Lect 9 Electric Potential

Since $\nabla \times \vec{E} = 0$ (or $\oint \vec{E} \cdot d\vec{l} = 0$) for electro-static potential, we can express $\vec{E} = -\nabla V$, where V is a scalar function: electric potential

$$V(r) = -\int_{O}^{r} \vec{E} \cdot d\vec{r}$$, O is a reference point, which can

choosen as an arbitary point. Different choice of O gives the same $V(r)$ up to a const.

In many cases, but not always, we choose O to the infinity. The potential difference between 2 point a and b, is independent of the reference point O

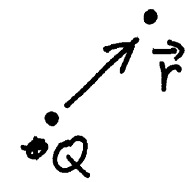
$$\boxed{V(a) - V(b) = -\int_{b}^{a} \vec{E} \cdot d\vec{l} = \int_{b}^{a} \nabla V \, d\vec{l}}$$

Comment: ① $V(\vec{r})$ is a scaler function, which is much simpler to use than $\vec{E}$. Actually, $\vec{E}$ is a special vector field with $\nabla \times \vec{E} = 0$. The vector form of $\vec{E}$ is not most maximum advatage form.

② For other formalism of mechanics and quantum mechanics, the concept of force is no longer important. Potentials, scalor or vector, are much more convenient.

Since electric fields satisfy super-position principle, the potentials also satisfy the super-position principle. $V = V_1 + V_2 + \cdots$

1) potential of a point charge Q

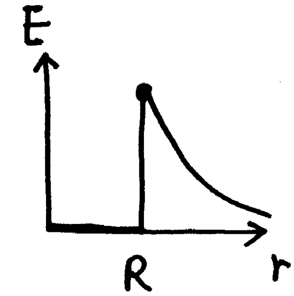


$$V(r) = -\int_{+\infty}^{r} d\vec{r}\cdot\vec{E} = \int_{r}^{+\infty} dr\,\frac{Q}{r^2} = \frac{Q}{r}$$

2) potential of a uniformaly charge ~~surfa~~ sphere with radius R and total charge Q.

outside the sphere, $\vec{E} = \frac{Q}{r^2}\hat{e}_r$

inside the sphere $\oint \vec{E}\cdot d\sigma = 0 \Rightarrow \vec{E} = 0$



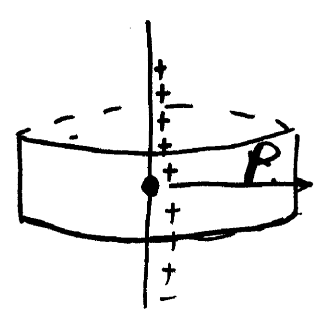
$$\Rightarrow V(r) = -\int_{+\infty}^{r} d\vec{r}\cdot\vec{E} = \int_{r}^{+\infty} d\vec{r}\cdot\vec{E}$$

for $r > R \Rightarrow V(r) = \frac{Q}{r}$

for $r < R \Rightarrow V(r) = \int_r^R 0\cdot dr + \int_R^{+\infty} dr\,E = \frac{Q}{R}$.

3) Potential of a uniformally charged line with charge density λ

① we can use Gauss's law and symm analysis to get electric field. Use cylindrical coordinate



① $\vec{E}$ is along the radial direction, and only depends on ρ. Can you provide the symm analysis? (translation + rotation + reflection

$\Rightarrow \oint \vec{E}\cdot d\vec{a} = 4\pi Q \Rightarrow E\cdot 2\pi\rho h = 4\pi h\lambda \Rightarrow \vec{E} = \frac{2\lambda}{\rho}\hat{e}_\rho$

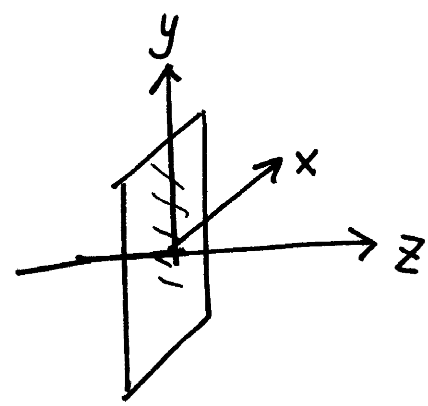
if we can the reference point at $\rho = \rho_0$

$$\Rightarrow V = -\int_{\rho_0}^{\rho} d\vec{r}\cdot\vec{E} = \int_{\rho}^{\rho_0} \frac{2\lambda}{\rho}\cdot d\rho = 2\lambda \ln\frac{\rho_0}{\rho}$$

For two points with radiams ρ_A and ρ_B

$$V_A - V_B = 2\lambda \ln \rho_B/\rho_A .$$

4) potential of a charged plane with the charge density σ.



we have $\vec{E}(\vec{r}) = \begin{cases} 2\pi\sigma\,\hat{z} & \text{for } z>0 \\ -2\pi\sigma\,\hat{z} & \text{for } z<0. \end{cases}$

Set $V(z=0) = 0$,

$$\Rightarrow V(z) = -\int_0^z d\vec{r}\cdot\vec{E} = -\int_0^{|z|} dz\ 2\pi\sigma = -2\pi\sigma|z|$$

§ Poisson and Laplace equation

$$\left.\begin{aligned} &\vec{E} = -\nabla V \Leftarrow \nabla\times\vec{E} = 0 \\ &\nabla\cdot\vec{E} = 4\pi\rho \end{aligned}\right\}$$ basic equations of electro-statics

$\Rightarrow \boxed{\nabla^2 V = -4\pi\rho} \Leftarrow$ Poisson equation.

if $\rho = 0 \Rightarrow \nabla^2 V = 0 \Leftarrow$ Laplace equation.

for a charge distribution $\rho(r)$, we know the electric field distribution $\vec{E}(\vec{r}) = \int \frac{\rho(r')\, d^3r'}{|\vec{r}-\vec{r}'|^3} (\vec{r}-\vec{r}')$

$$\Rightarrow \boxed{V(\vec{r}) = -\int_{+\infty}^{\vec{r}} d\vec{r}' \cdot \vec{E}(\vec{r}') = \int \frac{\rho(r')}{|\vec{r}-\vec{r}'|} d^3r'}$$

we can also check $\nabla^2 \frac{1}{r} = -\nabla \cdot \frac{\hat{e}_r}{r^2} = -4\pi\, \delta(\vec{r})$

$$\Rightarrow \nabla^2 V(r) = \int \rho(\vec{r}')\, \nabla^2 \frac{1}{|\vec{r}-\vec{r}'|} d^3r' = -4\pi \int \rho(\vec{r}')\, \delta(\vec{r}-\vec{r}')\, d^3r'$$

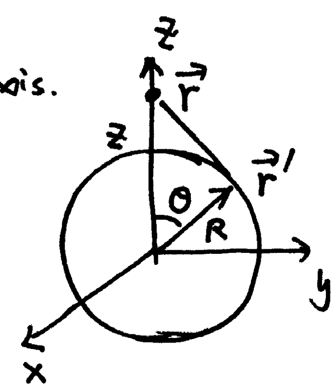
$$= -4\pi \rho(r)$$

Ex: find electric potential for uniformly charge sphere with Radial R and charge Q:

$\sigma = \frac{Q}{4\pi R^2}$. we use $V(\vec{r}) = \int \frac{\sigma}{|\vec{r}-\vec{r}'|} da$

without loss of generality, we set $\vec{r}$ along the z-axis.

$\vec{r} = z\, \hat{e}_z$



$$\Rightarrow V(\vec{r} = z\hat{e}_z) = \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \frac{\sigma \cdot R^2 \sin\theta\, d\theta\, d\varphi}{\sqrt{z^2 + R^2 - 2zR\cos\theta}}$$

$$= 2\pi\sigma \int_{-1}^{1} d\cos\theta \frac{R^2/\sqrt{2zR}}{\sqrt{\frac{z^2+R^2}{2zR} - \cos\theta}} \quad \leftarrow \int \frac{dx}{\sqrt{a-x}} = -2\sqrt{a-x} + C$$

$$\Rightarrow V(\vec{r}) = 2\pi\sigma \frac{R^2}{\sqrt{2zR}} (-z) \left[\sqrt{\frac{z^2+R^2}{2zR} - 1} - \sqrt{\frac{z^2+R^2}{2zR} + 1} \right]$$

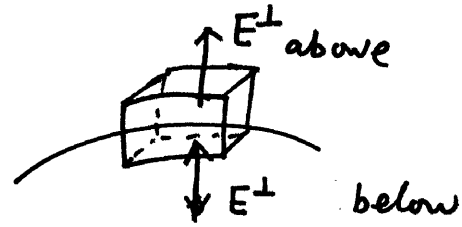
$$= \frac{4\pi\sigma R^2}{2zR} \Big[|z+R| - |z-R| \Big] = \begin{cases} \frac{4\pi\sigma R^2}{2zR} \left(z+R-(z-R) \right) & \text{for } (z>R) \\ \frac{4\pi\sigma R^2}{2zR} \left(z+R \cancel{+} -(R-z) \right) & \text{for } (z<R) \end{cases}$$

$$= \begin{cases} \frac{Q}{z} & \text{for } z>R \\ \frac{Q}{R} & \text{for } z<R. \end{cases}$$

§ boundary condition :

$$\boxed{\begin{cases} \nabla^2 V = -4\pi\rho(\vec{r}) & \text{differential equation} \\ + \text{ boundary conditions.} \end{cases} \\ \text{a complete problem}}$$

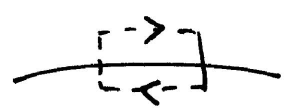
consider a boundary of a surface, which carry a surface charge density σ. ←



$$(E_{\perp,\text{above}} - E_{\perp,\text{below}}) \cdot A = 4\pi A \cdot \sigma$$

$$\Rightarrow \boxed{E_{\perp,\text{above}} - E_{\perp,\text{below}} = 4\pi\sigma} \leftarrow \text{ discontinuity}$$

$$\oint \vec{E} \cdot d\vec{l} = 0 \Rightarrow \boxed{E_{\parallel,\text{above}} = E_{\parallel,\text{below}}.}$$



E satisfied 1rst diff Eq

V : 2nd diff Eq.

if we use potential $\Rightarrow$ $\boxed{V_{\text{above}} - V_{\text{below}} = -\int_{\text{below}}^{\text{above}} E \cdot dl = 0}$

but $E_\perp = -\frac{\partial V}{\partial n}$

$$\Rightarrow \boxed{\left.\frac{\partial V}{\partial n}\right|_{\text{above}} - \left.\frac{\partial V}{\partial n}\right|_{\text{below}} = -4\pi\sigma}$$