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QUINTET PAIRING AND NON-ABELIAN VORTEX STRING IN SPIN-3/2 COLD ATOMIC SYSTEMS

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We study the s-wave quintet Cooper pairing phase ($S_{\text{pair}} = 2$) in spin-3/2 cold atomic systems and identify various novel features which do not appear in spin-1/2 pairing systems. A single quantum vortex is shown to be energetically less stable than a pair of half-quantum vortices. The half-quantum vortex exhibits the global analogue of the non-Abelian Alice string and $SO(4)$ Cheshire charge in gauge theories. The non-Abelian half-quantum vortex loop enables topological generation of quantum entanglement.

Keywords: Cold atom physics; half-quantum vortex; Cheshire charge; quintet pairing.

1. Introduction

Optical traps and lattices open up a whole new direction in the study of strongly correlated large spin systems by using cold atoms with hyperfine multiplets. In spin-1 bosonic systems (e.g., ^{23}Na and ^{87}Rb), spinor condensations, spin textures and nematic orders have generated a great deal of attention.[1–8] On the other hand, large spin fermions also exhibit many exciting novel features. For instance, the multi-particle clustering instability, i.e., a multi-particle counterpart of Cooper paring, is not allowed in spin 1/2 systems due to Pauli's exclusion principle, but is possible in large spin systems.[8–11] Furthermore, large spin fermions offer a unique playground to study high symmetries which do not appear in usual condensed matter systems. We have proved that spin-3/2 fermionic systems with contact interactions, which can be realized by atoms such as ^{132}Cs, ^{9}Be, ^{135}Ba, ^{137}Ba and ^{201}Hg, enjoy a generic $SO(5)$ symmetry for the continuum model with s-wave scattering interactions and

the lattice Hubbard model, whose exactness is regardless of dimensionality, lattice geometry and external potentials.[12,19] Such a high symmetry without fine-tuning is rare in both condensed matter and cold atom many-body systems. The important consequences of this symmetry are systematically investigated, including the protected degeneracy in collective excitations, the absence of the quantum Monte-Carlo sign problem, the four-fermion quartetting superfluidity, and even stronger quantum magnetic fluctuations than spin-1/2 systems,[9,13–18] which has been summarized in a review article.[19]

On the other hand, important progress has been made in the fault-tolerant topological quantum computation.[20–23] The key idea is that by using non-Abelian statistics in two dimensions, particles can be entangled in a robust way against local disturbances. The promising candidate systems to implement topological quantum computation include the non-Abelian quantum Hall states with fermions at the filling $\nu = 5/2$[24] and bosons at $\nu = 1$,[25,26] and also the $p_x + ip_y$ pairing state of spinless fermions.[27,28] In this paper, we show that due to the $SO(5)$ symmetry in spin-3/2 cold atomic systems,[9,12] the s-wave quintet Cooper pairing state ($S_{\mathrm{pair}} = 2$) in such systems provides another opportunity to topologically generate quantum entanglement between the particle and the non-Abelian half-quantum vortex (half-quantum vortex) loop.

The half-quantum vortex in superfluids with the spin degree of freedom is an exotic topological defect as a global analogue of the Alice string in gauge theories.[29–32] The half-quantum vortex loop can possess spin quantum number which is an example of the Cheshire charge phenomenon. An Abelian version of the global Alice string and Cheshire charge exists in the triplet superfluid of the ^{3}He-A phase,[33–37] where the spin $SU(2)$ symmetry is broken into the $U(1)$ symmetry around the z-axis. A remarkable property is that both quasi-particles and spin wave excitations reverse the sign of their spin quantum numbers s_z when going through the half-quantum vortex loop. Meanwhile, the half-quantum vortex loop also changes s_z to maintain spin conservation. However, due to the Abelian nature of this $U(1)$ Cheshire charge, no entanglement is generated in this process.

In this article, we investigate the non-Abelian Alice string and the topological generation of quantum entanglement through the non-Abelian Cheshire charge in spin-3/2 systems. The quintet Cooper pairing order parameters in the polar basis form a 5-vector of the $SO(5)$ symmetry group. The ground state exhibits the polar condensation where the $SO(5)$ symmetry is broken into $SO(4)$.[12] This allows the half-quantum vortex loop to possess the non-Abelian $SO(4)$ Cheshire charge, in contrast to the $U(1)$ Cheshire charge in the ^{3}He-A phase. We also explore the high symmetry effects on collective spin excitations and the structure of the half-quantum vortex line as a π-disclination in the spin channel. We show that by driving the fermion quasiparticle (or spin-wave impulse) through the half-quantum vortex loop, quantum entanglement between them is topologically generated. This effect has a potential application in the topological quantum computation.

2. Quintet Pairing

The spin-3/2 system is the simplest one to support the s-wave quintet pairing with total spin $S_T = 2$. In such a system with contact interactions, a hidden $SO(5)$ symmetry[12] arises as follows: the four-component spinor

$$\psi = \left(c_{\frac{3}{2}}, c_{\frac{1}{2}}, c_{-\frac{1}{2}}, c_{-\frac{3}{2}}\right)^T \tag{1}$$

forms the spinor representations of the $SU(4)$ group which is the unitary transformation of the four-component spinor. Each of the four component contributes the same to the kinetic energy which is explicitly $SU(4)$ symmetric. Generally speaking, the interactions break this symmetry to a lower level. From the view of the spin $SU(2)$ group, the interactions can be classified into the total spin 0(singlet), 1 (triplet), 2 (quintet), 3(septet) channels. For the contact interactions, say, s-wave scattering, only total spin singlet and quintet channel are allowed as required by the Pauli's exclusion principle. Interestingly, the spin $SU(2)$ singlet channel can also be interpreted as an $SO(5)$ singlet, and the spin $SU(2)$ quintet channel can be interpreted as an $SO(5)$ vector channel. This $SO(5)$ group only lies in the particle-hole channel as a subgroup of the $SU(4)$ group. Thus, the remaining symmetry with the s-wave scattering interaction is $SO(5)$ without fine-tuning of parameters.

We denote the interaction strength in the spin singlet and triplet channels as g_0 and g_2, respectively. We consider the case of $g_2 < 0$ where the quintet channel Cooper pairing dominates, and further neglect the interaction in the singlet channel. The mean field Hamiltonian reads

$$H_{MF} = \int d^D r \left\{ \sum_{\alpha = \pm \frac{3}{2}, \pm \frac{1}{2}} \psi_\alpha^\dagger(r) \left(\frac{-\hbar^2 \nabla^2}{2M} - \mu \right) \psi_\alpha(r) + \sum_{a=1\sim 5} \chi_a^\dagger(r) \Delta_a(r) + h.c. \right.$$

$$\left. - \frac{1}{g_2} \Delta_a^*(r) \Delta_a(r) \right\}, \tag{2}$$

with D the spatial dimension, μ the chemical potential, and M the atom mass. Δ_a is proportional to the ground state expectation value of the quintet pairing operators χ_a by

$$\Delta_a(r) = g_2 \langle \chi_a(r) \rangle, \tag{3}$$

where $a = 1 \sim 5$. The five χ_a operators are the spin channel counterparts of the five atomic d-orbitals $(d_{xy}, d_{xz}, d_{yz}, d_{3z^2-r^2}, d_{x^2-y^2})$, and transform as a 5-vector under the $SO(5)$ group. Explicitly, they are expressed as

$$\chi_a^\dagger(r) = -\frac{i}{2} \psi_\alpha^\dagger(r) (\Gamma^a R)_{\alpha\beta} \psi_\beta^\dagger(r), \tag{4}$$

where the five 4×4 Dirac $\Gamma^a (a = 1 \sim 5)$ matrices are defined as

$$\Gamma^1 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}, \quad \Gamma^{2,3,4} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix}, \quad \Gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$
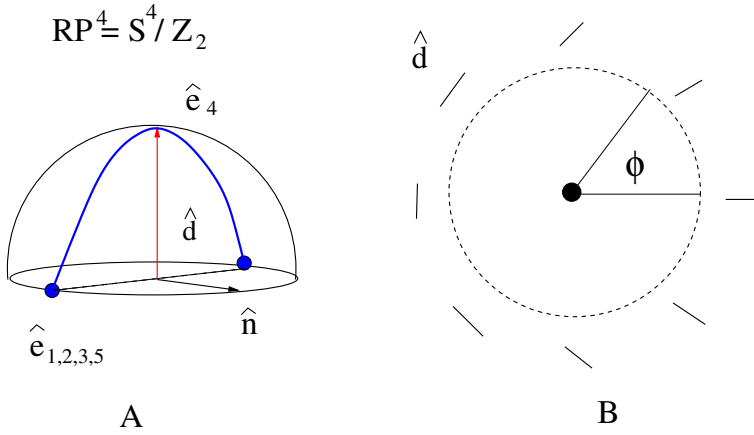
Fig. 1. (A) The Goldstone manifold of $\hat{d}$ is a 5D hemisphere RP^4. It contains a class of non-contractible loops as marked by the solid curve. (B) The π-disclination of $\hat{d}$ as a half-quantum vortex. Assume that $\hat{d} \parallel \hat{e}_4$ at $\phi = 0$. As the azimuthal angle ϕ goes from 0 to 2π, $\hat{d}$ is rotated at the angle of $\phi/2$ around any axis $\hat{n}$ in the S^3 equator spanned by $\hat{e}_{1,2,3,5}$.

which satisfy the anti-commutation relation as $\{\Gamma^a, \Gamma^b\} = 2\delta_{ab}$, and $R = \Gamma^1\Gamma^3$ is the charge conjugation matrix.[12]

This $SO(5)$ symmetry leads to new interesting results about the pairing structure in the ground state and the corresponding Goldstone modes. Within the BCS theory, Ref. 38 showed that the ground state of Eq. (2) is an $SO(3)$ polar condensate without noticing the hidden $SO(5)$ symmetry. We conclude here that the ground state is generically an $SO(5)$ polar condensate. The order parameters can be parameterized as $\Delta_a = |\Delta|e^{i\theta}d_a$, where θ is the $U(1)$ phase, $\hat{d} = d_a\hat{e}_a$ is a 5D unit vector, and $\hat{e}_a(a = 1 \sim 5)$ form a set of basis for the internal spin space. Rigorously speaking, $\hat{d}$ is a directionless director instead of a true vector because Δ_a's contain a Z_2 gauge symmetry of

$$\hat{d} \to -\hat{d}, \qquad \theta \to \theta + \pi. \tag{5}$$

Thus the Golstone manifold is $[SO(5) \otimes U(1)]/[SO(4) \otimes Z_2] = RP^4 \otimes U(1)$, where RP^4 is a 5D hemisphere instead of an entire S^4 sphere as depicted in Fig. 1(A).

The general Ginzburg-Landau free energy without the gradient term for quintet pairing has been given in Ref. 38 with three independent quartic terms. In the special spin-3/2 case with the SO(5) symmetry, it can be simplified into a more convenient SO(5) invariant form with only two quartic terms

$$F_{GL} = \int d^3r \left\{ \gamma \boldsymbol{\nabla}\Delta_a^* \cdot \boldsymbol{\nabla}\Delta_a + \alpha(T)\Delta_a^*\Delta_a + \frac{\beta_1}{2}(\Delta^*a\Delta_a)^2 + \frac{\beta_2}{2} \sum_{1 \le a < b \le 5} L_{ab}^2 \right\},$$

$$\tag{6}$$

where

$$L_{ab} = \frac{\Delta_a^* \Delta_b - \Delta_b^* \Delta_a}{\sqrt{2}} \,. \tag{7}$$

The first quartic term describes the density interaction among Cooper pairs while the second one describes the spin interaction among them. The polar-like state is favorable to make $L_{ab} = 0$ at $\beta_2 > 0$, while the axial state is favorable at $\beta_2 < 0$. Around the critical temperature T_c, the parameters $\alpha, \beta_{1,2}$ were calculated from the Gor'kov expansion in Ref. 38

$$\alpha = -\frac{1}{2}\frac{dn}{d\epsilon}\left(1 - \frac{T}{T_c}\right), \qquad \beta_1 = \beta_2 = \frac{1}{2}\frac{dn}{d\epsilon}\frac{7\zeta(3)}{8\pi^2 T_c^2} \,, \tag{8}$$

where n is the particle density, $dn/d\epsilon$ is the density of states at the Fermi level, and $\zeta(x)$ is the Riemann zeta function. The coefficient of γ can also be calculated as

$$\gamma = \frac{n\hbar^2}{4M}\frac{7\zeta(3)}{8\pi^2 T_c^2} \,. \tag{9}$$

3. Gross-Pitaevskii Equation

At zero temperature, the Ginzburg-Landau free energy fails. The low energy degree of freedom is described by an effective Gross-Pitaevskii equation for the Cooper pairs. In addition to the usual phonon mode, four branches of spin wave modes carrying the spin quantum number $S = 2$ arise because of the spontaneous symmetry breaking from $SO(5)$ to $SO(4)$. In other words, they can be called "spin nematic waves". For small fluctuations, the spin wave modes decouple from the phase mode. For the purpose of describing collective excitations, Cooper pairs can be treated as composite bosons. This treatment gives a good approximation to the phonon mode in the neutral singlet BCS superfluid.[39,40]

Here we generalize this method to the quintet pairing by using a phenomenological Hamiltonian for spin-2 bosons

$$H_{\text{eff}} = \int d^D r \left\{ \frac{\hbar^2}{4M} \sum_{1 \le a \le 5} \nabla \Psi_a^\dagger \nabla \Psi_a + \frac{1}{2\chi_\rho}(\Psi_a^\dagger \Psi_a - \rho_0)^2 \right.$$

$$\left. + \frac{1}{2\chi_{sp}} \sum_{1 \le a < b \le 5} (\Psi_c^\dagger L_{cd}^{ab} \Psi_d)^2 \right\} \,, \tag{10}$$

where Ψ_a's are the boson operators in the polar basis, the equilibrium Cooper pair density ρ_0 is half of the particle density ρ_f, χ_ρ and χ_{sp} are proportional to the compressibility and $SO(5)$ spin susceptibility, respectively. We define the $SO(5)$ generators in the 5×5 vector representation as

$$L_{cd}^{ab} = i(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) \,. \tag{11}$$

The Landau parameters in the $SO(5)$ symmetric Fermi liquid theory for spin-3/2 systems can be decomposed into three sectors as $SO(5)$ scalar F_l^s (density),

vector F_l^v(spin-quadrapole density), tensor F_l^t (spin and spin-octupole densities), where l denotes the quantum number of orbital angular momentum.[12,19] Taking into account the Fermi liquid correction,

$$\chi_\rho = \frac{N_f}{4(1+F_0^s)}, \qquad \chi_{sp} = \frac{N_f}{4(1+F_0^t)}, \tag{12}$$

where N_f is the fermion density of states at the Fermi energy.

We introduce $\rho(r)$ as the Cooper pair density and $l_{ab}(r)$ as the $SO(5)$ spin density, and parameterize $\Psi_a = \sqrt{\rho_0}e^{i\theta}d_a$. Using the standard commutation rules between l_{ab} and $\hat{d}_a$, ρ and θ, we arrive at

$$\partial_t l_{ab} = \frac{\hbar^2\rho_{sp}}{2M}(d_a\nabla^2 d_b - d_b\nabla^2 d_a), \quad \chi_{sp}\partial_t d_a = -l_{ab}d_b,$$

$$\chi_\rho\partial_t^2\theta - \frac{\hbar^2\rho_s}{2M}\nabla^2\theta = 0, \tag{13}$$

where ρ_{sp} is the superfluid density and ρ_s is the spin superfluid density. At $T = 0K$ in a Galilean invariant system, ρ_s is just $\rho_f/2$, while ρ_{sp} receives Fermi liquid corrections[41] as

$$\frac{\rho_{sp}}{\rho_s} = \frac{1+F_1^v/3}{1+F_1^s/3} \tag{14}$$

where F_1^v is the Landau parameter in the $SO(5)$ vector channel.[12] The spin wave and sound velocities are obtained as

$$v_{sp} = \sqrt{\frac{\rho_0}{2\chi_{sp}M}}, \qquad v_s = \sqrt{\frac{\rho_0}{2\chi_\rho M}}, \tag{15}$$

respectively.

4. Half-quantum Vortex

The fundamental group of the GS manifold is

$$\pi_1(RP^4 \otimes U(1)) = Z \otimes Z_2. \tag{16}$$

The Z_2 feature gives rise to the existence of the half-quantum vortex as a stable topological defect as depicted in Fig. 1(B). As we move along a loop enclosing the half-quantum vortex, the π phase mismatch in the θ field is offset by a π-disclination in the d-field, thus Δ_a's are maintained single-valued. Energetically, a single quantum vortex is less favorable than a pair of half-quantum vortices. From Eq. (10), the static energy function can be written as

$$E = \int d^D r \frac{\hbar^2}{4M}\{\rho_s(\nabla\theta)^2 + \rho_{sp}(\nabla\hat{d})^2\}. \tag{17}$$

The energy density per unit length of a single quantum vortex is $E_1 = (\hbar^2/4M)$ $\times\rho_s\log(L/a)$, while that of two isolated half-quantum vortices is $E_2 = \hbar^2/8M(\rho_s + \rho_{sp})\log(L/a)$. Although at the bare level $\rho_s = \rho_{sp}$, ρ_{sp} receives considerable Fermi

liquid correction and strong reduction due to quantum fluctuations in the 5D internal space. Generally speaking, the relation,

$$\rho_{sp} < \rho_s \,, \tag{18}$$

holds in terms of their renormalized values. Then a single quantum vortex is fractionalized into a pair of half-quantum vortices. In the presence of rotation, the half-quantum vortex lattice should appear instead of the usual single quantum vortex lattice. As a result of the doubling of vortex numbers, their vortex lattice constants differ by a factor of $\sqrt{2}$.

The half-quantum vortex was also predicted in the ^{3}He-A phase, where $\hat{d}$ is a 3D vector defined for spin-1 Cooper pairs. However, the dipole locking effect favors the d-vector aligned along the fixed direction of the l-vector, i.e., the direction of the p-wave orbital angular momentum. As a result, the two half-quantum vortices are linearly confined by a string of the mismatched d and l-vectors. In contrast, the orbital part of the quintet pairing is s-wave, no dipole locking effect exists.

5. SO(4) Cheshire Charge

The single half-quantum vortex line behaves like the Alice string because a quasi-particle changes its spin quantum number after it adiabatically moves around the half-quantum vortex once. For example, in the ^{3}He-A phase, a quasi-particle with spin $\uparrow$ flips its spin to $\downarrow$ up to a $U(1)$ Berry phase. The half-quantum vortex in the quintet superfluid behaves as a non-Abelian generalization with the $SU(2)$ Berry phase. Without loss of generality, we assume that $\hat{d}$ is parallel to $\hat{e}_4$ at the azimuthal angle $\phi = 0$. As ϕ changes from 0 to 2π, $\hat{d}$ is rotated at the angle of $\phi/2$ in the plane spanned by $\hat{e}_4$ and $\hat{n}$, where $\hat{n}$ is a unit vector perpendicular to $\hat{e}_4$, i.e., a vector located in the S^3 sphere spanned by $\hat{e}_{1,2,3,5}$. We define such a rotation operator as $U(\hat{n}, \phi/2)$. When U acts on an $SO(5)$ spinor, it takes the form of

$$U\left(\hat{n}, \frac{\phi}{2}\right) = \exp\left\{-i\frac{\phi}{2}\frac{n_b \Gamma^{b4}}{2}\right\} \tag{19}$$

where $\Gamma^{b4} = i[\Gamma^b, \Gamma^4]/2$ are $SO(5)$ generators in the 4×4 spinor representation; when U acts on an $SO(5)$ vector, it behaves as

$$U(\hat{n}, \phi/2) = \exp\left\{-i\frac{\phi}{2}n_b L^{b4}\right\} \tag{20}$$

where L^{ab}'s are the $SO(5)$ generators in the 5×5 vector representation explicitly defined in Eq. (11). The resulting configuration of $\hat{d}$ is

$$\hat{d}(\hat{n}, \phi) = U(\hat{n}, \phi/2)\hat{d}(\hat{n}, 0) = \cos\frac{\phi}{2}\hat{e}_4 - \sin\frac{\phi}{2}\hat{n} \,. \tag{21}$$

As fermionic quasi-particles circumscribe around the vortex line adiabatically, at $\phi = 2\pi$ fermions with $S_z = \pm 3/2$ are rotated into $S_z = \pm 1/2$ and *vice versa*. For convenience, we change the basis Ψ for the fermion wavefunction to

$(|3/2\rangle, |-3/2\rangle, |1/2\rangle, |-1/2\rangle)^T$. After taking into account the π phase winding of θ, Ψ transforms by

$$\Psi_a \rightarrow \Psi'_a = iU(\hat{n}, \pi)_{\alpha\beta}\Psi_\beta = \begin{pmatrix} 0 & W \\ W^\dagger & 0 \end{pmatrix}_{\alpha\beta} \Psi_\beta \qquad (22)$$

where W is an $SU(2)$ Berry phase depending on the direction of $\hat{n}$ on the S^3 sphere as

$$W(\hat{n}) = \begin{pmatrix} n_3 + in_2 & -n_1 - in_5 \\ n_1 - in_5 & n_3 - in_2 \end{pmatrix}. \qquad (23)$$

The non-conservation of spin in this adiabatic process is not surprising because the $SO(5)$ symmetry is completely broken in the configuration depicted in Fig. 1(B).

A more interesting but related concept is the Cheshire charge, which means that a pair of the half-quantum vortex loop can carry $SO(4)$ spin quantum numbers. An intersection between the half-quantum vortex loop and a perpendicular plane is depicted in Fig. 2, where $\phi_{1,2}$ are respect to the vortex and anti-vortex cores respectively. Without loss of generality, we assume $\hat{d}(\mathbf{r}) \rightarrow \hat{\mathbf{e}}_4$ as $r \rightarrow \infty$ where an $SO(4)$ symmetry generated by $\Gamma_{ab}(a, b = 1, 2, 3, 5)$ is preserved. In analogy to Fig. 1(B), the $\hat{d}$ vector is described by the difference between two azimuthal angles $\Delta\phi = \phi_2 - \phi_1$ as

$$\hat{d}(\hat{n}, \Delta\phi) = \cos\frac{\Delta\phi}{2}\hat{e}_4 - \sin\frac{\Delta\phi}{2}\hat{n}, \qquad (24)$$

where $\hat{n}$ again is a unit vector on the S^3 equator. This classical configuration is called a phase-sharp state denoted as $|\hat{n}\rangle_{vt}$. Because the above $SO(4)$ symmetry is only broken within a small region around the half-quantum vortex loop, quantum
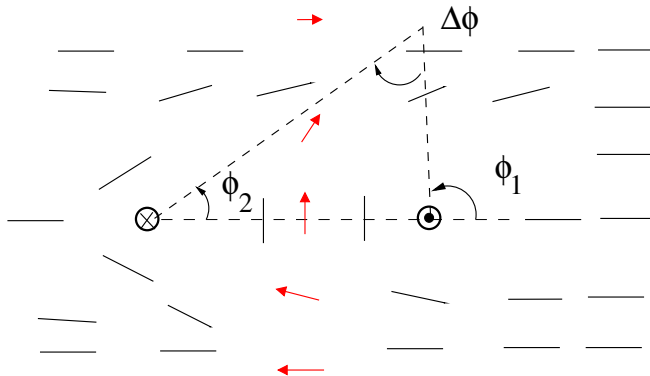

Fig. 2.   The configuration of a π-disclination pair or loop described by Eq. (24). $\phi_{1,2}$ and $\Delta\phi$ are azimuthal angles and $\hat{d}(\mathbf{r}) \parallel \hat{\mathbf{e}}_4$ as $\mathbf{r} \rightarrow \infty$. After a fermion passes the half-quantum vortex loop, the components with $S_z = \pm 3/2$ change to $S_z = \pm 1/2$ and *vice versa* with an $SU(2)$ matrix defined in Eq. (22).

fluctuations of $\hat{n}$ dynamically restore the $SO(4)$ symmetry as described by the Hamiltonian

$$H_{rot} = \sum_{a,b=1,2,3,5} \frac{M_{ab}^2}{2I}, \qquad M_{ab} = i(\hat{n}_a \partial_{\hat{n}_b} - \hat{n}_b \partial_{\hat{n}_a}), \tag{25}$$

with the moment of inertial

$$I = \chi_{sp} \int d^D r \rho_0 \sin^2 \frac{\Delta\phi}{2}. \tag{26}$$

Thus the zero modes $|\hat{n}\rangle_{vt}$ are quantized into the global $SO(4)$ Cheshire charge states, which are a non-Abelian generalization of the $U(1)$ case in the ^{3}He-A phase.[37] The global Cheshire charge density is localized around the half-quantum vortex loop. In contrast, the Cheshire charge in gauge theories is non-localized.[32]

The $SO(4)$ algebra can be grouped into two commutable sets of $SU(2)$ generators as

$$T_1(T_1') = \frac{1}{4}(\pm\Gamma_{35} - \Gamma_{12}), \qquad T_2(T_2') = \frac{1}{4}(\pm\Gamma_{31} - \Gamma_{25}),$$

$$T_3(T_3') = \frac{1}{4}(\pm\Gamma_{23} - \Gamma_{15}). \tag{27}$$

$T_{1,2,3}$ and $T'_{1,2,3}$ act in the subspaces spanned by $|\pm 3/2\rangle$ and $|\pm 1/2\rangle$, respectively. $SO(4)$ representations are denoted by $|T, T_3; T', T_3'\rangle$, i.e., the direct-product of representations of two $SU(2)$ groups. The half-quantum vortex loop in the $SO(4)$ Cheshire charge eigenstates is defined as

$$|TT_3; T'T_3'\rangle_{vt} = \int_{\hat{n} \in S^3} d\hat{n} F_{TT_3; T'T_3'}(\hat{n}) |\hat{n}\rangle_{vt}, \tag{28}$$

where $F_{TT_3; T'T_3'}(\hat{n})$ are the S^3 sphere harmonic functions. Thus $|TT_3; T'T_3'\rangle_{vt}$ is the non-Abelian generalization of the usual number-sharp state in $U(1)$ theories.

6. Generation of Entanglement

When a particle passes the half-quantum vortex loop, $\Delta\phi$ changes from 0 to 2π. The conservation of the $SO(4)$ spin is ensured by exciting the Cheshire charges and generating quantum entanglement between the particle and the half-quantum vortex loop. We demonstrate this process explicitly through a concrete example, with the initial state $|i\rangle$ made up from a zero charged half-quantum vortex loop and a quasiparticle with $S_z = 3/2$ as

$$|i\rangle = \int_{\hat{n} \in S^3} d\hat{n} |\hat{n}\rangle_{vt} \otimes \left(u c_{\frac{3}{2}}^\dagger + v c_{-\frac{3}{2}} \right) |\Omega\rangle_{qp}, \tag{29}$$

where $|\Omega\rangle_{qp}$ is the vacuum for Bogoliubov particles. For each phase-sharp state $|\hat{n}\rangle_{vt}$, the particle changes spin according to Eq. (22) in the final state $|f\rangle$. The
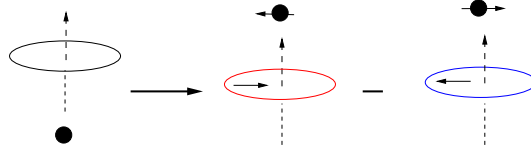
Fig. 3.   The topological generation of quantum entanglement. The initial state $|i\rangle$ is a product state of a zero charged half-quantum vortex loop and a quasiparticle with $S_z = 3/2$, both of which are a singlet of the $SU(2)$ group of T' acting in the subspace spanned by $|\pm 1/2\rangle$. The final state $|f\rangle$ is an EPR state of the T' group as described by Eq. (31).

superposition of the non-Abelian phase gives

$$
\begin{aligned}
|f\rangle &= \int_{\hat{n} \in S^3} d\hat{n} \big\{ u \big(W_{11}^\dagger c_{\frac{1}{2}}^\dagger + W_{21}^\dagger c_{-\frac{1}{2}}^\dagger \big) + v \big(W_{12}^T c_{\frac{1}{2}} + W_{22}^T c_{-\frac{1}{2}} \big) \big\} \qquad |\hat{n}\rangle_{vt} \otimes |\Omega\rangle_{qp} \\
&= \int_{\hat{n} \in S^3} d\hat{n} (\hat{n}_3 - i\hat{n}_2) |\hat{n}\rangle_{vt} \otimes \big(u c_{\frac{1}{2}}^\dagger + v c_{-\frac{1}{2}} \big) |\Omega\rangle_{qp} \\
&\quad - \int_{\hat{n} \in S^3} d\hat{n} (\hat{n}_1 - i\hat{n}_5) |\hat{n}\rangle_{vt} \otimes \big(u c_{-\frac{1}{2}}^\dagger - v c_{\frac{1}{2}} \big) |\Omega\rangle_{qp} \,,
\end{aligned}
\tag{30}
$$

as depicted in Fig. 3. In terms of the $SO(4)$ quantum numbers, $|i\rangle$ is a product state of $|00;00\rangle_{vt} \otimes |(1/2)(1/2);00\rangle_{qp}$, and $|f\rangle$ is

$$
\left| \frac{1}{2}\frac{1}{2}; \frac{1}{2}\frac{-1}{2} \right\rangle_{vt} \otimes \left| 00; \frac{1}{2}\frac{1}{2} \right\rangle_{qp} - \left| \frac{1}{2}\frac{1}{2}; \frac{1}{2}\frac{1}{2} \right\rangle_{vt} \otimes \left| 00; \frac{1}{2}\frac{-1}{2} \right\rangle_{qp} \,.
\tag{31}
$$

In the channel of (T', T'_3), the final state is exactly an entangled Einstein-Podolsky-Rosen (EPR) pair made up from the half-quantum vortex loop and the quasi-particle. We note that this mechanism of generating the quantum entanglement is entirely topological, dependent only on whether the trajectory of the quasi-particle lies inside or outside of the half-quantum vortex loop. In contrast, the half-quantum vortex loop in ^{3}He-A system only exhibits the $U(1)$ Cheshire charge, thus the final state is still a product state without the generation of entanglement.

Similarly, the entanglement between a spin wave impulse and the half-quantum vortex loop can also be generated. We consider the four local transverse bases $\hat{e}_b (b = 1, 2, 3, 5)$ at $\Delta\phi = 0$. Assume that the initial state made up from the half-quantum vortex loop and the spin wave is

$$
|i'\rangle = \int_{\hat{n} \in S^3} d\hat{n} |\hat{n}\rangle_{vt} \otimes (\hat{e}_1 + i\hat{e}_5)_{sw} \,,
\tag{32}
$$

where spin wave impulse carries $S_z = 2$. For each phase-sharp state $|\hat{n}\rangle_{vt}$ of the half-quantum vortex, the frame bases at $\Delta\phi = 2\pi$ transform to $\hat{e}_a \to \hat{e}_a - 2\hat{n}(\hat{e}_a \cdot \hat{n})$. Thus the entanglement is generated in the final state $|f'\rangle$ as

$$
\begin{aligned}
|f'\rangle &= \int_{\hat{n} \in S^3} d\hat{n} |\hat{n}\rangle_{vt} \otimes (\hat{e}_1 + i\hat{e}_5)_{sw} \\
&\quad - 2 \int_{\hat{n} \in S^3} d\hat{n} (n_1 + in_5) n_b |\hat{n}\rangle_{vt} \otimes \hat{e}_{b,sw} \,.
\end{aligned}
\tag{33}
$$

7. Conclusion

Recently, Bose condensation of the ^{174}Yb atom and sympathetic cooling between ^{174}Yb and the fermionic atom of ^{171}Yb[42,43] have been achieved. Their electron configurations are the same as the Ba atoms except an inside full-filled $4f$ shell, thus the spin-3/2 systems of ^{135}Ba and ^{137}Ba can be possibly realized in the near future. At the present time, scattering lengths of these two Ba atoms are not available. However, considering the rapid developments in this field, we are optimistic about the realization of the quintet pairing state and the associated non-Abelian topological defects.

We briefly discuss here the factors that limit the life time of the entanglement which comes from spin decoherence. As shown above, the generation of entanglement only depends on whether the particle trajectory penetrates the half-quantum vortex loop or not, but does not depend on the detail of how it penetrates the vortex loop. Thus this process is topological and robust. However, spin decoherence does come from the interaction between particles and vortex loops with the bulk low energy collective excitations. The quintet pairing states have gapless spin-wave excitations. The particle spin and the Chesire charge of the half-quantum vortex loop can flip when spin-waves scatter with them, which is the leading order spin decoherence mechanism. Nevertheless, spin-waves are Goldstone particles which only interact with other excitations through the derivative coupling, i.e., the coupling constant vanishes at long wave length limit. At low temperatures, only long wavelength spin waves are excited, thus their spin decoherence effect is small.

In summary, we have studied the quintet pairing state in spin 3/2 fermionic systems with the $SO(5)$ symmetry, including its Goldstone modes and the non-Abelian topological defects. The non-Abelian Berry phase effect and the Cheshire charge behavior are analyzed in detail. The topological mechanism of generating the quantum entanglement between quasi-particles and the half-quantum vortex loop could be useful for topological quantum computation.

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