^3 k}{3\epsilon_0} \\ D_1=\frac{k}{3\epsilon_0}(b^3-a^3) \end{cases}$

Putting the non-vanishing coefficients B_0, D_0, A_1, B_1, D_1 in ① and ②

$$V_1=\frac{k}{3\epsilon_0}r\cos\theta+\frac{aV_0}{r}-\frac{a^3k}{3\epsilon_0 r^2}\cos\theta,\quad a\leq r\leq b$$

$$V_2=\frac{aV_0}{r}+\frac{k}{3\epsilon_0}(b^3-a^3)\frac{\cos\theta}{r^2},\quad r\geq b$$

Problem 4-2

$$\int \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0} = \frac{1}{\epsilon_0}\int_0^r \rho(r)\, 4\pi r'^2 dr'$$

$$\Rightarrow E \times 4\pi r^2 = \frac{1}{\epsilon_0}\int_0^r \rho(r)\, 4\pi r'^2 dr', \quad \rho(r') = \frac{q}{\pi a^3} e^{-2r/a}$$

$$\Rightarrow E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\left[1 - e^{-2r/a}\left(1 + 2\frac{r}{a} + 2\frac{r^2}{a^2}\right)\right].$$

Let the proton be shifted by the external field E_e by a distance d.

$$E_e = \frac{q}{4\pi\epsilon_0}\frac{1}{d^2}\left[1 - e^{-2d/a}\left(1 + 2\frac{d}{a} + 2\frac{d^2}{a^2}\right)\right]$$

$$\simeq \frac{q}{4\pi\epsilon_0}\frac{4}{3a^2}(qd) = \frac{p}{3\pi\epsilon_0 a^3} \equiv \frac{1}{\alpha} p$$

$$\Rightarrow \alpha = 3\pi\epsilon_0 a^3$$

Problem 4-7

$$\vec{N} = \vec{p}\times\vec{E}$$

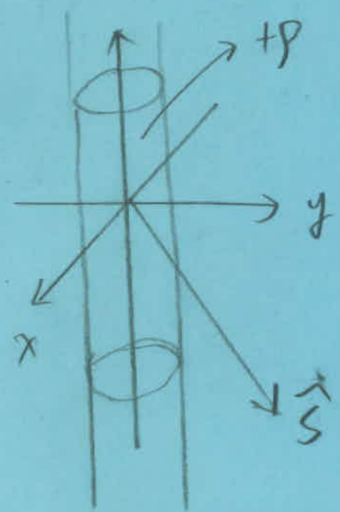
$$\Rightarrow N = pE\sin\theta$$

$$U = \int N d\theta = \int pE\sin\theta\, d\theta = -pE\cos\theta = -\vec{p}\cdot\vec{E}$$

Problem 4-13

Consider a cylinder of uniform charge density $+\rho$

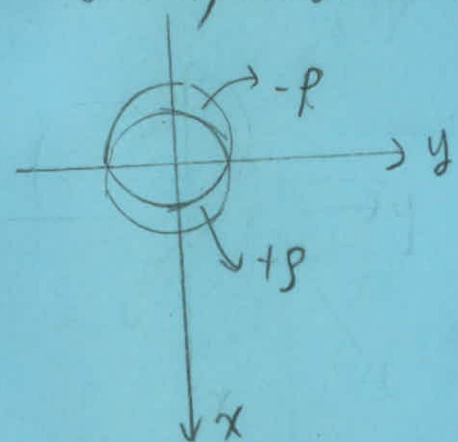
Using Gauss's law, you can obtain the $\vec{E}$ field



$$\vec{E}_+ = \begin{cases} \dfrac{\rho s}{2\epsilon_0}\hat{s}, & s<a \\ \dfrac{\rho a^2}{2\epsilon_0}\dfrac{\hat{s}}{s}, & s>a \end{cases}$$

The cylinder in this problem can be thought as two uniform cylinders with $\pm\rho$ separated by a small distance d.

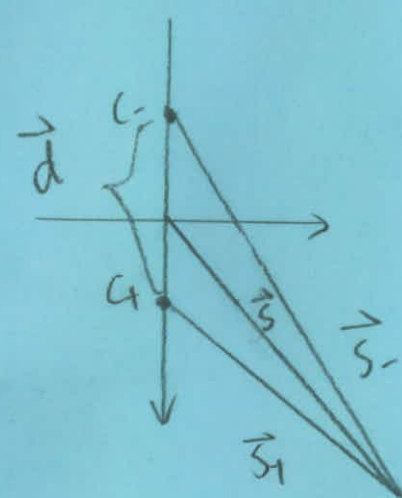
$$\rho\vec{d} = \vec{p}$$



The $\vec{E}$ field is

$$\vec{E} = \vec{E}_+ + \vec{E}_-$$

$$= \begin{cases} \left[\dfrac{\rho \vec{s}_+}{2\epsilon_0} + \left(\dfrac{-\rho\vec{s}_-}{2\epsilon_0}\right)\right], & s<a \\ \left[\left(\dfrac{\rho a^2}{2\epsilon_0}\dfrac{\hat{s}_+}{s_+}\right) + \left(\dfrac{-\rho a^2}{2\epsilon_0}\dfrac{\vec{s}_-}{s_-}\right)\right], & s>a \end{cases}$$



$$\vec{s}_+ = \vec{s} - \frac{\vec{d}}{2}$$

$$\vec{s}_- = \vec{s} + \frac{\vec{d}}{2}$$

$$\Rightarrow \vec{E} = \begin{cases} \frac{\rho}{2\epsilon_0}(-\vec{d}), & s<a \\ \frac{\rho a^2}{2\epsilon_0}\left(\frac{\hat{s}_+}{s_+} - \frac{\hat{s}_-}{s_-}\right), & s>a \end{cases}$$

$$\boxed{\begin{aligned} \frac{\hat{s}_+}{s_+} &= \frac{\vec{s}_+}{s_+^2} = \frac{\vec{s} - \frac{\vec{d}}{2}}{s^2 - \vec{d}\cdot\vec{s} + \frac{d^2}{4}} \\ &= \frac{1}{s}\cdot\frac{\hat{s} - \frac{\vec{d}}{2s}}{1 - \frac{\vec{d}\cdot\hat{s}}{s} + \frac{d^2}{4s^2}} \\ &\approx \frac{1}{s}\left(\hat{s} - \frac{\vec{d}}{2s}\right)\left(1 + \frac{\vec{d}\cdot\hat{s}}{s}\right) \approx \frac{1}{s}\left[\hat{s} + \left(\frac{\vec{d}\cdot\hat{s}}{s}\right)\hat{s} - \frac{\vec{d}}{2s}\right] \\ \frac{\hat{s}_-}{s_-} &\approx \frac{1}{s}\left[\hat{s} - \left(\frac{\vec{d}\cdot\hat{s}}{s}\right)\hat{s} + \frac{\vec{d}}{2s}\right] \end{aligned}}$$

$$\Rightarrow \vec{E} = \begin{cases} \frac{-\rho}{2\epsilon_0}\vec{d}, & s<a \\ \frac{\rho a^2}{2\epsilon_0}\frac{1}{s}\left[2\left(\frac{\vec{d}\cdot\hat{s}}{s}\right)\hat{s} - \frac{\vec{d}}{s}\right], & s>a \end{cases}$$

$$\boxed{\rho\vec{d} = \vec{P}}$$

$$= \begin{cases} \frac{-\vec{P}}{2\epsilon_0}, & s<a \\ \frac{a^2}{2\epsilon_0}\frac{1}{s^2}\left[2(\vec{P}\cdot\hat{s})\hat{s} - \vec{P}\right], & s>a. \end{cases}$$

Problem 4-8

$$U = -\vec{P}_1 \cdot \vec{E}_2$$

$\vec{P}_1 \downarrow$ dipole 1 $\qquad \vec{E}_2 \searrow$ dipole 2.

$$\vec{E}_2 = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(\vec{P}_2\cdot\hat{r})\hat{r} - \vec{P}_2\right]$$

$$\Rightarrow \quad U = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[\vec{P}_1\cdot\vec{P}_2 - 3(\vec{P}_1\cdot\hat{r})(\vec{P}_2\cdot\hat{r})\right].$$

Problem 4-10

(a) $\sigma_b = \vec{P}\cdot\hat{n} = \vec{P}\cdot\hat{r} = kR$

$$\rho_b = -\nabla\cdot\vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 kr) = -3k$$

(b) $\int \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$

For $r < R$, $Q_{enc} = \int_0^r -3k\,4\pi r'^2\,dr' = -4\pi r^3 k$

$$\Rightarrow \quad E \times 4\pi r^2 = \frac{1}{\epsilon_0}(-4\pi r^3 k) \Rightarrow E = -\left(\frac{k}{\epsilon_0}\right) r$$

$$\vec{E} = -\left(\frac{k}{\epsilon_0}\right)\vec{r}$$

For $r > R$, $Q_{enc} = 0$

$$\Rightarrow \quad \vec{E} = 0$$

Problem 4-17

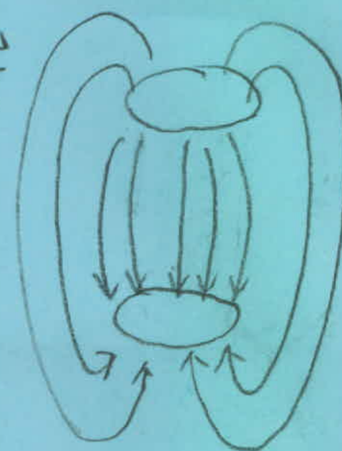
$\vec{P}$ is uniform.

$\sigma_b = \vec{P} \cdot \hat{n} \Rightarrow$

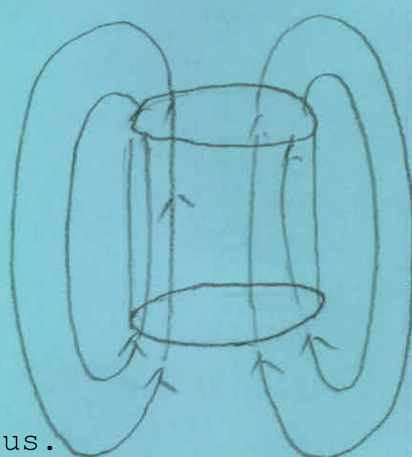
$\sigma_b = +P$

$\sigma_b = -P$

$\Rightarrow$ $\vec{E}$



$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$



$\vec{\nabla} \cdot \vec{D} = 0$

$\Rightarrow \vec{D}$ is continuous.

~~continous.~~

Problem 4.16

(a) $\vec{E} = \vec{E}_0 + \vec{E}_p$, where $\vec{E}_p$ is the field by a sphere

$$\vec{E}_{-p} = \frac{\vec{P}}{3\epsilon_0} \quad (\text{eq } 4.14)$$

$$\Rightarrow \vec{E} = \vec{E}_0 + \frac{\vec{P}}{3\epsilon_0}$$

$$\vec{D} = \epsilon_0 \vec{E} + 0 = \epsilon_0 \vec{E}_0 + \frac{1}{3}\vec{P} = D_0 - \frac{2}{3}\vec{P}$$

(b) $\vec{E} = \vec{E}_0 + \vec{E}_{-p}$

$$= \vec{E}_0 + 0 = \vec{E}_0$$

$$\vec{D} = \epsilon_0 \vec{E} = \epsilon \vec{E}_0 = \vec{D}_0 - \vec{P}$$

(c) $\vec{E} = \vec{E}_0 + \vec{E}_{-p}$

$$= \vec{E}_0 + \frac{\vec{P}}{\epsilon_0}$$

$$\vec{D} = \epsilon_0 \vec{E} + 0 = \epsilon_0 \vec{E} + \vec{P} = \vec{D}_0$$