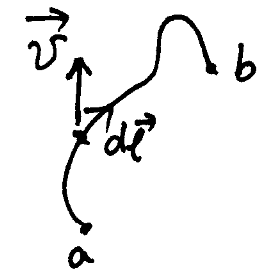
Lect 3 Integral Calculus

§ Line integrals — path



$$\int_{a\,P}^{b} \vec{v} \cdot d\vec{l}$$

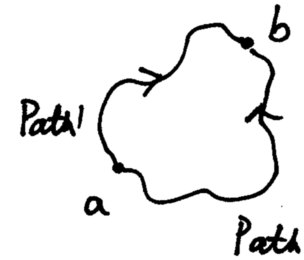
if the path forms a closed loop → $\oint_P \vec{v} \cdot d\vec{l}$.

Example: work done by following a path $W = \int_a^b \vec{F} \cdot d\vec{l}$

Generally speaking, the integral $\int_a^b \vec{v} \cdot d\vec{l}$ depends on the path.

For a special class of vector field, such integrals are path-independent, which means for a closed loop, the integral $= 0$.

$$\int_{a\,P_1}^{b} \vec{v} \cdot d\vec{l} = \int_{a\,P_2}^{b} \vec{v} \cdot d\vec{l}$$



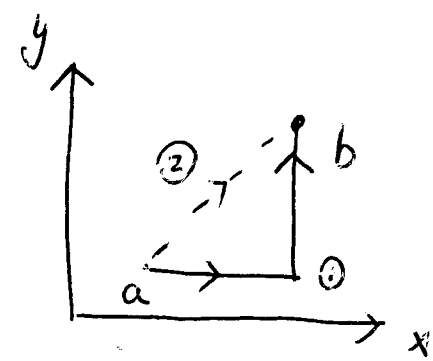
$$\Rightarrow \oint_{P_1 - P_2} \vec{v} \cdot d\vec{l} = 0$$

For force field satisfying this properties, we call it conservative force.

The static gravity, eletrostatic force, .. etc are conservative force!

Example: $\vec{v} = y^2 \hat{x} + 2x(y+1)\hat{y}$,

$a(1,1)$, $b(2,2)$



following path 1

$$\int \vec{v}\cdot d\vec{l} = \int_{(1,1)\to(2,1)} v_x\,dx + \int_{(2,1)\to(2,2)} v_y\,dy = \int_1^2 dx + \int_1^2 4(y+1)\,dy$$

$$= 1 + (2y^2+4y)\Big|_1^2 = 1 + (16-6) = 11$$

following path 2 : $\int \vec{v}\cdot d\vec{l} = \int_1^2 v_x dx + \int_1^2 v_y dy$

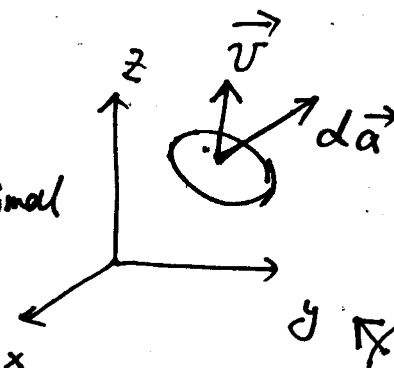
$$= \int_1^2 x^2 dx + \int_1^2 2y(y+1)\,dy = \frac{x^3}{3}\Big|_1^2 + \frac{2y^3}{3} + y^2\Big|_1^2$$

plug in $x=y$

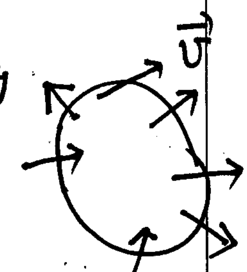
$$= \frac{8-1}{3} + \frac{8-1}{3}\times 2 + 4-1 = 7+3 = 10$$

if following ① and reverse ② and come back to a $\Rightarrow \oint \vec{v}\cdot d\vec{l} = 1$

§2. surface integrals



$\int_S \vec{v}\cdot d\vec{a}$, $d\vec{a}$ is an infinitesimal area, the direction is along the normal direction.



for a close surface

$\oint \vec{v}\cdot d\vec{a}$, the normal direction: from inside to outside.

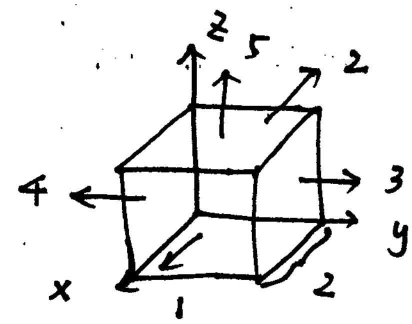
If $\vec{v}$ describe a flow of a liquid, then $\oint \vec{v}\cdot d\vec{a}$ is the flux.

Ex 1.7 $\vec{v} = 2xz\,\hat{x} + (x+2)\,\hat{y} + y(z^2-3)\,\hat{z}$

a cube with edge length 2.

the 5 - surfaces (except the bottom)

form a big surface, calculate $\int \vec{v}\cdot d\vec{a}$



$\int \vec{v}\cdot d\vec{a} = \int_1 + \int_2 + \cdots \int_5$

$\int_1 \vec{v}\cdot d\vec{a} = \int v_x \, dy dz = \int 4z \, dy dz = \int_0^2 dy \int_0^2 4z dz = 2 \cdot 2z^2 \Big|_0^2 = 16$

set $x=2$

$\int_2 \vec{v}\cdot d\vec{a} = -\int v_x \, dy dz = 0$ set $x=0$

$\int_3 \vec{v}\cdot d\vec{a} = \int v_y \, dx dz = \int_0^2 (x+2) dx \int_0^2 dz = \left(\frac{x^2}{2} + 2x\right)\Big|_0^2 \cdot 2 = 12$

set $y=2$

$\int_4 \vec{v}\cdot d\vec{a} = -\int v_y \, dx dz$ set $y=2$. $= -\int_0^2 (x+2) dx \int_0^2 dz = -12$

$\int_5 \vec{v} d\vec{a} = \int v_z \, dx dy$ set $z=2$ $= \int_0^2 \int_0^2 y \, dx dy = 2 \cdot \frac{y^2}{2}\Big|_0^2 = 4$

$\Rightarrow \int_1 + \cdots \int_5 = 16 + 4 = 20$

§ volume integrals $\int_{\mathcal{V}} T \, d\tau$ where $d\tau = dx\,dy\,dz$

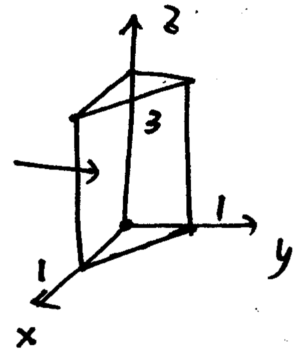
T is a scalar function

sometimes we also calculate volume integral of vectors

$$\int \vec{v}\, d\tau = \left(\int v_x \, d\tau\right)\hat{x} + \left(\int v_y \, d\tau\right)\hat{y} + \left(\int v_z \, d\tau\right)\hat{z}$$

Ex 1.8 :

$T = xyz^2$, $\int_{\mathcal{V}} T \, d\tau$ for the prism



$$\int_0^3 dz \int_0^{1-x} dy \int_0^1 dx \quad xyz^2$$

$$= \int_0^3 z^2 dz \int_0^1 x\,dx \int_0^{1-x} y\,dy = \left.\frac{z^3}{3}\right|_0^3 \cdot \int_0^1 dx \cdot x \, \frac{(1-x)^2}{2}$$

$$= \frac{9}{2} \cdot \int_0^1 dx \; (x - 2x^2 + x^3) = \frac{9}{2} \cdot \left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right] = \frac{9}{2} \cdot \frac{1}{12} = \frac{3}{8}$$