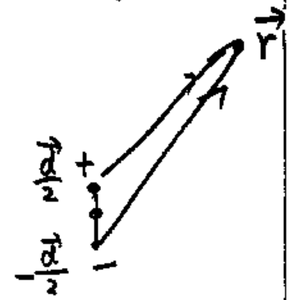
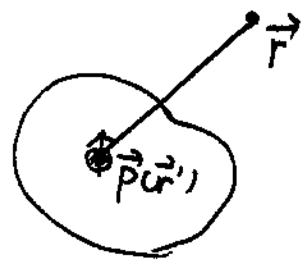
Lect 18 Field of a Polarized Object

Suppose that we know the distribution of $\vec{P}(\vec{x})$, how is the eletric field /potential generated by $\vec{P}(\vec{x})$? we will study such a question.



For a single dipole
$$V(\vec{r}) = \frac{q}{|\vec{r}-\vec{d}/2|} - \frac{q}{|\vec{r}+\frac{\vec{d}}{2}|}$$
$$= q\,\nabla\frac{1}{r}\cdot(-\vec{d}) = -\vec{P}\cdot\nabla\frac{1}{r}$$
$$= \frac{\vec{P}\cdot\hat{r}}{r^2}$$

Then we apply to a distribution of $\vec{P}(\vec{r})$, by replacing $\vec{P}$ with $\vec{P}(r)d\tau$



$$\Rightarrow\quad V(\vec{r}) = \int_V \frac{(\vec{r}-\vec{r}')\cdot\vec{P}(\vec{r}')}{|\vec{r}-\vec{r}'|^3}\,d\tau'$$
$$= \int \vec{P}(\vec{r}')\cdot\nabla'_{r'}\frac{1}{|\vec{r}-\vec{r}'|}\,d\tau'$$
$$= \int \frac{-\nabla_{r'}\cdot\vec{P}(\vec{r}')}{|\vec{r}-\vec{r}'|}\,d\tau' + \int \nabla'\left(\frac{\vec{P}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right)d\tau'$$
$$= \int d\tau' \;\frac{-\nabla_{r'}\cdot\vec{P}(\vec{r}')}{|\vec{r}-\vec{r}'|} + \oint_S \frac{\vec{P}(\vec{r}')\cdot d\vec{a}'}{|\vec{r}-\vec{r}'|}$$

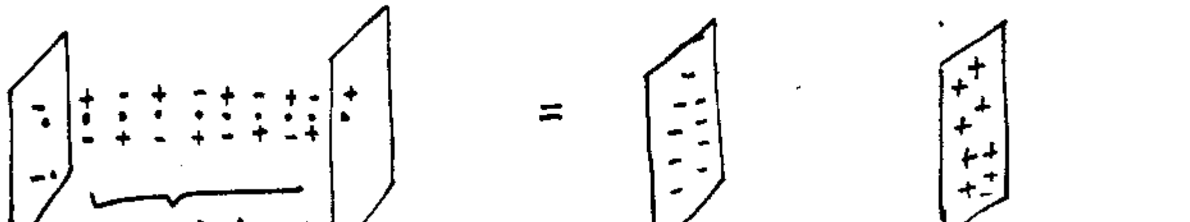
we can intepreate it as $V(\vec{r})$ is generated by a body charge density and surface charge density

bound charges

$$V(r) = \int d\tau'\,\frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} + \oint_S \frac{\sigma(\vec{r}')\,da'}{|\vec{r}-\vec{r}'|}, \quad \text{where } \boxed{\begin{aligned}\rho(\vec{r}') &= -\nabla\cdot\vec{P}(r)\\ \sigma(\vec{r}') &= \vec{P}(r)\cdot\hat{n}\end{aligned}}$$

Interpretation of bound charges

$\longrightarrow z$



canceled out

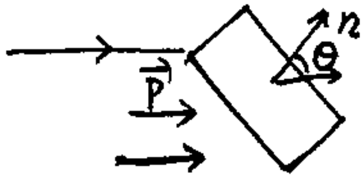
uniform $\vec{P}$

$\rho = \nabla \cdot \vec{P} = 0$

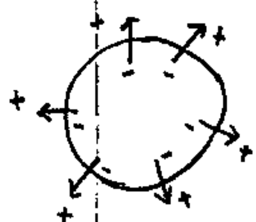
$\sigma = \vec{P} \cdot (-\hat{z}) = -P$

$\sigma = \vec{P} \cdot \hat{z} = P$

if for



$\Rightarrow \sigma_b = \vec{P} \cdot \hat{n} = P\cos\theta$



$Q = \int \rho_b \, d\tau = -\oint \vec{P} \cdot da \Rightarrow \rho_b = -\nabla \cdot \vec{P}$

Exercise: electric field distribution of a uniformly charged ball

Solution: $\rho_b = -\nabla \cdot \vec{P} = 0$

$\sigma_b = \vec{P} \cdot \hat{n} = P\cos\theta.$

As shown before, this can be mapped to the solution of Laplace equation $\nabla^2 V = 0$, subject to the boundary condition $\sigma(\theta) = P\cos\theta$ on the sphere. Only the $l=1$ component exist as

$$V(r,\theta) = \begin{cases} \dfrac{4\pi P}{3} r\cos\theta & r \le R \\ \dfrac{4\pi P}{3} \dfrac{R^3}{r^2}\cos\theta & r \ge R. \end{cases}$$

check

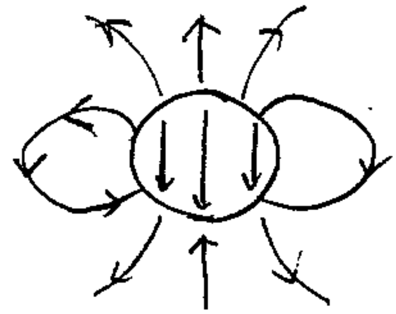
$$\Rightarrow \left.\frac{\partial V}{\partial r}\right|_{r=R^+} - \left.\frac{\partial V}{\partial r}\right|_{r=R^-} = \frac{4\pi P}{3} \cdot 3\cos\theta = 4\pi\sigma$$

$$\Rightarrow \sigma(\theta) = P\cos\theta.$$

The field inside the ball is uniform $\vec{E} = -\nabla V = -\frac{4\pi}{3}\vec{\mathbb{P}}$.

The potential outside is a dipole-like $V = \dfrac{\vec{p}\cdot\hat{r}}{r^2}$ with $\vec{p} = \frac{4}{3}\pi R^3 \vec{\mathbb{P}}$.

The field at $r > R$ $\quad \vec{E}_{dip}(r,\theta) = \dfrac{p}{r^3}(2\cos\theta\,\hat{e}_r + \sin\theta\,\hat{e}_\theta)$



* Justification of using $\left.\begin{array}{l}\rho_b = -\nabla\cdot\vec{\mathbb{P}}\\ \sigma_b = \vec{p}\cdot\hat{n}\end{array}\right\}$

to calculate the field. Clearly it's fine to use (ρ_b, σ_b) to describe the effects of induced charge for points outside the material.

$\vec{\mathbb{P}}$ ~~is~~ can only be defined as an average over a micro-scopic-ly large but macroscopically small area.

$$\vec{\mathbb{P}}(\vec{r}) = \sum_{r_i \text{ close to } \vec{r}} \vec{P}_i / \Delta V .$$

$\underline{\Delta V \text{ is large}}$ so that $\vec{\mathbb{P}}(\vec{r})$ is smooth by average the micro-scopic $\vec{P}_i$, which fluctuates strongly

and is also small so that $\vec{\mathbb{P}}(\vec{r})$ doesn't vary much.

For points outside the material, naturally it is fine to use the smooth ρ_b and σ_b to represent the effect of media.

However, for points inside the media, microscopicly $\vec{E}_{mi}$ can be very large if $\vec{r}$ is close to electrons, but it fluctuats strongly. We are not ~~interested~~ interested in its detailed distribution. Instead, we use a coarse average definition $\vec{E}(\vec{r}) = \int_{\text{close to } \vec{r}} \vec{E}_{mi}(\vec{r}')d\vec{r}' / \Delta V$. this macroscopic field is what we are interested

This smoothrized fields $\vec{E}(\vec{r})$ are what really describid by σ_b and ρ_b.

We will introduce a new quantity to describle electricity in the media.

* electric Displacement

$$\vec{D} = \vec{E} + 4\pi\vec{P}$$

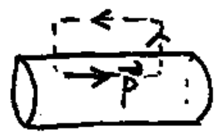
$$\nabla\cdot\vec{D} = \nabla\cdot\vec{E} + 4\pi\nabla\cdot\vec{P} = 4\pi(\rho_f + \rho_b) - 4\pi(\rho_b) = 4\pi\rho_f$$

so $\vec{D}$ is determined purely from ρ_f.

The Gauss's law for $\vec{D}$: $\oint \vec{D}\cdot d\vec{a} = 4\pi Q_f$.

however, even for electro-statics, $\nabla\times\vec{D} = 4\pi\,\nabla\times\vec{P} \neq 0$ in general.

there's no potential for $\vec{D}$.

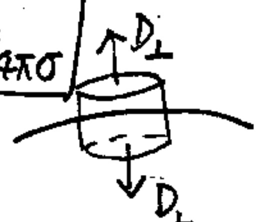


$\oint \vec{P}\cdot d\vec{l} \neq 0$

boundy conditions for $\vec{D}$.

$$\boxed{D_{\perp,\text{above}} - D_{\perp,\text{below}} = 4\pi\sigma_f}.$$

c.f. $\boxed{E_{\perp,\text{above}} - E_{\perp,\text{below}} = 4\pi\sigma}$



The // component of D does n't vanish in general

$$D^{\parallel}_{\text{above}} - D^{\parallel}_{\text{below}} = E_{\parallel\,\text{above}} - E_{\parallel\,\text{below}} + 4\pi(\vec{P}_{\text{above}\,\parallel} - \vec{P}_{\text{below}\,\parallel})$$

$$= 4\pi(\vec{P}_{\text{above}\,\parallel} - \vec{P}_{\text{below}\,\parallel}),$$ but this no useful.

we use $\boxed{E_{\parallel,\text{above}} = E_{\parallel,\text{below}}}$

examp: Prob 4.15 a thick spherical shell with inner radius a, outer radius b is made of dielectric material with a "frozen-in" polarization

$$p(r) = \frac{k}{r}\hat{r},$$ where k is a constant. we find the charge distribution and electric field distributions in the 3 regions.

1° we computer the bound charge density directly

$$\rho_b(r) = -\nabla\cdot p(r) = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{k}{r}\right) = -\frac{k}{r^2}$$

$$\sigma_b(r=a) = -\vec{P}\cdot\hat{n} = -\frac{k}{a}$$

$$\sigma_b(r=b) = +\vec{P}\cdot\hat{n} = \frac{k}{b}$$

check the total bound charge $4\pi\left[\int_a^b \rho_b(r)\, r^2 dr - \frac{k}{a}\cdot a^2 + \frac{k}{b}\cdot b^2\right] = 0.$

we can apply Gauss's theorem, draw a sphere around the center and calculate the electric flux

1° for $r < a$. $\oint \vec{E}\cdot da = 0 \Rightarrow E(r) = 0$

2° for $a < r < b$ $\oint \vec{E}_{(r)}\cdot da = \sigma_a\cdot 4\pi a^2 + \int_a^r \rho(r)\, r^2 dr\cdot 4\pi$

$$= (4\pi)^2\left[-ka - k\int_a^r dr\right] = -4\pi r \quad \Rightarrow \vec{E}(r) = -\frac{4\pi}{r}\times\hat{r}$$

3° for $r > b$ $\frac{1}{4\pi}\oint \vec{E}\cdot da = \sigma_a\cdot 4\pi a^2 + \int_a^r \rho(r)\, r^2 dr\cdot 4\pi + \sigma_b\, 4\pi b^2 = 0$

$$\Rightarrow \vec{E}(r) = 0$$

2° We can get the same electric field distribution by use D. Since there're no free charge, $\oint \vec{D}\cdot d\vec{a} = 0$. Due to the spherical symmetry, $\vec{D}$ is along the radial direction. $\Rightarrow \vec{D}(r) = 0$ everywhere.

at $r<a$ or $r>b$, it's vacum $\Rightarrow \vec{E}(r) = \vec{D}(r) = 0$

at $a<r<b \Rightarrow \vec{D} = \vec{E} + 4\pi\vec{P} = 0 \Rightarrow \vec{E} = -4\pi\vec{P} = -4\pi\frac{k}{r}\hat{r}$.