MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics
String Theory (8.821) – Prof. J. McGreevy – Fall 2008

Problem Set 3

Conformal invariance, large N, geometry of AdS

Due: Tuesday, October 21, 2008 or so.

1. Show that the conformal algebra of $\mathbb{R}^{p,q}$, with generators $P_\mu, C_\mu, M_{\mu\nu}, D$ and commutators given in lecture, is isomorphic to $SO(p+1, q+1)$.

2. [much more optional than others]

   Show that the generator of an infinitesimal conformal transformation in $d > 2$ must be at-most quadratic in x. That is, convince yourself that (for $d > 2$)

   $$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} \eta_{\mu\nu} \partial \cdot \epsilon$$

   implies that all third derivatives of ϵ must vanish. [I don't actually know if there is a simple way to do this.]

3. [more optional than others]

   In a generic matrix quantum field theory, give some kind of proof by induction on the number of vertices and propagators that a planar vacuum diagram always contributes an amplitude of order N^2.

4. **Useful coordinates in AdS**

   Lorentzian AdS_{p+2} is the locus

   $$-L^2 = \eta_{ab} X^a X^b \equiv -(X^{p+2})^2 - (X^0)^2 + \sum_{i=1}^{p+1} (X_i)^2 \qquad (\star)$$

   inside $R^{p+1,2}$ with metric $ds^2 = \eta_{ab} dX^a dX^b$.

   a) Show that in the *global coordinates* defined by

   $$\begin{aligned} X^{p+2} &= L \cosh\rho \sin\tau \\ X^0 &= L \cosh\rho \cos\tau \end{aligned}$$

$$X^i = L\sinh\rho\Omega_i, \qquad \sum_{i=1}^{p+1}\Omega_i^2 = 1$$

the induced metric on the locus ($\star$) becomes

$$ds^2 = L^2\left(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho\ d\Omega_p^2\right)$$

with $d\Omega_p^2 = d\Omega_i d\Omega_i$ the metric on the unit p-sphere, S^p.

b) The *Poincaré patch* coordinates x^μ, u are defined via the relations

$$\begin{aligned} X^{p+2} + X^{p+1} &= u \\ -X^{p+2} + X^{p+1} &= v \\ X^\mu &= \frac{ux^\mu}{L} \quad . \end{aligned}$$

Using the defining equation to eliminate v in terms of the other variables, show that the induced metric takes the form

$$ds^2 = L^2\frac{du^2}{u^2} + \frac{u^2}{L^2}dx^\mu dx_\mu.$$

This metric looks nicer if we introduce $z \equiv \frac{L^2}{u}$:

$$ds^2 = L^2\frac{dz^2 + dx^\mu dx_\mu}{z^2}.$$

The boundary of AdS is at $z = 0$. Think about what part of the AdS space is not covered by these coordinates. Hint: z runs only over positive values, since at $z \to \infty$, the timelike killing vector $\frac{\partial}{\partial x^0}$ becomes null.

c) Starting from the global coordinates, introduce $r \equiv L\sinh\rho$. Show that the metric takes the form

$$ds^2 = -Hdt^2 + H^{-1}dr^2 + r^2 d\Omega_p^2$$

with $H \equiv 1 + \frac{r^2}{L^2}$.

With the metric in this form, the angular part of the Einstein tensor $G_{\theta\theta}$ is linear in H and is proportional to $\partial_r(r^p\partial_r H)$. Using this information write down the metric for Schwarzschild-AdS_{p+2}, *i.e.* the spacetime that contains a black hole and is asymptotic to AdS at infinity.

5. **Stereographic projection coordinates** [more optional]

Let's think about the euclidean-signature hyperbolic space:

$$H_d = \{-X_d^2 + \sum_{i=1}^{d} X^i X^i = -L^2\} \subset \mathbb{R}^{d,1}$$

with metric $ds^2 = -dX_d^2 + \sum_{i=1}^d dX^i dX^i$. Introduce new coordinates ξ by

$$X^i = \xi^i \frac{2L^2}{L^2 - r^2}, \quad r \equiv \sqrt{\xi^i \xi^i}.$$

a) These coordinates arise by considering the line from a point P on the hyperboloid H_d to the point $Q = (-L, \vec{0})$; this line intersects the plane $X^{d+1} = 0$ at some point P' whose position in the i direction is ξ^i. Try to parse this last sentence, *i.e.* draw a figure.

b) Show that the induced metric on the hyperbolic space is

$$ds^2 = \frac{4L^2 d\xi^i d\xi^i}{(1 - r^2)^2}$$

6. **Geodesics in AdS**

a) Given a stationary observer at fixed ρ in global AdS, how long does it take (on his clock) for a radially-directed light-ray (*i.e.* a massless geodesic) to leave him, hit the boundary of AdS, and come back? (Assume that the observer is living in a circumstance where there are reflecting boundary conditions on the electromagnetic field.)

b) Examine the behavior of a massive geodesic in AdS (with zero angular momentum on the spatial sphere). Show that the motion is bounded away from the boundary of AdS.

7. **Volumes and areas are not so different in AdS**

In a fixed-τ slice of the global coordinates, consider the region $\rho \leq \bar{\rho}$. Show that

$$\lim_{\bar{\rho} \to \infty} \frac{A(\bar{\rho})}{V(\bar{\rho})}$$

is finite, where A and V are the surface area and volume inside the specified region.