Lect 12 Laplace's equation

$$V(\vec{r}) = \int \frac{\rho(\vec{r}')\,dr'}{|\vec{r}-\vec{r}'|}$$ which is correct but is often useless.

To complicated to do real calculations.

→ $\nabla\cdot\vec{E} = 4\pi\rho$ and $\vec{E} = -\nabla V \Rightarrow -\nabla^2 V = 4\pi\rho$.

In the case of $\rho = 0$, we have $\boxed{\nabla^2 V = 0}$ ← Laplace Eq.

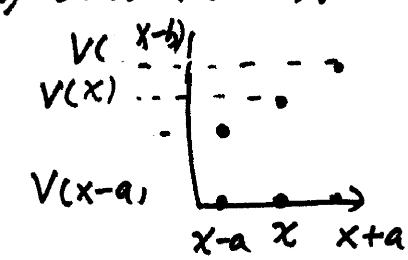
- Laplace Eq in 1D

$$d^2V/dx^2 = 0 \quad \Rightarrow \quad V(x) = C_1 x + C_2 \text{ : linear function.}$$

for boundry condition $V(a) = V_a$, $V(b) = V_b$

$$\Rightarrow V(x) = \frac{V_b - V_a}{b-a}(x-a) + V_a .$$

Properties ① $V(x)$ is the average of $V(x+a)$ and $V(x-a)$.



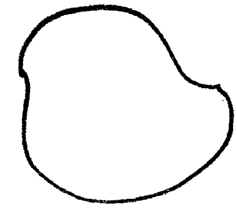
x is the average of $x+a$ and $x-a$, and

or $$\boxed{V\left(\frac{a+b}{2}\right) = \frac{1}{2}\left(V(a) + V(b)\right)}$$

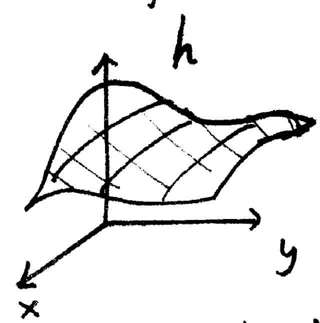
② in the domain of $\frac{d^2V}{dx^2} = 0$, $V(x)$ has no maximal and minimal. If $V(x_0)$ is a local maximal, we have for a sufficitly small ε, $V(x_0) > V(x_0 \pm \varepsilon)$, $\Rightarrow V(x_0) > \frac{1}{2}\left(V_0(x_0+\varepsilon) + V_0(x_0-\varepsilon)\right)$

Thus it is contradictary to ①.

• 2D $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = 0.$



We need know the values of $V(x,y)$ along the boundry. Special techniqes are needed for an arbitary shape.



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$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$$

① Properties: The value of V at (x,y) is the average of those around (x,y).

For example for a circle around (x_0, y_0)

$$V(x_0, y_0) = \frac{1}{2\pi R} \oint_{\text{circle}} V\, dl \,.$$

let us check. <u>It's not a proof</u>. If the radius is small ϵ.

$$\left.\begin{array}{l} x = x_0 + \epsilon\cos\theta \\ y = y_0 + \epsilon\sin\theta \end{array}\right\} \Rightarrow V(x,y) = V(x_0,y_0) + \left.\frac{\partial V}{\partial x}\right|_{x_0,y_0} \epsilon\cos\theta + \left.\frac{\partial V}{\partial y}\right|_{x_0 y_0} \epsilon\sin\theta$$
$$+ \frac{1}{2}\left.\frac{\partial^2 V}{\partial x^2}\right|_{x_0,y_0} \epsilon^2\cos^2\theta + \frac{1}{2}\left.\frac{\partial^2 V}{\partial y^2}\right|_{x_0,y_0} \epsilon^2\sin^2\theta + \left.\frac{\partial^2 V}{\partial x\partial y}\right|_{x_0,y_0} \epsilon^2\cos\theta\sin\theta$$

$$\oint dl = R\int_0^{2\pi} d\theta$$

$$\Rightarrow \frac{1}{2\pi R}\cdot 2\pi R\, V(x_0,y_0) + \frac{1}{2\pi R}\cdot\frac{2\pi}{2}\cdot\frac{1}{2}\epsilon^2\left[\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right] = V(x_0,y_0)$$

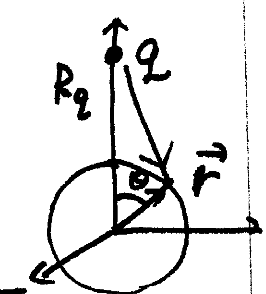
② $V(x)$ has no maximal and minimal in the regin of $\nabla^2 V = 0$.

· 3D

· The average value of V around a sphere at ~~the~~ center $\vec{r}_0$, is the same as $V(\vec{r}_0)$. Thus V has no maximals and minimals. The maximals and minimals have to be located on boundies.

we can check for a potential generated by a point charge. Without loss of generality, we put the charge at $(0.0, z)$ and the sphere at the origin.

This sphere must not include q inside!!



$$V(\vec{r}) = \frac{q}{|\vec{R}_q - \vec{r}|} = \frac{q}{\sqrt{z^2+R^2-2zR\cos\theta}}$$

$$\int d\sigma\, V(\vec{r}) = R^2 \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\varphi\, \frac{q}{\sqrt{2zR}} \cdot \frac{1}{\sqrt{\frac{z^2+R^2}{2zR} - \cos\theta}}$$

$$= \frac{R^2 \cdot 2\pi q}{\sqrt{2zR}} \int_{-1}^{1} dx \frac{1}{\sqrt{a-x}} \quad \text{where} \quad a = \frac{z^2+R^2}{2zR}.$$

$$\int dx \frac{1}{\sqrt{a-x}} = -2\sqrt{a-x} \Rightarrow \int_{-1}^{1} dx \frac{1}{\sqrt{a-x}} = 2\left[\sqrt{a+1} - \sqrt{a-1}\right]$$

$$\Rightarrow \int d\sigma\, V(r) = \frac{2\pi R^2 q}{\sqrt{2zR}} \cdot 2\left[\sqrt{\frac{z^2+R^2}{2zR}+1} - \sqrt{\frac{z^2+R^2}{2zR}-1}\right] = \frac{4\pi R^2 q}{(\sqrt{2zR})^2}\left[(z+R)-(z-R)\right]$$

$$= \frac{4\pi R^2 q}{2zR} \cdot 2R = 4\pi R^2 \frac{q}{z} \Rightarrow \boxed{\frac{1}{4\pi R^2}\int d\sigma\, V(r) = \frac{q}{z}}$$

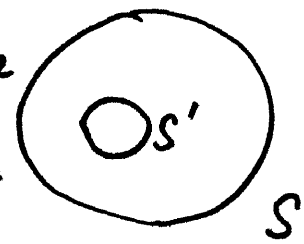
by the linear super position, for all the charge outside the sphere, we have $\frac{1}{4\pi R^2}\int_{\text{sphere}} d\sigma\, V(\vec{r}) = V(\vec{r}_0)$.

No charge inside the sphere.

Boundary anditions and uniqueness theorems

Theorem

① If the V is specified on the boundary surface S, the Laplace equation has a unique soulution inside the volume surrounded by S.



Actually, we can have more than one boundary. The proof has no much difference. Suppose that we have more than one solution V_1 and V_2, satisfying

$$\begin{cases} \nabla^2 V_i = 0 \\ V_i(\vec{r}) = V_0(\vec{r}) \quad \text{for } \vec{r} \in S \end{cases} \qquad i=1, \text{and } 2.$$

Then define $V_3 = V_1 - V_2$, we have

$$\begin{cases} \nabla^2 V_3 = 0 \\ V_3(\vec{r}) = 0 \quad \text{for } \vec{r} \in S. \end{cases}$$

we have

$$0 = \int d^3\vec{r}\; V_3 \nabla^2 V_3 = \int d^3\vec{r}\; \nabla\cdot(V_3 \nabla V_3) - \int d^3\vec{r}\; (\nabla V_3)^2$$

$$= \underbrace{\oint_S}_{\text{Surface}} d\vec{a}\; \underbrace{V_3}_{=0} \nabla V_3 - \int d^3\vec{r}\; (\nabla V_3)^2 \Rightarrow \int d^3\vec{r}\; (\nabla V_3)^2 = 0$$

thus $\nabla V_3 = 0$

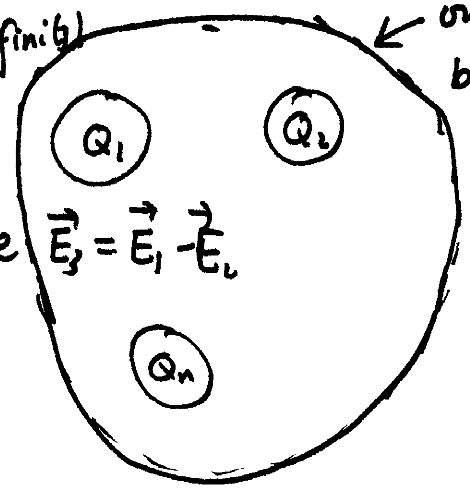
$$\left.\begin{array}{l} \Rightarrow V_3 \text{ is a constant,} \\ V_3 = 0 \text{ on boundary} \end{array}\right\} \Rightarrow V_3 = 0 \text{ or } V_1 = V_2.$$

Even inside the volume there exist charges, if the distribution $\rho(r)$ is fixed, then the uniqueness theorem ① also works. The proof is similar:

$$\left.\begin{array}{l} \nabla^2 V_1 = -4\pi\rho \\ \nabla^2 V_2 = -4\pi\rho \end{array}\right\} \Rightarrow \nabla^2(V_1 - V_2) = 0 \text{ and } V_1 - V_2 = 0 \text{ on boundary.}$$

$\Rightarrow V_1 - V_2 = 0$ everywhere.

② Conductors: In a volume surrounded by conductors, inside such a volum there exists a charge distribution $\rho(\vec{r})$, the electric field distribution is unique if the total charge Q_i on each conductor C_i is given. (The outer boundry can be a conductor with total charge, Q, or at infinity)



Proof: if there exist two different field distributions $\vec{E}_1$ and $\vec{E}_2$, and define $\vec{E}_3 = \vec{E}_1 - \vec{E}_2$

$$\left.\begin{array}{l} \nabla\cdot\vec{E}_1 = 4\pi\rho \\ \nabla\cdot\vec{E}_2 = 4\pi\rho \end{array}\right\} \Rightarrow \nabla(\vec{E}_3) = \nabla(\vec{E}_1 - \vec{E}_2) = 0.$$

$$\left\{\begin{array}{l} \oint \vec{E}_1\cdot d\vec{a}_i = 4\pi Q_i \\ \oint \vec{E}_2\cdot d\vec{a}_i = 4\pi Q_i \end{array}\right. \Rightarrow \oint \vec{E}_3 \cdot d\vec{a}_i = 0$$

for each conductor.

Then $\nabla\cdot(V_3 E_3) = V_3(\nabla\cdot\vec{E}_3) + \vec{E}_3\cdot\nabla V_3 = -\vec{E}_3^{\,2}$

$$\int_V \nabla\cdot(V_3 E_3)\,d\tau = \sum_i \oint_S V_3 \vec{E}_3\cdot d\vec{a}_i = -\int_V E_3^2\,d\tau$$

for each conductor, V_3 is a constant. $\Rightarrow \oint V_3 \vec{E}_3\cdot d\vec{a}_i$
$= V_1 - V_2$
$= V_3 \oint \vec{E}_3\cdot d\vec{a}_i = 0$

if the outer boundary at infinity $V_3 = 0$ there.

$$\Rightarrow \int E_3^2 \, d\tau = 0 \quad \Rightarrow \quad \vec{E}_3 = \vec{E}_1 - \vec{E}_2 = 0.$$

Examples:

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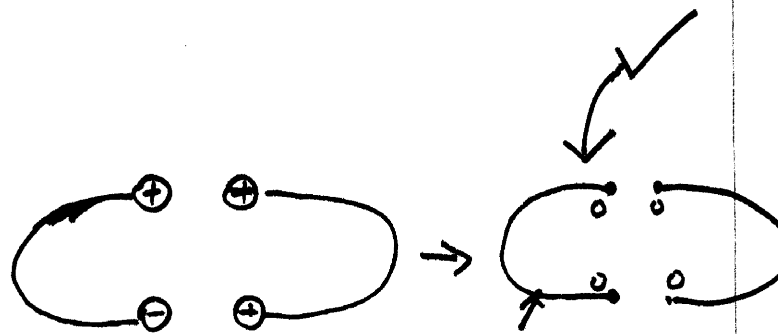
(1)

what's the equilibrium config after connecting the charges by conducting wires



×, wrong.

wires are conductors, total charges are zero.