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Correction of satellite tracking data for an arbitrary tropospheric profile

John W. Marini

Goddard Space Flight Center
Greenbelt, Maryland 20771

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A procedure is given for deriving elevation-error and range-error correction equations in a form suitable for use in the rapid processing of satellite tracking data. The refractivity of the troposphere is assumed to have spherical symmetry, but may have any given profile that does not depart greatly from standard. When the procedure was tested for numerical accuracy by application to an exponential profile, the corrections calculated agreed with those obtained by ray tracing to 0.3% or better over a range of surface refractivity from 200 to 450 and a range of radiowave arrival angles from horizontal to vertical.

1. INTRODUCTION

Range and elevation-angle errors caused in satellite tracking data by tropospheric refractivity (and the refractivity of the remainder of the lower nonionized atmosphere) can be calculated by ray-tracing methods [*Thayer,* 1967], or by numerical integration [*Blake,* 1968]; but the compution time required is excessive for the routine processing of large quantities of data. Formulas simple enough to be used for such processing have been published, but the necessary simplification has been obtained either by assuming an exponential profile [*Thayer,* 1961; *Rowlandson and Moldt,* 1969], or by neglecting path curvature [*Hopfield,* 1969], or both [*Freeman,* 1962]. Neglecting path curvature is mathematically equivalent to retaining the first term of a series expansion in powers of the surface refractivity of the assumed profile, a procedure that is not accurate at low elevation angles.

The derivation of formulas specifically intended for the correction of satellite tracking data, however, can be accomplished without making either of these simplifying assumptions if advantage is taken of the nature of the data collected.

2. SATELLITE TRACKING DATA

The satellites tracked are ordinarily above the region, which extends from the ground to about 70 km in altitude, where almost all the radio-ray bending caused by the nonionized atmosphere of the earth takes place. Most integrals occurring in the formulation of the corrections therefore may be extended to infinity with negligible error.

In a typical pass of a satellite over a ground station, the satellite might be under observation for several minutes. During this period, tracking measurements are taken periodically at a rate, perhaps, of one set of data per second. To simplify the formulas used to correct the large volume of data generated, it is possible to take advantage of the circumstance that the corrections are exercised repeatedly, using varying values of elevation angle and range, but with a fixed atmospheric profile. Initial or 'pre-pass' calculations that involve only the atmospheric conditions at the time of the satellite pass and that are independent of satellite position may be lengthy without causing a significant percentage increase in the total computer time per satellite pass. For efficiency in calculation, therefore, the mathematical formulation of the corrections should be such that quantities functionally dependent on the parameters of the atmospheric profile are separated from those dependent on elevation and range.

Such a separation can be effected by expansions in rational functions of the sine of the elevation angle and negative powers of the range. The coefficients in these expansions will depend on the atmospheric profile alone, and can be calculated in advance of the satellite pass.

3. GEOMETRY AND NOTATION

The geometry involved is shown in Figure 1. The earth is considered spherical with a radius a nomi-

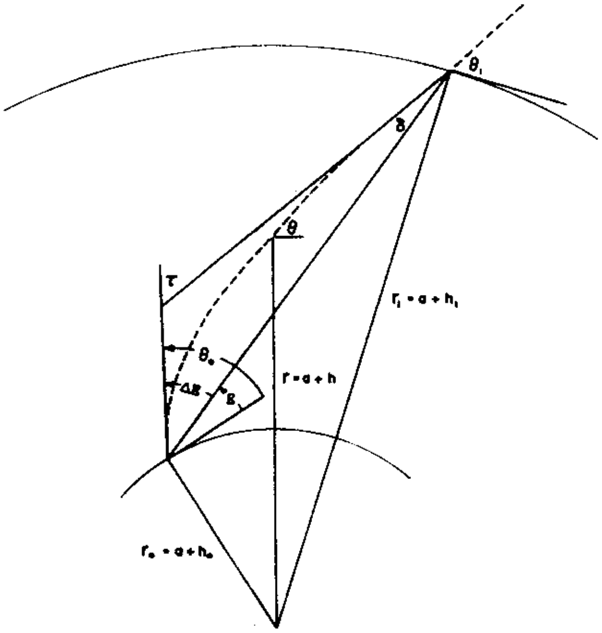

Fig. 1. Satellite and tracking station geometry.

nally equal to 6373 km. The tracking station is located at a distance r_0 from the center of the earth and at a height h_0 above sea level. The satellite is at a distance r_1 from the center of the earth and at a height h_1. The radio-ray path between the satellite and the station is shown as a dotted line. The distance between the center of the earth and a given point on the ray path is r, the height of the point above sea level is h, and the elevation of the ray path at the point is θ.

The angle of arrival is the angle θ_0 above the local horizontal at the ground station. The angle ΔE is the elevation error, or difference between the angle of arrival and the true elevation angle E of the satellite with respect to the ground station

$$\Delta E = \theta_0 - E \tag{1}$$

and τ is the total bending of the radio path. The refractive index n is assumed to depend only on the height h above the surface of the earth. The radio refractivity

$$N(h) = 10^6[n(h) - 1] \tag{2}$$

will be used in normalized form, and a normalized height above the tracking station will be employed. Taking the refractive index and the refractivity at the tracking station to be

$$n_0 = n(h_0) \tag{3}$$

and

$$N_0 = N(h_0) \tag{4}$$

respectively, and taking the effective height of the troposphere above the tracking station to be

scale height

$$H = \frac{1}{N_0} \int_{h_0}^{\infty} N(h)\, dh \tag{5}$$

the normalized height above the tracking station

$$x = (h - h_0)/H. \qquad 0 \le x < \infty \tag{6}$$

is defined. In terms of this variable, the refractivity may be written as

$$N(h) = N_0 f(x) \tag{7}$$

where the normalized profile $f(x)$, which will be abbreviated as f, is

$$f = N(h_0 + Hx)/N_0 \tag{8}$$

which is equal to unity at $x = 0$

$$f(0) = 1. \tag{9}$$

and integrates to unity

$$\int_0^{\infty} f\, dx = 1 \tag{10}$$

from the tracking station upward. In the particular case of an exponential profile, f is equal to exp $(-x)$.

The radio range, or electrical distance along the ray path is designated as R_e,

$$R_e = \int_{r_0}^{r_1} \frac{n}{\sin\theta}\, dr \tag{11}$$

the geometrical distance along the ray path is

$$R_g = \int_{r_0}^{r_1} \frac{1}{\sin\theta}\, dr \tag{12}$$

and the straight-line distance or slant range is R. The range error is then the difference

$$\Delta R = R_e - R \tag{13}$$

The parameters

$$p = (2H/r_0)^{1/2} \tag{14}$$

$$q = 10^{-6} N_0 r_0/H = 2 \times 10^{-6} N_0/p^2 \tag{15}$$

$$Q = q \cos^2\theta_0 \tag{16}$$

the normalized sine of the angle of arrival

$$\alpha = (1/p) \sin\theta_0 \tag{17}$$

the normalized sine of the elevation angle

$$\beta = (1/p)\sin E \quad (18)$$

and the normalized inverse range

$$\rho = pr_0/R \quad (19)$$

will be used. Typical values are $p = 0.05$ and $q = 0.25$. The value of α ranges from 0 on the horizon to about 20 for a wave arriving vertically downward.

4. INPUTS TO THE CORRECTION FORMULAS

The desired form of the correction formulas for the elevation error ΔE and the range error ΔR depend on the intended use. In the first case, which applies, for example, to a tracking radar, it is assumed that measured values of the angle of arrival θ_0 and the radio range R_e are available. The quantities ΔE and ΔR should then be given as functions of the variables θ_0 and R_e.

In the second case the satellite ephemeris is assumed to be known from previous tracking and orbit determination. In this case the true slant range R and the true elevation angle E will be known quite accurately as functions of time, and one or both of the corrections ΔE and ΔR may be needed to provide accurate predictions of the angle of arrival and radio range either for acquisition or for comparison with values to be measured. The latter comparisons are used to improve the satellite ephemeris by an iterative process that minimizes some weighted function of the differences observed. Here ΔE and ΔR need to be expressed as functions of E and R.

In both cases above, the percentage difference between R and R_e is small. R_e is at worst about 200 meters larger than R, while R has been taken to be at least 70 km. Consequently the distinction between these quantities may be neglected in their use as inputs to the error formulas. The distinction between E and θ_0, however, must be retained at lower elevation angles.

5. ELEVATION ERROR WITH ANGLE OF ARRIVAL KNOWN

A procedure for determining the elevation error has been given by *Bean and Thayer* [1959]. The bending τ is first calculated by integration, and the angle δ is next calculated from the geometry of Figure 1 using the known value of τ. The elevation error is then given by

$$\Delta E = \tau - \delta \quad (20)$$

The usual approximation for the bending is [*Thayer*, 1961]

$$\tau = 10^{-6} N_0 \cos\theta_0 I(\alpha)/p \quad (21)$$

where the bending integral is defined as

$$I(\alpha) = \int_0^\infty \frac{-f'}{[x + \alpha^2 - Q(1-f)]^{1/2}}\,dx \quad (22)$$

in which it has been assumed that the satellite height h_1 is great enough to permit the upper limit to be extended to infinity.

An approximate equation giving δ (and therefore ΔE) in terms of τ has been given by *Rowlandson and Moldt* [1969]

$$\Delta E \cong 10^{-3} N_0 (1/p) \cos\theta_0 [I(\alpha) - \rho L(\alpha)] \text{ mrad} \quad (23)$$

with

$$L(\alpha) = 1 - \alpha I(\alpha) + \tfrac{1}{4} q I^2(\alpha) \quad (24)$$

This equation is simpler than the exact expression, and the approximations used to obtain it are consistent with those employed elsewhere in this paper.

6. A DIGRESSION ON THE APPROXIMATIONS USED

The quantities $10^{-6} N_0 \sim 300 \times 10^{-6}$ and $p^2 \sim 0.0025$ are neglected compared with unity. However, the quantity $q \sim 0.25$ cannot be so neglected, even though it contains the surface refractivity as a factor.

The quantity Q that appears in the radical in (22) is not independent of satellite position because of its dependence on θ_0; this would lead to complication of the formulas to be derived. Fortunately Q may be replaced by q with negligible error, since the term neglected thereby is small compared to the square of α

$$q \sin^2\theta_0 (1 - f) \ll \alpha^2 = (1/p^2)\sin^2\theta_0 \quad (25)$$

It will also be found that the same approximation can be made elsewhere in the formulas to be derived; for example, in the asymptotic expansions that follow, in which the quantity neglected by replacing Q by q in one term of the expansion is small compared with the preceding term.

Because the approximation is used in (22), it has also been used in (24) and in similar polynomial expressions to preserve accuracy when α is large, i.e., to obtain the correct asymptotic expansion of the polynomial expression.

7. EXPANSIONS OF THE BENDING INTEGRAL

To use the formula (23) for the elevation error, a rapid method for calculating the bending integral (22) is needed. Since α is as large as 20 at high elevation angles, it is natural to expand (22) in powers of $1/\alpha$. This can be accomplished by a formal binomial expansion of the radical in (22) after the square of α has been factored out (followed by an integration by parts of the factors composed of f' and powers of f). The expansion obtained is

$$I(\alpha) \sim (1/\alpha) - I_1(1/\alpha)^3 + I_2(1/\alpha)^5 - \cdots \qquad (26)$$

where

$$I_1 = (\tfrac{1}{2})[1 - (\tfrac{1}{2})q] \qquad (27)$$

and

$$I_2 = \frac{3}{4}\left[\int_0^\infty xf\,dx - q\left(1 - \frac{1}{2}\int_0^\infty f^2\,dx\right) + \tfrac{1}{6}q^2\right] \qquad (28)$$

This expansion is asymptotic if $q = 0$ and $f = \exp(-x)$ [*Abromowitz and Stegun*, 1964]; and the development proceeds formally under the assumption that the expansion provides a valid approximation in general when α is large, although this has not been investigated.

Use of the first term of (26) in (21) results in the familiar

$$\tau \sim 10^{-3}N_0 \cot\theta_0 \qquad \text{mrad} \qquad (29)$$

which holds at high elevation angles.

Equation 26 suffers from the usual deficiency of an asymptotic expansion; it is not useful at small values of the argument α. There is, it happens, a procedure [*Wall*, 1948] for converting a divergent series such as (26) into a continued fraction expansion that converges, at least when $q = 0$ and $f = \exp(-x)$, for all $\alpha > 0$. The expansion diverges at $\alpha = 0$, however, and converges only slowly when α is near zero. Rather than apply the procedure directly, therefore, the integral $I(\alpha)$ is expanded, with α small, as

$$I(\alpha) = i_0 - i_1\alpha + \cdots \qquad (30)$$

where

$$i_0 = I(0) = \int_0^\infty \frac{-f'}{[x - q(1-f)]^{1/2}}\,dx \qquad (31)$$

and, differentiating (22), substituting $z = x/\alpha^2$, and letting α approach zero

$$i_1 = -I'(0) = -2f'(0)/[1 + qf'(0)] \qquad (32)$$

Noting that (26) approximates $I(\alpha)$ when α is large, and that (30) approximates $I(\alpha)$ when α is small, the approximation of $I(\alpha)$ over the entire range of α is accomplished by means of a ratio of polynomials in α. The coefficients of the polynomials are chosen in such a way that the expansion of the ratio in inverse powers of α agrees with the leading terms of (26) on the one hand, and its expansion in ascending powers of α agrees with the leading terms of (30) on the other hand. This method of approximation insures accuracy if α is either large or small. Accuracy with intermediate values is obtained by the inclusion of a sufficient number of terms from each series expansion. The number of terms used here—three from (26) and two from (30)—is not necessarily optimum, but worked out well when the method was applied to an exponential profile. It is evident that the method requires a certain degree of smoothness in the profile if an accurate approximation is to be obtained with this number of terms.

8. FORM OF THE APPROXIMATION

Consider a rational function of α, $F(\alpha; F_1, F_2, f_0, f_1)$, that depends on four parameters F_1, F_2, f_0 and f_1 and is expressed in the form of a continued fraction

$$F(\alpha; F_1, F_2; f_0, f_1) \equiv \cfrac{1}{\alpha + \cfrac{\hat{f}_1}{\alpha + \cfrac{\hat{f}_2}{\alpha + \cfrac{\hat{f}_3}{\alpha + \hat{f}_4}}}} \qquad (33)$$

where the intermediate constants $\hat{f}_1$, $\hat{f}_2$, $\hat{f}_3$, and $\hat{f}_4$ are calculated from the set of parameters F_1, F_2, f_0, and f_1 using in sequence

$$\hat{f}_1 = F_1 \qquad (34)$$

$$\hat{f}_2 = (F_2/\hat{f}_1) - \hat{f}_1 \qquad (35)$$

$$\hat{f}_3 = \hat{f}_2 \bigg/ \left[f_0^2\hat{f}_1\left(1 + \frac{\hat{f}_1}{\hat{f}_2}\right) - (1 + f_1\hat{f}_1)\right] \qquad (36)$$

$$\hat{f}_4 = f_0\hat{f}_1\hat{f}_3/\hat{f}_2 \qquad (37)$$

On clearing the denominator of equation 33 of fractions, and expanding the resulting fractional form by long division in descending powers of α

$$F(\alpha; F_1, F_2; f_0, f_1) = (1/\alpha) - F_1(1/\alpha)^3 + F_2(1/\alpha)^5 - \text{const}\,(1/\alpha)^7 \cdots \qquad (38)$$

If the long division is carried out using ascending powers of α

$$F(\alpha; F_1, F_2; f_0, f_1) = f_0 - f_1 \alpha + \text{const}\, \alpha^2 \cdots \quad (39)$$

Thus the function $F(\alpha; F_1, F_2, f_0, f_1)$ is well suited to approximate $I(\alpha)$ provided that the parameters F_1, F_2, f_0, and f_1 are chosen as I_1, I_2, i_0, and i_1, respectively; i.e., $I(\alpha)$ is approximately equal to $F(\alpha; I_1, I_2; i_0, i_1)$. Note that these latter parameters, equations 27, 28, 31, and 32, depend only on the refractivity through the parameter q and the profile $f(x)$. Their calculation is independent of satellite position, and numerical evaluation prior to each satellite pass need not be ruled out. If f is a given model profile, moreover, the integrals can be evaluated analytically if the functional form of $f(x)$ permits; or, otherwise, the integrals may be evaluated numerically and curve or surface fitted empirically.

9. FORMULA FOR RANGE ERROR

The range error (13) may be written as the sum of the difference between the electrical and geometric distances along the ray path and the difference between the geometric distance along the ray path and the slant range

$$\Delta R = (R_e - R_g) + (R_g - R) \quad (40)$$

The first term in (40), the difference along the ray path, is, after division by r_0

$$(R_e - R_g)/r_0 = (1/r_0) \int_0^\infty [10^{-6} N(h)/\sin\theta]\, dr \quad (41)$$

$$\doteq (\tfrac{1}{2}) 10^{-6} N_0 p\, J(\alpha)$$

where

$$J(\alpha) \equiv \int_0^\infty \frac{f}{[x + \alpha^2 - q(1-f)]^{1/2}}\, dx \quad (42)$$

Although the geometrical difference $(R_g - R)$ between the ray and straight-line paths is considerably smaller than the electrical difference $(R_e - R_g)$ along the ray path, it is not always negligible. In Appendix A an expression for this difference is derived. Substituting (A5) and (41) into (40), the expression for the range error is

$$\Delta R/r_0 = \tfrac{1}{2} 10^{-6} N_0 p [M(\alpha) - \tfrac{1}{2} p Q L^2(\alpha)] \quad (43)$$

where

$$M(\alpha) \equiv J(\alpha) + q[I(\alpha) - \tfrac{1}{2} K(\alpha) - \tfrac{1}{2}\alpha I^2(\alpha) + \tfrac{1}{12} q I^3(\alpha)] \quad (44)$$

with

$$K(\alpha) \equiv \int_0^\infty \frac{-2ff'}{[x + \alpha^2 - q(1-f)]^{1/2}}\, dx \quad (45)$$

The calculation of $J(\alpha)$ and $K(\alpha)$ can be carried out in the same manner as that of $I(\alpha)$, but it is more efficient to calculate $M(\alpha)$ directly. Thus, expanding the integrals on the right-hand side of (44) asymptotically

$$M(\alpha) \sim (1/\alpha) - M_1 (1/\alpha)^3 + M_2 (1/\alpha)^5 - \cdots \quad (46)$$

where

$$M_1 = \frac{1}{2}\left[\int_0^\infty xf\, dx - q\left(1 - \frac{1}{2}\int_0^\infty f^2\, dx\right)\right] \quad (47)$$

$$M_2 = \frac{3}{4}\left[\frac{1}{2}\int_0^\infty x^2 f\, dx - q\left(\frac{1}{6} + \int_0^\infty xf\, dx - \frac{1}{2}\int_0^\infty xf^2\, dx\right) + q^2\left(\frac{1}{2} - \frac{1}{2}\int_0^\infty f^2\, dx + \frac{1}{6}\int_0^\infty f^3\, dx\right)\right] \quad (48)$$

Expanding the right-hand side of (44) for small values of α

$$M(\alpha) = m_0 - m_1 \alpha + \cdots \quad (49)$$

with

$$m_0 = j_0 + q i_0 + (\tfrac{1}{12}) q^2 i_0^3 - (\tfrac{1}{2}) q k_0 \quad (50)$$

$$m_1 = j_1 + (\tfrac{1}{2}) q i_0^2 [1 + (\tfrac{1}{2}) q i_1] \quad (51)$$

where

$$j_0 = J(0) = \int_0^\infty \frac{f}{[x - q(1-f)]^{1/2}}\, dx \quad (52)$$

$$j_1 = 2/[1 + q f'(0)] \quad (53)$$

$$k_0 = K(0) = \int_0^\infty \frac{-2ff'}{[x - q(1-f)]^{1/2}}\, dx \quad (54)$$

$M(\alpha)$ is now calculated using (33)–(37); i.e., it is given by $F(\alpha; M_1, M_2; m_0, m_1)$.

10. NUMERICAL EXAMPLE USING AN EXPONENTIAL PROFILE

If an exponential refractivity profile is assumed, the normalized profile f becomes

$$f(x) = e^{-x} \quad (55)$$

The effective height H is equal to the reciprocal of the decay constant of the profile and, for convenience, was estimated from the refractivity at the ground station using the empirical formula [*Bean and Thayer*, 1959]

$$1/H = \ln \frac{N_0}{N_0 - 7.32 \exp (0.005577 N_0)} \qquad (56)$$

although this would not be the best choice in satellite tracking at high elevation angles. Equation 5 should be used instead, the value of the integral being estimated, for example, from pressure and water vapor measurements [*Hopfield*, 1971].

Most of the required integration, that in (28), (47), and (48), was performed directly. The integrals in (31) and (54), however, had to be handled numerically. The integral (31) was evaluated at 20 different values of q spread over the expected range $0.1 < q < 0.7$ using associated Gauss-Laguerre quadrature [*Concus et al.*, 1963] with $x^{1/2}$ factored out of the radical. The values obtained were then approximated by an exponential expression (a polynomial could have been used) using a least-squares fit:

$$i_0 = (\pi)^{1/2}(1 - 0.9206\, q)^{-0.4468} \pm 0.04\% \qquad 0 \le q \le 0.7 \qquad (57)$$

The integral k_0 was also closely approximated by a similar exponential expression. The equations for the corrections, some of which have been changed to unnormalized form, are collected in Appendix B.

The refractivity at the tracking station was taken to be $N_0 = 313$, with the tracking station at sea level, $r_0 = 6373$ km. The pre-pass calculations of Appendix B were performed, giving $H = 6.9513$ km, $p = 0.046706$, $q = 0.28696$, and the equation for the elevation error

$$\Delta E = 0.313 \cos \theta_0 [i - (6373/R)L] \quad \text{mrad} \qquad (58)$$

with

$$i = \cfrac{1}{\sin \theta_0 + \cfrac{0.00093424}{\sin \theta_0 + \cfrac{0.0021163}{\sin \theta_0 + \cfrac{0.0060511}{\sin \theta_0 + 0.11626}}}} \qquad (59)$$

$$L = 1 - i \sin \theta_0 + 0.0001565\, i^2 \qquad (60)$$

The equation for the range correction became

$$\Delta R = 0.0021757[m - (914.40/R)L^2 \cos^2 \theta_0] \text{ km} \qquad (61)$$

$$m = \cfrac{1}{\sin \theta_0 + \cfrac{0.00085599}{\sin \theta_0 + \cfrac{0.0021722}{\sin \theta_0 + \cfrac{0.0060788}{\sin \theta_0 + 0.11571}}}} \qquad (62)$$

The corrections (58) and (61) were calculated using twelve values of θ_0 from 0 to 900 mrad and two different values of the range R (in kilometers) at each value of θ_0. (These ranges are for satellites at heights with respect to the tracking station of 70 km and 475 km.) The computer output, in exponential form, is shown in Table 1. The corrections calculated using a double-precision ray-trace program are also listed for comparison.

The largest difference is about 0.3%. If desired, a final empirical adjustment of the coefficients can easily be made to reduce this error to less than 0.1% [*Marini*, 1971]. Similar checks were made at $N_0 = 200$ and $N_0 = 450$ with equivalent results.

The computer time required to perform the pre-pass calculations for Table 1, i.e. to obtain (58)–(62), was 2/60 sec (IBM 360/95). About the same amount of time was needed to compute the twenty-four pairs of corrections in Table 1; this represents a rate of about 700/sec. Programs designed for operational use should show improvements over these times.

11. CORRECTIONS USING KNOWN ELEVATION ANGLE

The formulas for the corrections ΔE and ΔR as explicit functions of E and R rather than θ_0 and R are more difficult to obtain. *Reichley* [1967] used a perturbation method to obtain the first two terms of the expansion of these corrections in powers of the surface refractivity. This, in the notation and with the approximations used here, is equivalent to an expansion in powers of the parameter q which may be as large as 0.64 at $N_0 = 450$. Such an expansion is accurate at large values of the elevation angles as may be determined by an examination of the asymptotic expansions, in which the higher powers of q appear only in the higher order terms, but more terms are needed if formulas numerically accurate at small

TABLE 1. Corrections calculated from arrival angle and range

		ΔE, mrad			ΔR, km		
θ_0, mrad	R, km	Equation 58	Ray-trace	Percent error	Equation 61	Ray-trace	Percent error
0	1020.5	1.109E01	1.109E01	0.00	1.018E-01	1.019E-01	-0.06
0	2587.7	1.262E01	1.262E01	0.00	1.038E-01	1.039E-01	-0.06
1	1011.6	1.079E01	1.079E01	0.00	9.856E-02	9.863E-02	-0.06
1	2578.9	1.227E01	1.227E01	0.00	1.004E-01	1.005E-01	-0.06
2	1002.9	1.050E01	1.051E01	-0.01	9.548E-02	9.555E-02	-0.07
2	2570.1	1.194E01	1.194E01	-0.01	9.717E-02	9.724E-02	-0.07
4	986.0	9.971E00	9.975E00	-0.04	8.980E-02	8.989E-02	-0.10
4	2553.1	1.131E01	1.132E01	-0.04	9.125E-02	9.134E-02	-0.10
8	953.8	9.033E00	9.043E00	-0.11	8.006E-02	8.018E-02	-0.16
8	2520.2	1.022E01	1.023E01	-0.10	8.113E-02	8.126E-02	-0.16
15	902.0	7.721E00	7.738E00	-0.22	6.691E-02	6.707E-02	-0.24
15	2466.2	8.693E00	8.710E00	-0.19	6.758E-02	6.774E-02	-0.24
30	805.6	5.817E00	5.834E00	-0.28	4.880E-02	4.893E-02	-0.27
30	2360.8	6.499E00	6.514E00	-0.22	4.908E-02	4.921E-02	-0.27
65	633.6	3.589E00	3.594E00	-0.13	2.900E-02	2.904E-02	-0.15
65	2147.2	3.965E00	3.969E00	-0.09	2.906E-02	2.911E-02	-0.15
100	512.0	2.547E00	2.548E00	-0.04	2.027E-02	2.029E-02	-0.10
100	1962.7	2.799E00	2.799E00	-0.02	2.030E-02	2.032E-02	-0.10
200	316.8	1.350E00	1.350E00	0.02	1.073E-02	1.073E-02	-0.08
200	1546.6	1.477E00	1.477E00	0.02	1.073E-02	1.074E-02	-0.08
400	174.9	6.616E-01	6.615E-01	0.03	5.556E-03	5.560E-03	-0.08
400	1046.4	7.234E-01	7.233E-01	0.02	5.556E-03	5.561E-03	-0.08
900	89.1	2.234E-01	2.233E-01	0.03	2.774E-03	2.776E-03	-0.08
900	593.8	2.443E-01	2.443E-01	0.02	2.774E-03	2.776E-03	-0.09

values of the elevation angle are to be obtained by this method. Accurate formulas can be obtained by expanding instead in powers of ρ, β, and $1/\beta$, and by using the method of the preceding paragraphs [*Marini*, 1971]; but the derivation of these formulas is complicated and is not given here. Instead, it is noted that the corrections can be calculated without the aid of these explicit formulas by using iteratively the formulas already derived.

If θ_0 in (58), (59), and (60) is replaced by $E + \Delta E$, then these equations are in the correct form for an iterative solution for ΔE when the value of E (and R) is given. An iterative solution for ΔE has the disadvantage that the iterative process has to be repeated for each satellite position. The speed with which (58)–(60) can be computed, however, makes the use of iteration feasible, especially since the convergence is rapid. Using Wegstein's method [*Lance*, 1960] it was found that (58) needed to be exercised no more than five times in the worst case to reproduce the corrections in Table 1 using given values of E instead of θ_0. To begin the iteration, a starting value of zero was used for ΔE except where the satellite was below the geometric horizon, in which case the starting value was taken to be $|E|$.

12. DISCUSSION

When ray-trace methods are used to calculate range and elevation-angle errors, integrals for the bending, electric distance along the ray path, and the ground range (or the sums to which these integrals are reduced by layering) must be calculated for each of the many data points of a satellite pass. The integrals, moreover, must be calculated with great precision since the range error is calculated by taking the difference between the electrical range and the slant range, two nearly equal numbers.

In the method given by *Thayer* [1967] the number of calculations required over a satellite pass is considerably reduced since the coefficients of the terms in the sums used to evaluate the integrals are independent of the elevation angle of the ray. The elevation angle at each layer must be calculated for each new ray, however, and the need for great precision remains.

The formulas derived here require, instead, the once-per-pass evaluation of the nine integrals occurring in (5), (28), (31), (48), (52), and (54). The relative accuracy in each integration need be only slightly greater than that expected in the final corrections; and, in the case of integrals occurring only in higher order terms, the relative accuracy can probably be further reduced without effecting the final accuracy. The final accuracy is, however, limited by the number of terms selected from the series used to form the corrections.

If a standard refractivity profile involving one or two parameters is used, the numerical integrations required can be reduced in number or eliminated altogether by direct integration or appropriate curve or surface fitting, as in the exponential example given.

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APPENDIX A
APPROXIMATION FOR THE GEOMETRICAL DIFFERENCE

Using Snell's law in polar coordinates [*Bean and Dutton*, 1966], R_g from (12) may be written as

$$R_g = \int_{r_0}^{r_1} \frac{(r + n_0^2 r_0^2 \cos^2 \theta_0 n'/n^3) - n_0^2 r_0^2 \cos^2 \theta_0 n'/n^3}{(r^2 - n_0^2 r_0^2 \cos^2 \theta_0/n^2)^{1/2}} \, dr \qquad \text{(A1)}$$

Integrating the first term, setting $n(r_1) = 1$, making use of the geometrical relationship

$$r_1 \sin \theta_1 = R \cos \delta + r_0 \sin (\theta_0 - \tau) \qquad \text{(A2)}$$

which follows from Figure 1, and using the exact integral for τ

$$\tau = -r_0 \cos \theta_0 \int_0^\infty \frac{n_0 n'}{n^2(r^2 - n_0^2 r_0^2 \cos^2 \theta_0/n^2)^{1/2}} \, dr \qquad \text{(A3)}$$

there results

$$\begin{aligned}(R_g - R)/r_0 &= -(R/r_0)(1 - \cos \delta) - (1 - \cos \tau) \sin \theta_0 \\ &\quad + (\tau - \sin \tau) \cos \theta_0 + \cos^2 \theta_0 \\ &\quad \cdot \int_0^\infty \frac{n_0(n - n_0)n'}{n^3(r^2 - n_0^2 r_0^2 \cos^2 \theta_0/n^2)^{1/2}} \, dr \qquad \text{(A4)}\end{aligned}$$

On approximating $\cos \delta$, $\sin \tau$, and $\cos \tau$, by the first two terms of their power series expansions and making the usual approximations in the integrals, there results

$$\begin{aligned}(R_g - R)/r_0 &= \tfrac{1}{2} 10^{-6} N_0 p Q[(I - \tfrac{1}{2} K - \tfrac{1}{2}\alpha I^2 + q I^3/12) \\ &\quad - (r_0 p/2R)(1 - \alpha I + \tfrac{1}{4} q I^2)^2] \qquad \text{(A5)}\end{aligned}$$

APPENDIX B
CORRECTION EQUATIONS FOR AN EXPONENTIAL PROFILE, ARRIVAL ANGLE KNOWN

$$H = 1/\ln [N_0/(N_0 - 7.32 e^{0.005577 N_0})] \qquad \text{km}$$

$$p = (2H/r_0)^{1/2}$$

$$q = 10^{-6} N_0 r_0/H$$

$$i_0 = \pi^{1/2}(1 - 0.9206q)^{-0.4468}$$

$$i_1 = 2/(1 - q)$$

$$I_1 = (\tfrac{1}{2})(1 - \tfrac{1}{2}q)$$

$$I_2 = 0.75[1 - 0.75q + (\tfrac{1}{6})q^2]$$

$$k_0 = (2\pi)^{1/2}(1 - 0.9408q)^{-0.4759}$$

$$m_0 = i_0[1 + q + (\tfrac{1}{12})q^2 i_0^2] - (\tfrac{1}{2})q k_0$$

$$m_1 = 2(1 + (\tfrac{1}{4})q i_0^2)/(1 - q)$$

$$M_1 = (\tfrac{1}{2})[1 - (\tfrac{3}{4})q]$$

$$M_2 = 0.75[1 - (25/24)q + (11/36)q^2]$$

$$i = F(\sin \theta_0; p^2 I_1, p^4 I_2; i_0/p, i_1/p^2)$$

$$L = 1 - i \sin \theta_0 + \tfrac{1}{2} 10^{-6} N_0 i^2$$

$$\Delta E = 0.001 N_0 \cos \theta_0 (i - r_0 L/R) \qquad \text{mrad}$$

$$m = F(\sin \theta_0; p^2 M_1, p^4 M_2; m_0/p, m_1/p^2)$$

$$\Delta R = 10^{-6} N_0 H[m - \tfrac{1}{2} 10^{-6} N_0 r_0^2 \cos^2 \theta_0 L^2/(RH)] \quad \text{km}$$

The function i is equal to $I(\alpha)/p$, and the intermediate constants in (34)–(37) associated with i can be obtained from those of $I(\alpha)$ by multiplying the first three by p^2 and the last by p.

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