Tidal Effects on Earth's Surface

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This treatment follows the conventions of F. D. Stacey's *Physics of the Earth*, and is largely an elaboration on this work.

1 Tidal Potential

The gravitational influence of the moon and sun, in combination with revolution about the respective centers of mass of these systems, produces a potential gradient across Earth's surface, giving rise to a tidal deformation of the earth. The full expression for the induced potential, limited to second-order, is:

$$W = -\frac{Gm}{R}\left(1 + \frac{1}{2}\frac{m}{M+m}\right) - \frac{Gma_0^2}{R^3}\left(\frac{3}{2}\cos^2\psi - \frac{1}{2}\right) - \frac{1}{2}\omega_L^2 a_0^2 \sin^2\theta \tag{1}$$

where M is the earth mass, m is the perturber mass, R is the center-to-center distance between Earth and the perturber, a_0 is the earth radius, ψ is the angle between the mass centers and the observer, ω_L is the (siderial) rate of revolution of the two bodies, and θ is the co-latitude of the observer referenced to Earth's center of mass.

The last, rotational term, simply adds to the rotational potential of the earth, adding one part in ~ 750 to the flattening of the earth (about 30 meters) in the case of the moon, but does not appear as a tidal perturbation. The second term is the term of importance, describing the tidal potential resulting in a prolate ellipsoidal deformation aligned along the center of mass line.

This important term is designated

$$W_2 = -\frac{Gma_0^2}{R^3}\left(\frac{3}{2}\cos^2\psi - \frac{1}{2}\right), \tag{2}$$

and it is the rotation of the earth within this potential that produces the observed tides. The term in parentheses is the second order Legendre polynomial, $P_2(\cos\psi)$.

The equipotential surface for a rigid Earth (which is not followed by the rigid surface) affected by the lunar tide would be vary between $(W_2/g)_{\rm max} = 0.358$ m and $(W_2/g)_{\rm min} = -0.179$ m for a total tidal amplitude of 0.538 m. The tidal potential from the sun is 0.45 times that from the moon.

2 Deformable Response

The reaction of a deformable body in the presence of a tidal potential is to reach an equilibrium whereby the surface describes an equipotential. But this redistribution of matter further modifies the potential, exaggerating the effect. A Love number, k, is introduced to account for this added effect:

$$V_{\text{total}} = W_2(1+k). \tag{3}$$

Let's say the earth's surface comes to fluid equilibrium with this modified potential. The radial displacement from the unperturbed surface is then roughly

$$\Delta r \approx \frac{V_{\text{total}}}{g} = \left(-\frac{a_0^2}{GM}\right)\left(-\frac{Gma_0^2}{R^3}\right)(1+k)\left(\frac{3}{2}\cos^2\psi - \frac{1}{2}\right), \tag{4}$$

which we characterize as $\Delta r = \alpha \mathrm{P}_2(\cos\psi)$, where

$$\alpha = \frac{m}{M}\left(\frac{a_0}{R}\right)^3 a_0(1+k). \tag{5}$$

The deformation part of the total potential is just

$$V_{\text{def}} = kW_2 = -\frac{Gma_0^2 k}{R^3}\left(\frac{3}{2}\cos^2\psi - \frac{1}{2}\right), \tag{6}$$

which, in a generalized axisymmetric potential expanded into spherical harmonics,

$$V_{\text{arb}} = -\frac{GM}{r}\left(1 - J_1\left(\frac{a_0}{r}\right)\mathrm{P}_1(\cos\psi) - J_2\left(\frac{a_0}{r}\right)^2 \mathrm{P}_2(\cos\psi) - \ldots\right), \tag{7}$$

corresponds to the $\mathrm{P}_2(\cos\psi)$ term, with

$$J_2 = -\frac{m}{M}\left(\frac{a_0}{R}\right)^3 k \tag{8}$$

at $r = a_0$.

We need to be able to relate J_2 to a given mass distribution, which in this case is a prolate ellipsoid with a surface described by

$$R_{\text{surf}} = a_0 + \alpha \mathrm{P}_2(\cos\psi). \tag{9}$$

To do this, we integrate the mass distribution, dM, to give us the total potential:

$$V = -\int \frac{GdM}{r\left[1 + \left(\frac{\rho}{r}\right)^2 - 2\left(\frac{\rho}{r}\right)\cos\psi\right]^{\frac{1}{2}}}, \tag{10}$$

where ρ is the internal radial coordinate being integrated over. Stacey's book expands this potential, keeping terms to order $1/r^2$, through which it is found that this expression reduces to

$$V = -\frac{GM}{r} - \frac{G}{2r^3}(A + B + C - 3I), \tag{11}$$

which is known as MacCullugh's formula, where A, B, and C are moments of inertia about the x, y, and z axes, respectively, and I is the moment of inertia about the axis to the observer at distance, r. If we place the moon (or sun) along the z axis, this reduces to a familiar form:

$$V = -\frac{GM}{r} + \frac{G}{r^3}(C - A)\left(\frac{3}{2}\cos^2\psi - \frac{1}{2}\right), \tag{12}$$

whereby we can associate J_2 from Equation 7 with the moments of inertia of the body:

$$J_2 = \frac{C - A}{M a_0^2}. \tag{13}$$

This can now be associated with k through Equation 8, but only after computing the moments of inertia of the prolate ellipsoid about its major and minor axes.

2.1 Uniform Density Case

Under the assumption of uniform density, the moments of inertia take the form

$$C = \int (x^2 + y^2) dM = \int r^2 \sin^2 \psi dM = \rho \int_{-1}^{1} d\mu \int_0^{2\pi} d\varphi \int_0^{R_{\rm surf}} r^4 (1 - \mu^2) dr, \tag{14}$$

and

$$A = \int (y^2 + z^2) dM = \int r^2 (\cos^2 \psi + \sin^2 \varphi \sin^2 \psi) dM = \frac{\rho}{2} \int_{-1}^{1} d\mu \int_0^{2\pi} d\varphi \int_0^{R_{\rm surf}} r^4 (1 + \mu^2) dr, \tag{15}$$

after implicitly performing the integration over φ in Equation 15, and with $\mu \equiv \cos\psi$ in both. The expression for the upper limit of the r integrals is found in Equation 9, which just becomes $a_0 + \frac{\alpha}{2}(3\mu^2 - 1)$. These integrals are simple to perform, amounting to just polynomials, though of very high order. However, because $\alpha \ll a_0$, many higher order terms (with α^2 or higher) can be ignored. The moments of inertia become

$$C = \frac{2}{5} M a_0 (a_0 - \alpha), \text{ and } A = \frac{2}{5} M a_0 (a_0 + \frac{1}{2}\alpha), \tag{16}$$

so that $C - A = -\frac{3}{5} M a_0 \alpha$, and

$$J_2 = -\frac{3}{5} \frac{\alpha}{a_0}. \tag{17}$$

Now using Equation 5 to replace α / a_0 above, and associating this with the identification of J_2 in Equation 8, we find that

$$J_2 = -\frac{3}{5} \frac{m}{M} \left(\frac{a_0}{R}\right)^3 (1 + k) = -\frac{m}{M} \left(\frac{a_0}{R}\right)^3 k, \tag{18}$$

leading to a solution for k:

$$k = \frac{3}{2}, \text{ uniform density}. \tag{19}$$

2.2 Real Earth Density Profile

Of course the real earth is not uniform in density. It is, however, in fluid equilibrium with the rotational potential. So we can compute the k value of the fluid earth with the real density profile based on its observed response to the rotational potential.

There are two approaches, delivering similar answers. The first notes that the rotational potential,

$$V_{\rm rot} = -\frac{1}{2} \omega^2 r^2, \tag{20}$$

swings by $\frac{1}{2}\omega^2 a_0^2$ from equator to pole. The shape-induced potential, as would be felt by a satellite not rotating with the earth's surface, is characterized by the $\mathrm{P}_2(\cos\theta)$ term in Equation 7, which has a corresponding equator-to-pole swing of

$$\Delta V_{\text{shape}} = GMJ_2\frac{1}{2a_0} + GMJ_2\frac{a_0^2}{c_0^2}, \tag{21}$$

where c_0 is the earth's polar radius. The ratio of these two potantial swings is defined as k, which computes to

$$k = \frac{2GMJ_2}{\omega^2 a_0^2}\left(\frac{1}{2a_0} + \frac{a_0^2}{c_0^3}\right) = 0.944, \tag{22}$$

with $J_2 = 1.082626 \times 10^{-3}$ being determined empirically.

The other method is more similar to the technique used in Section 2.1, with $\frac{3}{2}\alpha = a_0 - c_0$ as the peak-to-peak amplitude corresponding to the swing in the total potential, which is given by $V_{\text{total}} = (1+k)V_{\text{rot}} = \frac{1}{2}(1+k)\omega^2 a_0^2$. Using this approach, we have

$$\frac{3}{2}\alpha = a_0 - c_0 = \frac{V_{\text{tot}}}{g} = \frac{1}{2}\omega^2 a_0^2\left(\frac{a_0^2}{GM}\right)(1+k), \tag{23}$$

which produces an estimate for k of

$$k = 0.937, \tag{24}$$

matching the value of k given in Stacey's book for the fluid earth with the real density profile. I think the value of k in Equation 22 is the more reliable of the two, given that the earth's ellipticity is represented here, whereas the value of g in Equation 24 is the equatorial value, and not averaged over the geoid.

3 Earth's Tidal Response

The real earth does not have time to arrive at equilibrium with respect to the W_2 tidal potential it rotates underneath. As such, an additional Love number, h, describes how much deflection the earth body experiences relative to that which would be expected directly from the perturbing potential. Empirically, the h, k, and l Love numbers are seen to be:

$$h = 0.612, \tag{25}$$

$$k = 0.303, \tag{26}$$

$$l = 0.04. \tag{27}$$

The definitions of these numbers are somewhat slippery, so I offer here my best interpretation of what these numbers mean.

- h describes the height (radial) displacement attained by the (solid) surface relative to what would have been attained by a perfectly fluid body in response to the perturbing force, neglecting the additional potential generated by the redistribution of matter. Therefore a perfectly fluid body in tidal equilibrium has $h = 1 + k$ (see below). The value $h = 0.612$ is presumably for the solid earth, with a value nearer unity for the oceans.

- k is the ratio of the potential contributed by the tidally deformed body to that of the perturbing potential (i.e., W_2). A rigid body has $k = 0$ because there is no redistribution of matter. A uniform density fluid body has $k = 3/2$, and a fluid body with the earth's density profile has $k = 0.94$. The observed earth tidal response has $k = 0.303$ for the *solid* earth, and $k = 0.245$ for the earth-plus-oceans. I assume these are lower than the equilibrium value of 0.94 because only the low-density exterior of the earth responds to the transient tidal disturbance.
- l is the ratio of horizontal displacement observed to that which would be exhibited by a completely fluid surface. The fact the l is so low is good, but the horizontal displacement of a fluid earth would be comparable to the radial displacement, in the ballpark of 0.5 m. Thus horizontal tidal displacements register on the cm scale.

If I understand everything correctly, the height of the earth body tide is

$$\Delta r = h\frac{W_2}{g}, \tag{28}$$

which, for a perfectly fluid earth ($k = 0.94 \rightarrow h = 1.94$) would have a peak-to-peak amplitude of 1.04 m, and for empirical values has a 0.33 m amplitude.

I had much confusion over this for a while, partly owing to conflicting published information. Stacey's book claims that h reflects the deformation relative to that which would be achieved by a marine tide in tidal equilibrium. This implies that the deformation includes the response to the modified potential induced by the deformation itself (pulling on it's own bootstraps). This self-pulling process converges to an equilibrium indicated, for example, in Equation 18. If Stacey's claim is correct, then Δr would be equal to $h(1+k)W_2/g$, rather than the expression presented in Equation 28, such that $h = 1$ for a perfectly fluid body. But other evidence (including an explicit statement on the website: `http://www.treasure-troves.com/physics/LoveNumber.html`) says that $h = 1 + k$ for a perfect fluid body. This convention is consistent with some of the published derivations of observable quantities, which match, incidently, the corresponding expressions given in Stacey's book.

3.1 Effect on Surface Gravity

Now for the thing we really care about. Fortunately, gravimeters have been a primary tool for measuring tidal effects, so that the theoretical response is well developed. There's only a slight problem in that I do not fully follow the development. But hopefully this will get us mostly there.

The acceleration due to gravity can be expressed as the gradient of the potential at the location of interest. Let us then start by accounting for the total potential. First, there is the base potential from the rotating, oblate earth, V_0. Then a tidal potential (deforming potential) is introduced by the moon or sun, W_n (allowing for higher orders than $n = 2$). The tidally induced deformation generates additional potential, V_n, which we have previously characterized as k_nW_n. Then, because the surface moves in response by some amount ζ, the potential of the surface changes:

$$V = V_0 + W_n + V_n + \zeta_n \left.\frac{\partial V}{\partial r}\right|_{r=a}. \tag{29}$$

Here, the derivative is evaluated at the surface, where $r = a$ (formerly a_0), and

$$\zeta_n = h_n\frac{W_n}{g}, \tag{30}$$

as in Equation 28. It should be noted that a fluid body in tidal equilibrium would arrange such that the displacement, ζ_n, arrives at equipotential, cancelling the W_n and V_n terms, leaving $V = V_0$. The total gravity from the potential given in Equation 29 is

$$g = -\left.\frac{\partial V}{\partial r}\right|_{r=a} = -\left.\frac{\partial V_0}{\partial r}\right|_{r=a} - \left.\frac{\partial W_n}{\partial r}\right|_{r=a} - \left.\frac{\partial V_n}{\partial r}\right|_{r=a} - \zeta_n \left.\frac{\partial^2 V}{\partial r^2}\right|_{r=a}. \tag{31}$$

The first term on the right-hand side is just g_0, the ordinary surface gravity in the absence of tides. For the last term, the first order effect is from the V_0 part of the earth's potential, such that it is equivalent to $2g/r$. The form of W_n follows the form of W_2 in Equation 2:

$$W_n = Gm\frac{r^n}{R^{n+1}}P_n(\cos\psi), \tag{32}$$

such that the second term on the right in Equation 31 is nW_n/a. And here's the part I can't make complete sense of. In the book by Melchior, *The Tides of the Planet Earth*, an expression is given for V_n, the redistribution potential, as

$$V_n = \frac{4\pi G}{a^{n+1}} \int_0^a \rho \frac{\partial}{\partial r}\left(\frac{r^{n+3}}{2n+1}Y_n\right) dr, \tag{33}$$

where ρ is the density and Y_n is a spherical harmonic describing the distortion. If one ignores the dependence of the integral on the upper limit, a, and differentiates with respect to a (as is the effect of differentiating with respect to r and evaluating at a), one finds that the third term in Equation 31 amounts to $-(n+1)V_n/a$. The catch is twofold. First, I don't see why one can ignore the integral in the differentiation. Second, if $V_n = k_n W_n$ then this derivative would be nk_nW_n/a, which differs by a factor of $-n/(n+1)$ from the earlier expression. Yet it is the earlier expression which appears in the development of Melchior. Suspending disbelief and continuing with the analysis,

$$g = g_0 - \frac{nW_n}{a} + (n+1)\frac{V_n}{a} - \zeta_n\frac{2g}{a}, \tag{34}$$

so that the change in surface gravity, incorporating Equation 30, and $V_n = k_nW_n$, is

$$\Delta g = -\left(n - (n+1)k_n + 2h_n\right)\frac{W_n}{a}, \tag{35}$$

and observing that $\left.\partial W_n/\partial r\right|_{r=a} = nW_n/a$,

$$\Delta g = -\left(1 + \frac{2}{n}h_n - \frac{n+1}{n}k_n\right)\left.\frac{\partial W_n}{\partial r}\right|_{r=a}. \tag{36}$$

This form mimics the gravitational contribution directly from the deforming potential, W_n, with a coefficient differing from unity as a result of earth's deformation. For the only significant terms, $n = 2$ and $n = 3$ (moon only), this coefficient, δ_n, is

$$\delta_2 = 1 + h_2 - \frac{3}{2}k_2, \text{ and} \tag{37}$$

$$\delta_3 = 1 + \frac{2}{3}h_3 - \frac{4}{3}k_3. \tag{38}$$

The values of h_3 and k_3 are approximately the same as their second-order cousins.

Again, this analysis is not completely satisfying to me because I don't understand the treatment of V_n, as elaborated above. But Equation 37 is exactly the same expression that appears in Stacey's book, and that quoted by Ken in his e-mail. As long as Stacey didn't snag his equation (which appears only in the second edition—not the third) from Melchior's book, then one may justify placing increased confidence in these results.

The total peak-to-peak swing of the gravity signal from the moon is then about 190 μGal, equivalent to a 0.622 m peak-to-peak displacement from the earth center—roughly twice the displacement actually experienced.

With realistic values for h and k, $\delta_2 = 1.16$. For a rigid earth where h and k are zero, $\delta_2 = 1$. For a uniform density fluid body in tidal equilibrium, $\delta_2 = 1.25$. For a fluid body with Earth's density profile, $\delta_2 = 1.53$. These values are remarkably similar for a wide range of scenarios, which I can't say I would have predicted.

3.2 Relation to Gravimeter

One may develop an expression for the signal generated by the gravimeter that includes tides and other local effects. This may look like

$$\Delta g = -\delta_2 \left(\left. \frac{\partial W_{2\odot}}{\partial r} \right|_{r=a} + \left. \frac{\partial W_{2(}}{\partial r} \right|_{r=a} \right) - \delta_3 \left. \frac{\delta W_{3(}}{\partial r} \right|_{r=a} + \sum_i \left(\zeta_i \left. \frac{\partial g}{\partial r} \right|_{r=a} + \gamma_i(\zeta_i) \right), \tag{39}$$

where the parentheses as subscripts represent contributions from the moon. The last term encapsulates all the non-tidal effects on the gravimeter such as loading from the ocean, atmosphere, and ground water. This accomodates both the effect of the displacement, ζ_i, relative to the earth center of mass, and the direct gravitational attraction, γ_i, which ought to be strongly tied to ζ_i through some sort of "admittance". Insofar as the tidal terms on the left can be well modeled, the total contribution from the summation on the right is measured by the gravimeter. Other measurements, such as lunar range and barometric pressure can help break the degeneracy of the summation, with some assumption about the relation of γ_i and ζ_i.

The value of $\partial g/\partial r$ on the earth's surface amounts to 307 μGal m^{-1}, inverting to 3.25 mm μGal^{-1}. Thus sub-mm precision from gravimetry requires gravity measurements (and therefore tidal modeling) to the precision of about 0.1 μGal (1 Gal is 1 cm s^{-2}). Using the values of δ_n from Equation 36, and employing the empirical values of h and k, the moon contributes a total amplitude signal in Δg of 191 μGal in W_2, 5.5 μGal in W_3, and 0.07 μGal in W_4. The sun contributes 86 μGal in W_2, and 0.006 μGal in W_3. Therefore we only need consider those terms appearing explicitely in Equation 39.

Ken has a point in that without knowing the Love numbers to a fraction of a percent, one will not be able to subtract the exact tidal contribution from the measured Δg. We therefore need to think in terms of a simultaneous solution involving the LLR and gravimeter measurements, relying on the frequency domain to separate tidal responses from random, weather-related gravimeter responses.