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CORRECTION OF LASER RANGE TRACKING DATA FOR ATMOSPHERIC REFRACTION AT ELEVATIONS ABOVE 10 DEGREES

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ABSTRACT

A formula for correcting laser measurements of satellite range for the effect of atmospheric refraction is given. The corrections apply above 10° elevation to satellites whose heights exceed 70 km. The meteorological measurements required are the temperature, pressure, and relative humidity of the air at the laser site at the time of satellite pass.

The accuracy of the formula was tested by comparison with corrections obtained by ray-tracing radiosonde profiles. The standard deviation of the difference between the refractive retardation given by the formula and that calculated by ray-tracing was less than about 0.04% of the retardation or about 0.5 cm at 10° elevation, decreasing to 0.04 cm near zenith.

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CORRECTION OF LASER RANGE TRACKING DATA FOR ATMOSPHERIC REFRACTION AT ELEVATIONS ABOVE 10 DEGREES

INTRODUCTION

The correction of tracking data for atmospheric refraction has been exhaustively studied, and many correction formulas have been published [1-6]. For certain earth and ocean physics applications, however, position accuracies of better than a few centimeters are desirable [7], and these accuracies are much greater than required for most previous applications. Out of the work cited, only the approach given by Marini [3], and the expansion and integral evaluations of Saastamoinen [5,6] provide the desired accuracy at lower elevation angles (10° - 20°). In this report Saastamoinen's integral evaluations are incorporated into Marini's continued fraction form to provide relatively simple algorithms for correcting laser range-data using surface meteorological measurements.

REFRACTIVITY AT OPTICAL FREQUENCIES

There are a number of formulas [8-11] for the refractive index n of air and for the corresponding refractivity

$$N \equiv 10^6 (n - 1) \tag{1}$$

all of which have sufficient accuracy for use here. The formula employed is [12]

$$N = \left(287.604 + \frac{1.6288}{\lambda^2} + \frac{0.0136}{\lambda^4}\right)\left(\frac{P}{1013.25}\right)\left(\frac{1}{1 + 0.003661\,t}\right) - 0.055\left(\frac{760}{1013.25}\right)\left(\frac{e}{1 + 0.00366\,t}\right) \tag{2}$$

where

$\lambda \equiv$ wavelength of radiation in microns
$P \equiv$ atmospheric pressure in millibars
$e \equiv$ partial water vapor pressure in millibars
$t \equiv$ temperature in degrees Celsius

Because air is dispersive at optical frequencies, the group refractivity N_g is also required

$$N_g = \frac{d}{df}(fN) = N - \lambda \frac{dN}{d\lambda} \tag{3}$$

where f is the frequency. The expression for the group refractivity can be written as

$$N_g = 80.343\, f(\lambda)\frac{P}{T} - 11.3\,\frac{e}{T} \tag{4}$$

where

P = Total air pressure in millibars
e = Partial pressure of water vapor (mb)
T = Temperature (°K)

and

$$f(\lambda) \equiv 0.9650 + \frac{0.0164}{\lambda^2} + \frac{0.000228}{\lambda^4} \tag{5}$$

which, at the 0.6943 micron wavelength of the ruby laser becomes

$$f(0.6943) = 1.0000 \tag{6}$$

GEOMETRY AND NOTATION

The geometry of the satellite-tracking station configuration is shown in Figure 1. Spherical symmetry is assumed, i.e. the refractivity is taken to be a function of height only. The height h is measured from the tracking station upward. The subscript "0" designates quantities evaluated at the tracking station, the subscript "1", quantities evaluated at the satellite. The ray or phase path between tracking station and satellite is shown as a curved line. The true range R is the distance along the straight line connecting the tracking station and the satellite, and the true elevation angle E is the angle between this line and the horizontal at the station. The nominal earth radius used is r_e = 6378 km, and H is the height of the tracking station above sea level. The latitude of the tracking station is φ degrees above the equator.

EXPANSION FORMULA

The apparent range R_e between the ground station and the satellite as measured by a pulsed system is given by the integral of the group index of refraction along the phase path [13,14]

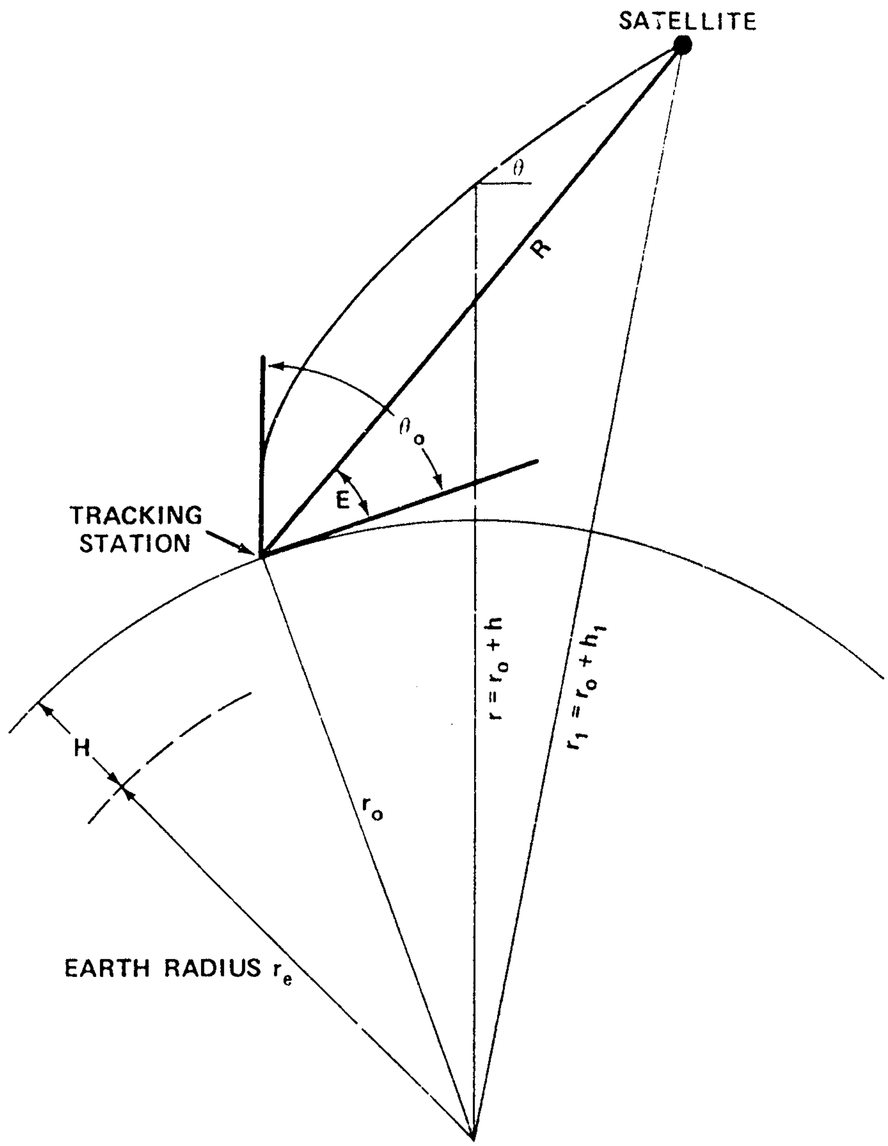

Figure 1. **Geometry**

$$R_e = \int_{r_0}^{r_1} \frac{n_g}{\sin\theta} \, dr \tag{7}$$

where the angle θ is given by Snell's law for a spherically stratified medium

$$n r \cos\theta = n_0 \, r_0 \cos\theta_0 \tag{8}$$

The correction sought is the difference between the measured and the true value of the range

$$\Delta R \equiv R_e - R \tag{9}$$

The expansion of ΔR in inverse powers of $\sin\theta_0$, following Marini [3] gives

$$\begin{aligned} \Delta R \doteq\ & 10^{-6} \int N_g \, dh \cdot \frac{1}{\sin\theta_0} \\ & - \left[\frac{10^{-6}}{r_0} \int h N_g dh - 10^{-12} N_0 \int N_g dh \right. \\ & \left. + 10^{-12} \int (N N_g - \frac{1}{2} N^2) \, dh \right] \cdot \frac{1}{\sin^3\theta_0} \\ & + \circ\circ\circ \end{aligned} \tag{10}$$

where the range of integration is from the tracking station ($h = 0$) upward to above the atmosphere ($h = \infty$). The terms containing the satellite range R that appear in reference [3] can be neglected, as shown in Appendix 1, because (10) is to be applied only where $E > 10^\circ$ and $h_1 > 70$ km.

The expansion (10) is not the most useful one for many orbit determination programs because the correction is expressed as a function of arrival angle θ_0, which may not even be measured, rather than as a function of elevation angle E, which is computed. To convert (10) to the desired form, the first term of the expansion of the angular correction is used

$$\theta_0 - E \doteq 10^{-6} N_0 \cot E \tag{11}$$

substituting (11) into (10), and making suitable approximations

$$\Delta R = \left[10^{-6} \int N_g \, dh \right] \cdot \frac{1}{\sin E}$$

$$- \left[\frac{10^{-6}}{r_0} \int h N_g dh + 10^{-12} \int (N N_g - \frac{1}{2} N^2) \, dh \right] \frac{1}{\sin^3 E} \qquad (12)$$

$$+ \circ \circ \circ$$

Equation (12) above is the expansion that provides the basis for the correction formula that is the subject of this report.

EVALUATION OF INTEGRALS

The evaluation of the integrals, appearing in (12), as functions of the pressure, temperature, and relative humidity of the surface air at the tracking station, has been treated by Saastamoinen [6]. For completeness, and because they differ in detail, our evaluations are given in Appendix 2. The results are

$$10^{-6} \int N_g \, dh \overset{\circ}{=} \frac{f(\lambda)}{f(\varphi, H)} [0.002357 P_0 + 0.000141 e_0] \qquad (13)$$

$$\frac{10^{-6}}{r_0} \int h N_g dh = f(\lambda) (1.084 \times 10^{-8}) P_0 \, T_0 \, K \qquad (14)$$

$$10^{-12} \int (N N_g - \frac{1}{2} N^2) \, dh = f(\lambda) (4.734 \times 10^{-8}) \frac{P_0^{\,2}}{T_0} \cdot \frac{2}{3 - 1/K} \qquad (15)$$

where

$$f(\varphi, H) = 1 - 0.0026 \cos 2\varphi - 0.00031 H \qquad (16)$$

and

$$K = 1.163 - 0.00968 \cos 2\varphi - 0.00104 \, T_0 + 0.00001435 P_0 \qquad (17)$$

CORRECTION FORMULA

The formula for calculating the range error ΔR from the satellite elevation E is obtained by approximating (12) by a continued fraction form

$$\Delta R = \frac{f(\lambda)}{f(\varphi, H)} \cdot \frac{A + B}{\sin E + \dfrac{B/(A+B)}{\sin E + 0.01}} \tag{18}$$

where

$$A = 0.002357 P_0 + 0.000141 e_0 \tag{19}$$

$$B = (1.084 \times 10^{-8}) P_0 T_0 K + (4.734 \times 10^{-8}) \frac{P_0^{\,2}}{T_0} \frac{2}{(3 - 1/K)} \tag{20}$$

$$K = 1.163 - 0.00968 \cos 2\varphi - 0.00104 T_0 + 0.00001435 P_0 \tag{21}$$

Here

ΔR = Range correction (meters)
E = True elevation of satellite
P_0 = Atmospheric pressure at the laser site (millibars)
T_0 = Atmospheric temperature at the laser site (degrees Kelvin)
e_0 = Water vapor pressure at the laser site (millibars)
$f(\lambda)$ = 1 for a ruby laser, and is given by (5) otherwise
$f(\varphi, H)$ = 1 for a laser site at 45° latitude and at sea level, and is given by (16) for sites at different latitudes φ and elevations H (in km)

The water vapor pressure e_0 may be calculated from a relative humidity measurement R_h (%)

$$e_0 = \frac{R_h}{100} \times 6.11 \times 10^{\frac{7.5\,(T_0 - 273.15)}{237.3 + (T_0 - 273.15)}} \tag{22}$$

In (18) the quantity 0.01 is an empirical constant that serves to compensate for the neglect of higher order terms. The divisor $f(\varphi, H)$ can be factored out of the series (12) and consequently the fraction (18) because the error thereby incurred in the second term of (12) is negligable. The use of the sum A + B where it appears in (18) instead of using A alone is an optional adjustment used to reduce at elevations near 90° a small bias that occurs in the expansion (12) because of approximations made in its derivation.

TEST OF ACCURACY

To test the accuracy of formula (18), which is based on surface measurements, range corrections obtained using the formula were compared with corrections obtained by ray-tracing radiosonde refractivity profiles. The ray-trace corrections are considered to have state-of-the-art accuracy, so that the differences between these corrections and those calculated from the simpler formulas represent the penalty paid for simplicity in calculation and measurement.

The data used in Figures 2-11 was obtained from the National Climatic Center at Asheville, North Carolina. It consists of radiosonde observations taken near Dulles Airport, Virginia, during the year 1967.

Using the procedure described in Appendices 3 and 4, 634 refractivity profiles were calculated up to a height of 1000 kilometers from the radiosonde observations. The calculated profiles were ray-traced [16] at arrival angles of 10°, 15°, 20°, 40°, and 80°, and the tropospheric errors in range and elevation angle were obtained. The histograms of these errors are shown in Figures 2,4,6,8, and 10. The correction formula (18) was applied using only surface data and the known elevation angle to obtain approximate tropospheric corrections. The differences between these algorithm corrections and the ray-trace corrections were calculated. The histograms of these differences is shown in Figures 3,5,7,9, and 11. The maximum bias of the error remaining after correction was -0.1 cm, and the maximum standard deviation was 0.49 cm at 10°, decreasing to 0.04 cm at 80°.

In addition formula (18) was compared with range corrections obtained by ray tracing (at arrived angles of 10°, 15°, 20°, 40°, and 80°) radiosonde refractivity profiles calculated at Jananarive (85 profiles), Fairbanks, Alaska (200 profiles), Athens, Georgia (200 profiles), Greensboro, North Carolina (200 profiles), and Nashville, Tennessee (135 profiles). The maximum standard deviation of the error in the algorithm at 10° was 1 centimeter and the maximum at 80° was 0.06 centimeters. The maximum mean error of the algorithm at 10° was 0.16 cm and the maximum at 80° was 0.07 cm.

CONCLUSIONS

An equation that corrects laser range data for atmospheric refraction using surface meteorological measurements has been derived, and a comparison made between the corrections calculated using this equation (equation 18) and the corrections calculated by ray-tracing through a radiosonde profile. The comparison (Figures 2-11) indicates that the differences between the corrections calculated be the two methods are negligible for practical applications. Hence accurate refraction correction of laser range data can be made without the requirement for radiosonde measurements or lengthy ray-tracing algorithms.

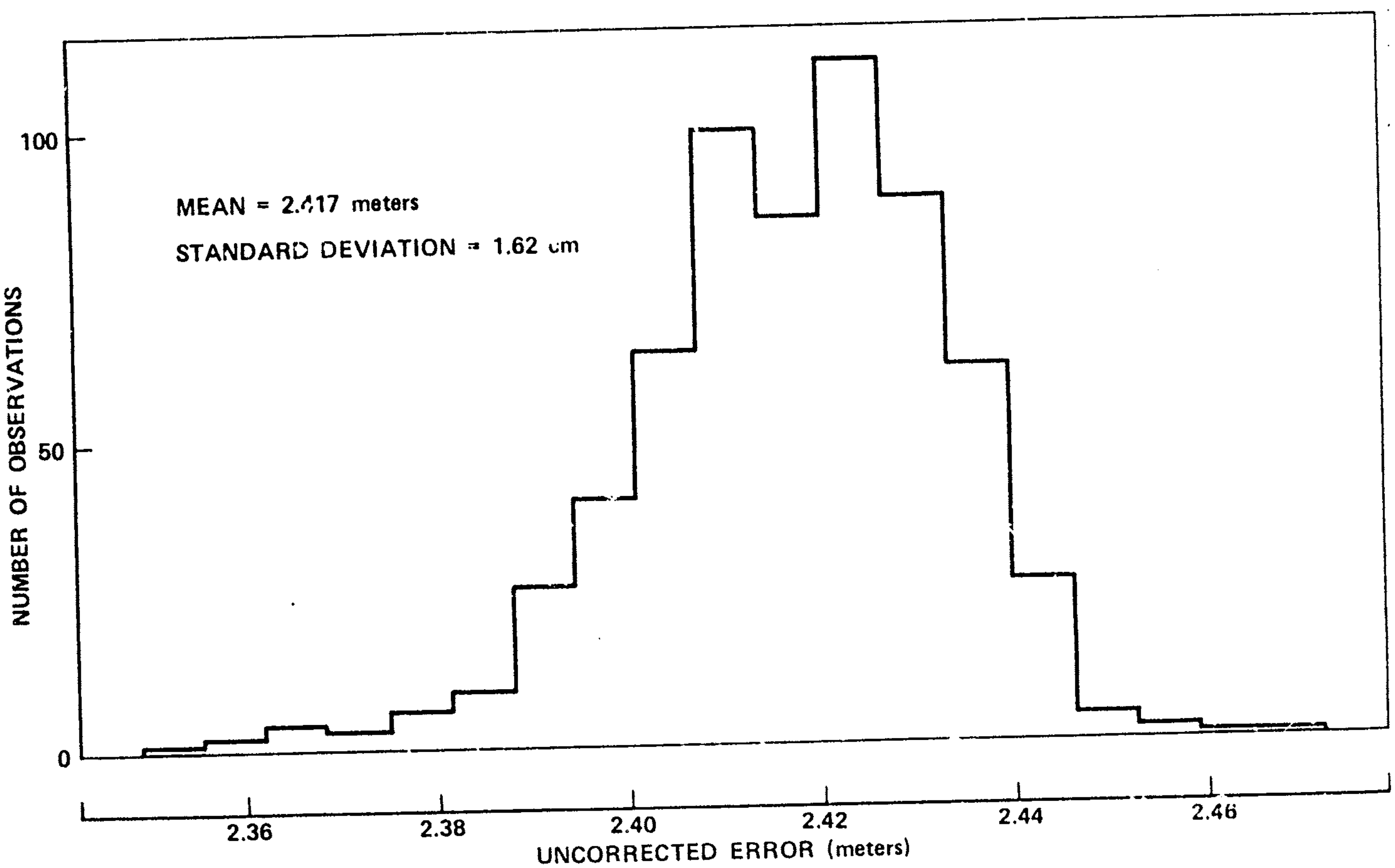

Figure 2. Tropospheric Range Error at About 30 Degrees Elevation for Laser Frequency of 0.6943 Microns

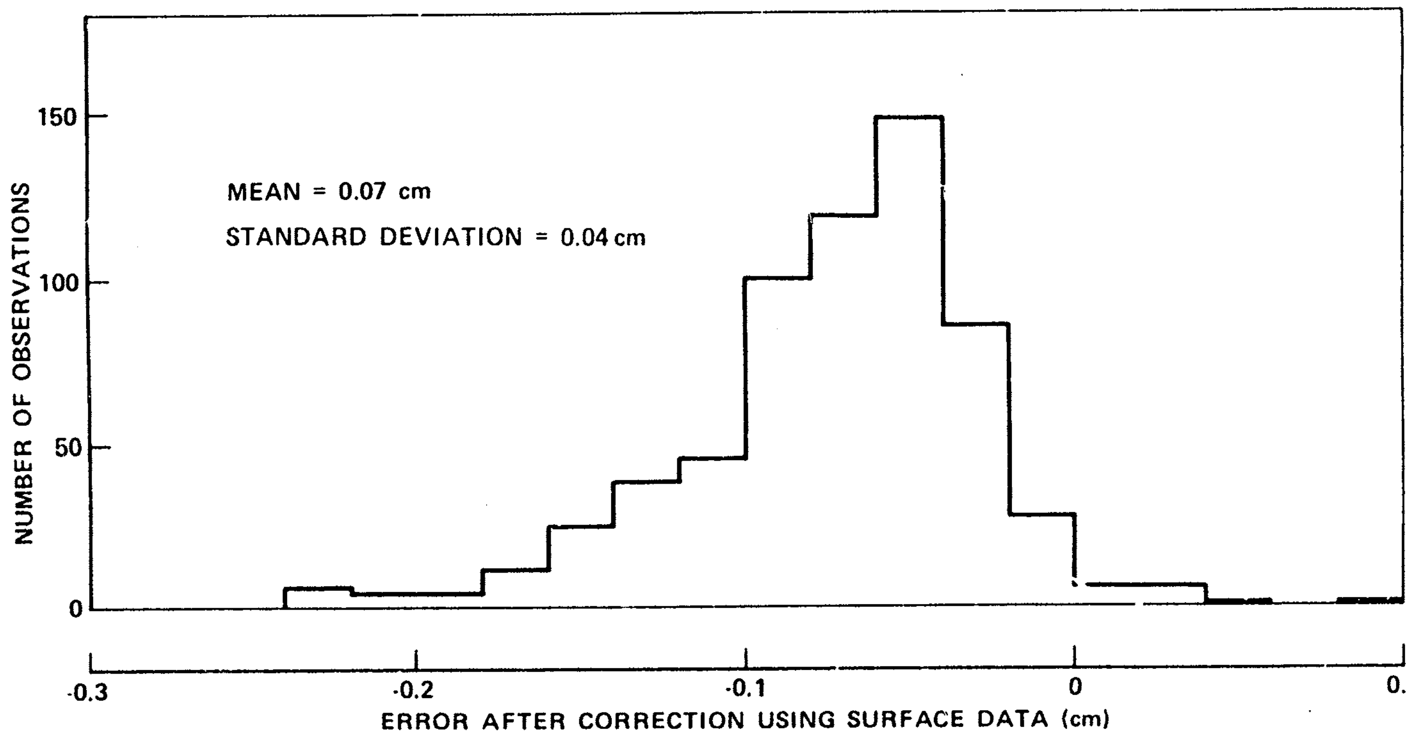

Figure 3. Tropospheric Range Error at About 80 Degrees Elevation for Laser Frequency of 0.6943 Microns

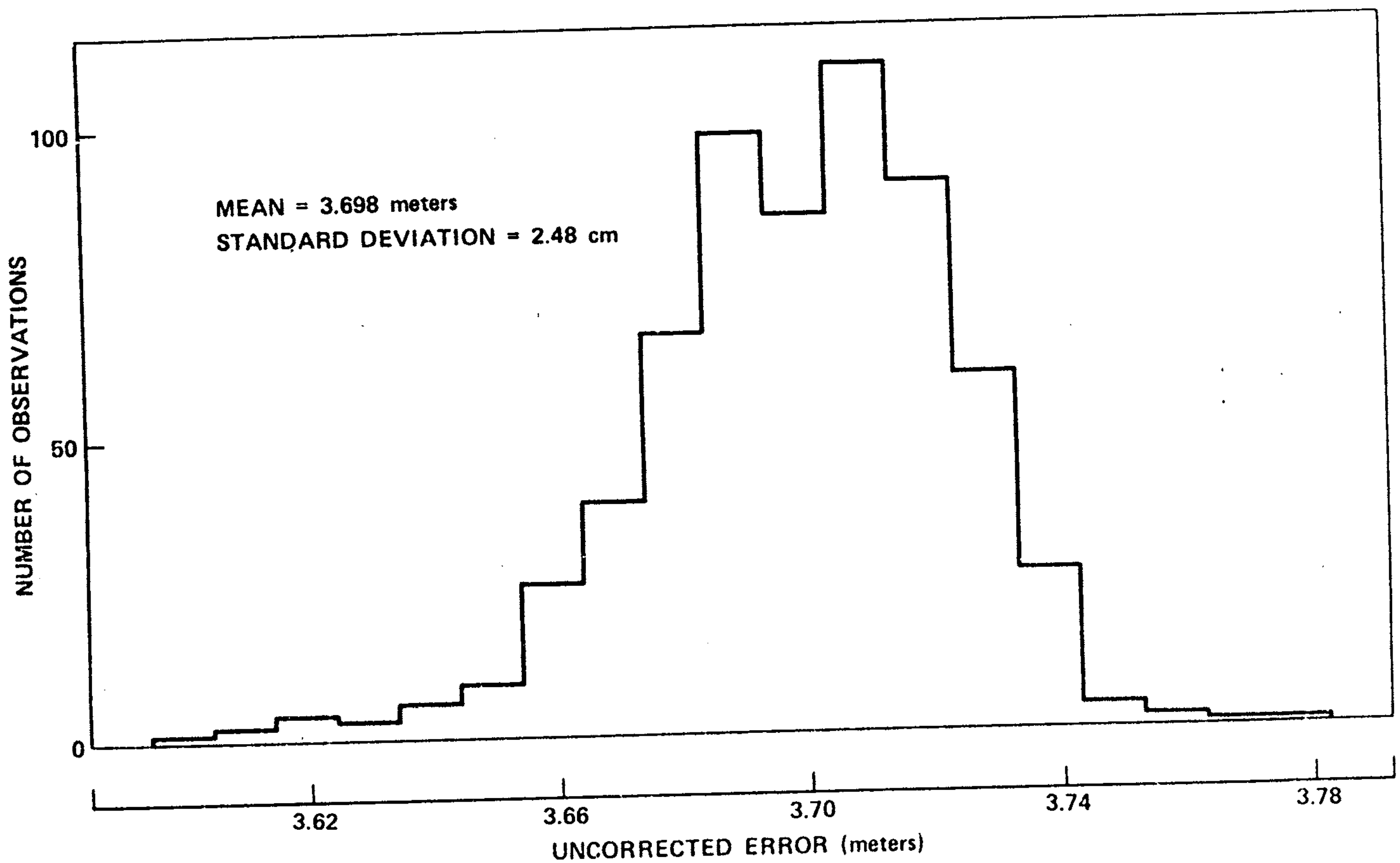

Figure 4. Tropospheric Range Error at About 40 Degrees Elevation for Laser Frequency of 0.6943 Microns

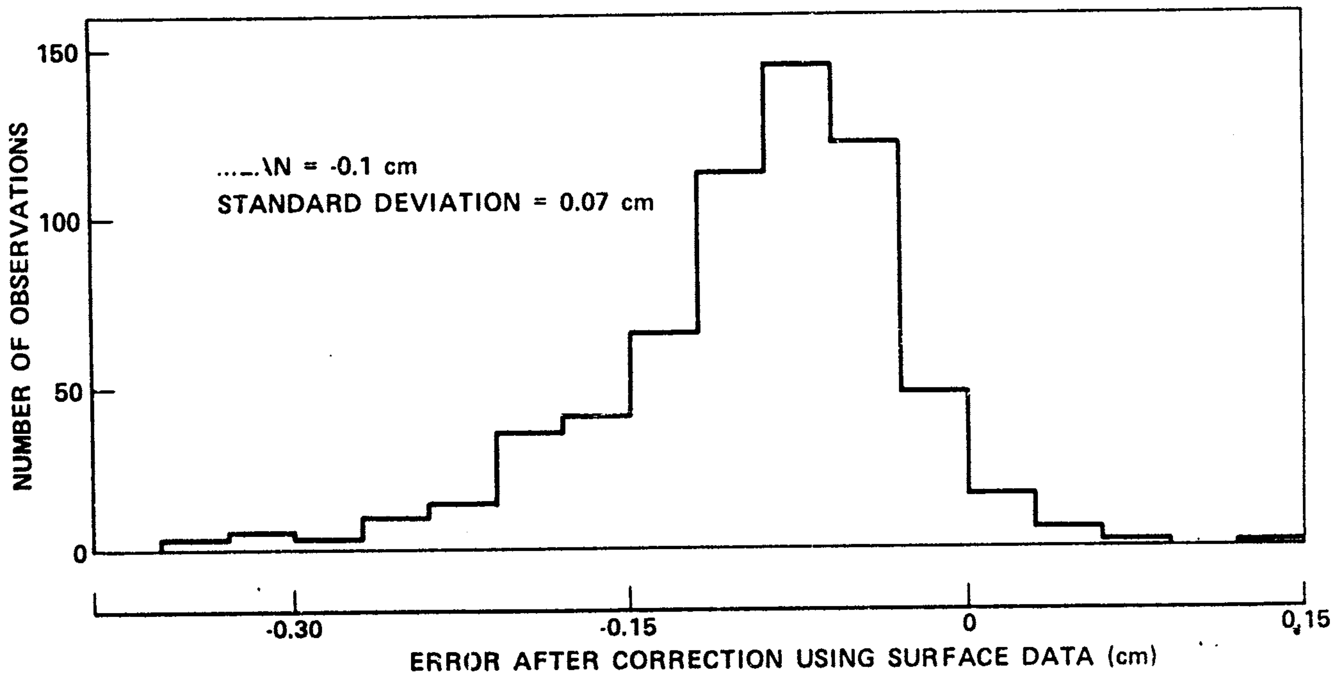

Figure 5. Tropospheric Range Error at About 40 Degrees Elevation for Laser Frequency of 0.6943 Microns

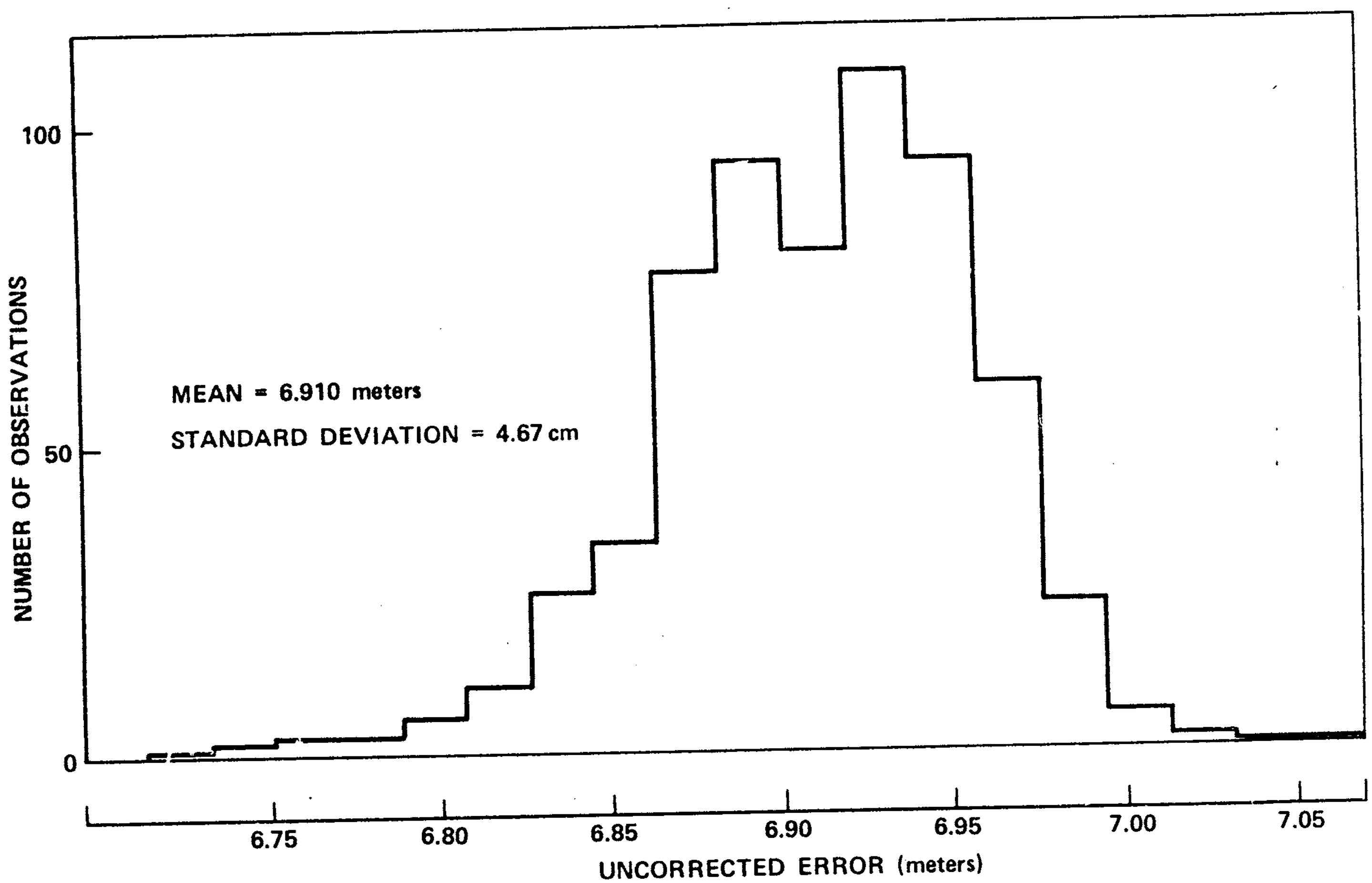

Figure 6. Tropospheric Range Error at About 20 Degrees Elevation for Laser Frequency of 0.6943 Microns

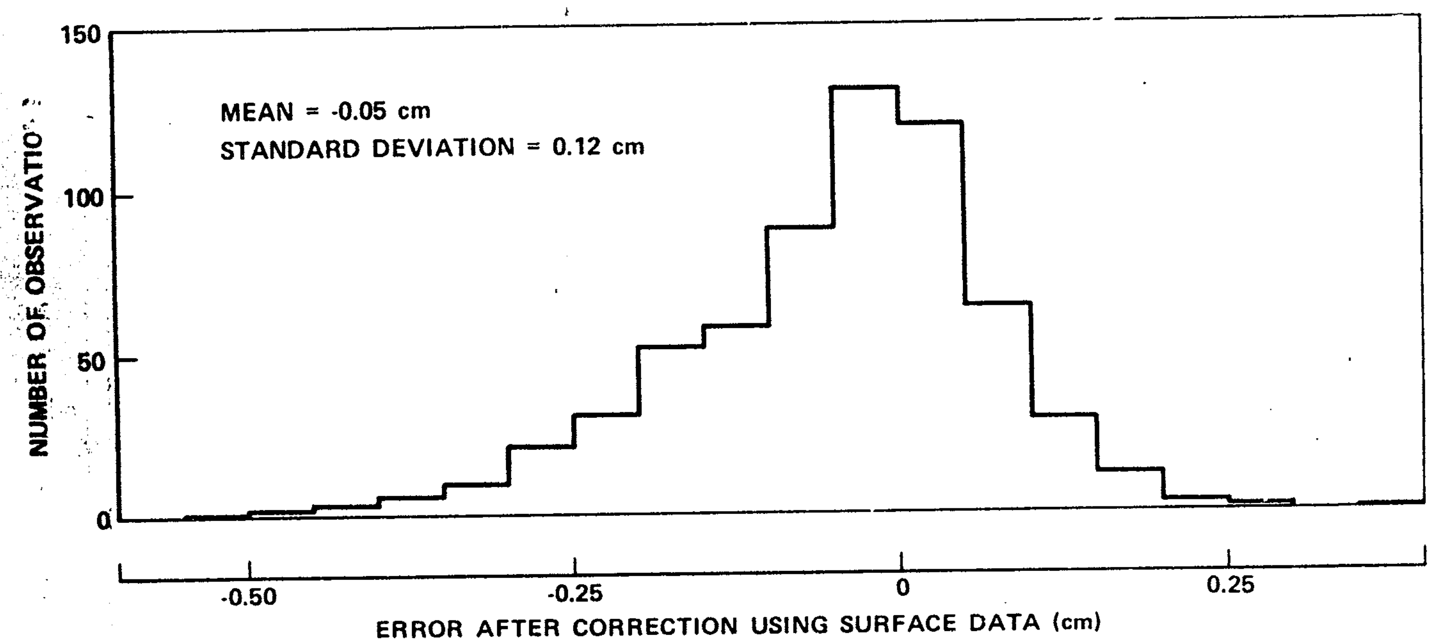

Figure 7. Tropospheric Range Error at About 20 Degrees Elevation for Laser Frequency of 0.6943 Microns

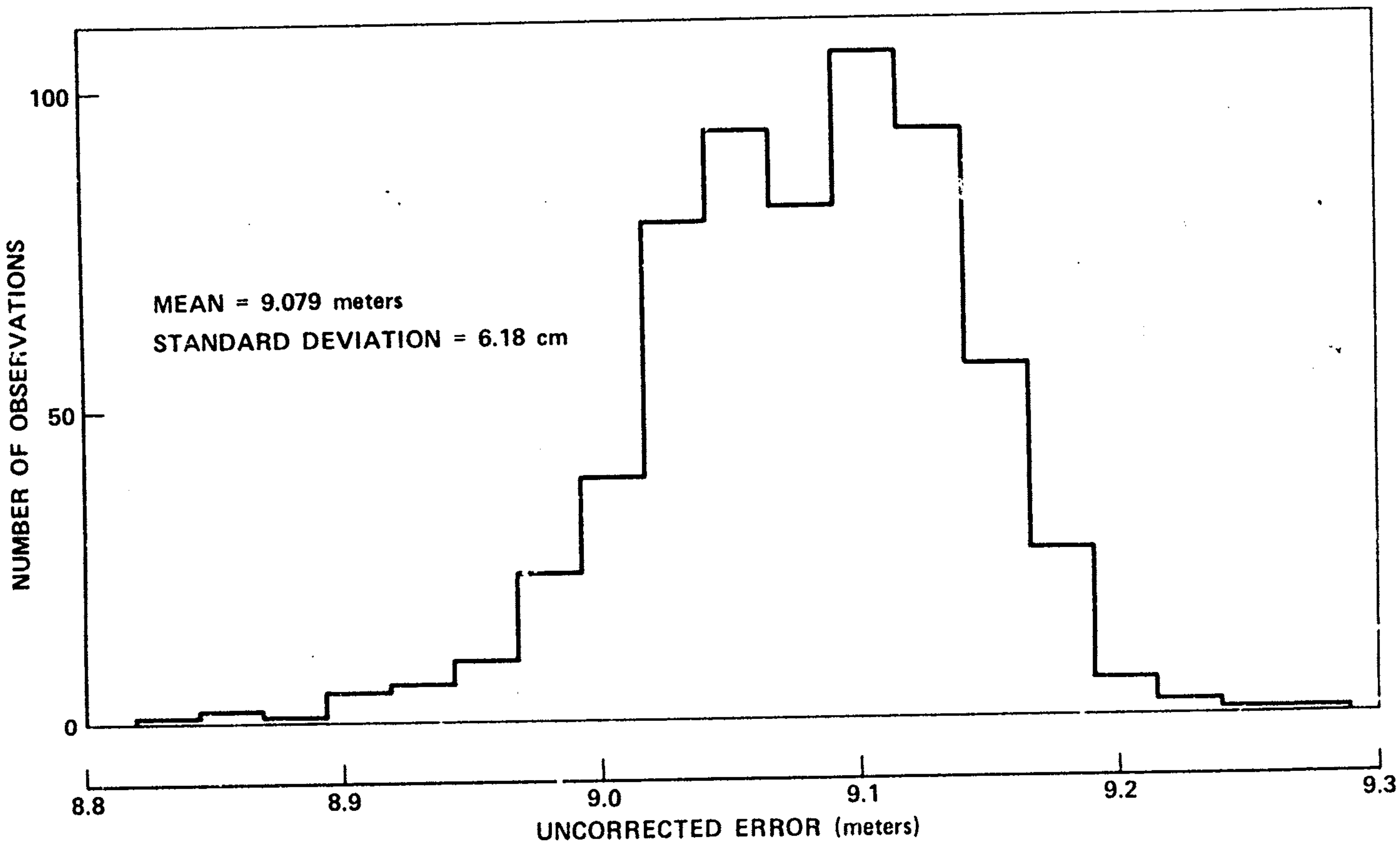

Figure 8. Tropospheric Range Error at About 15 Degrees Elevation for Laser Frequency of 0.6943 Microns

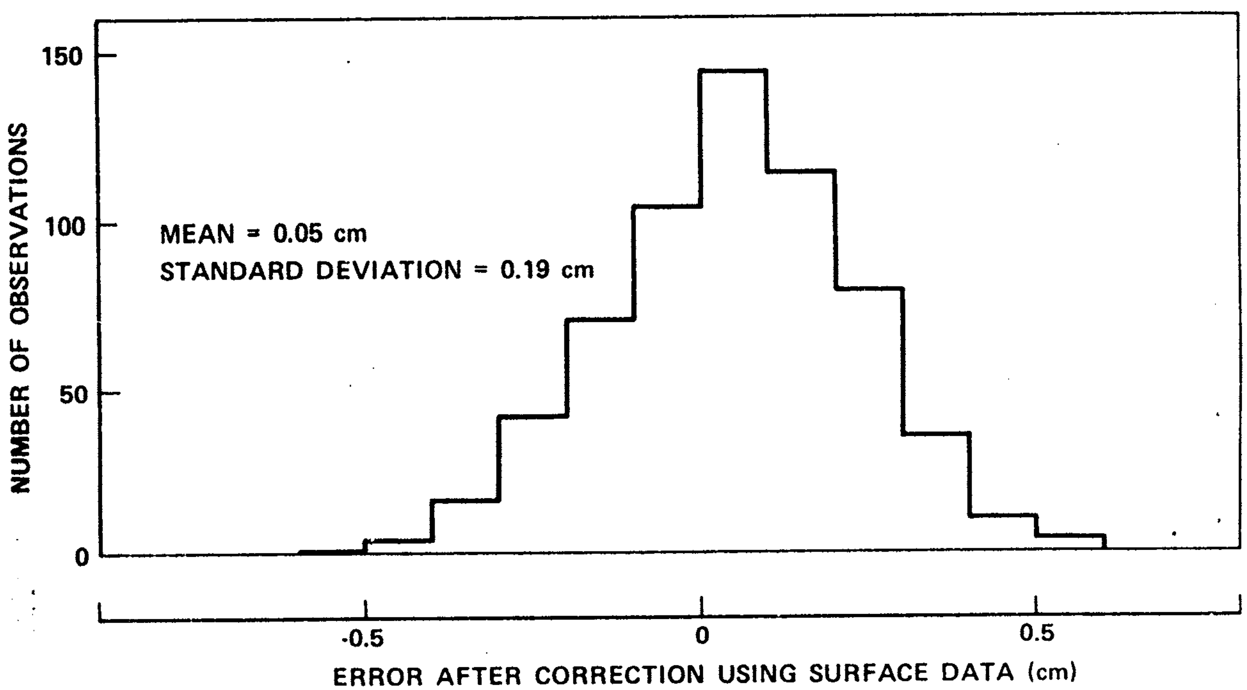

Figure 9. Tropospheric Range Error at About 15 Degrees Elevation for Laser Frequency of 0.6943 Microns

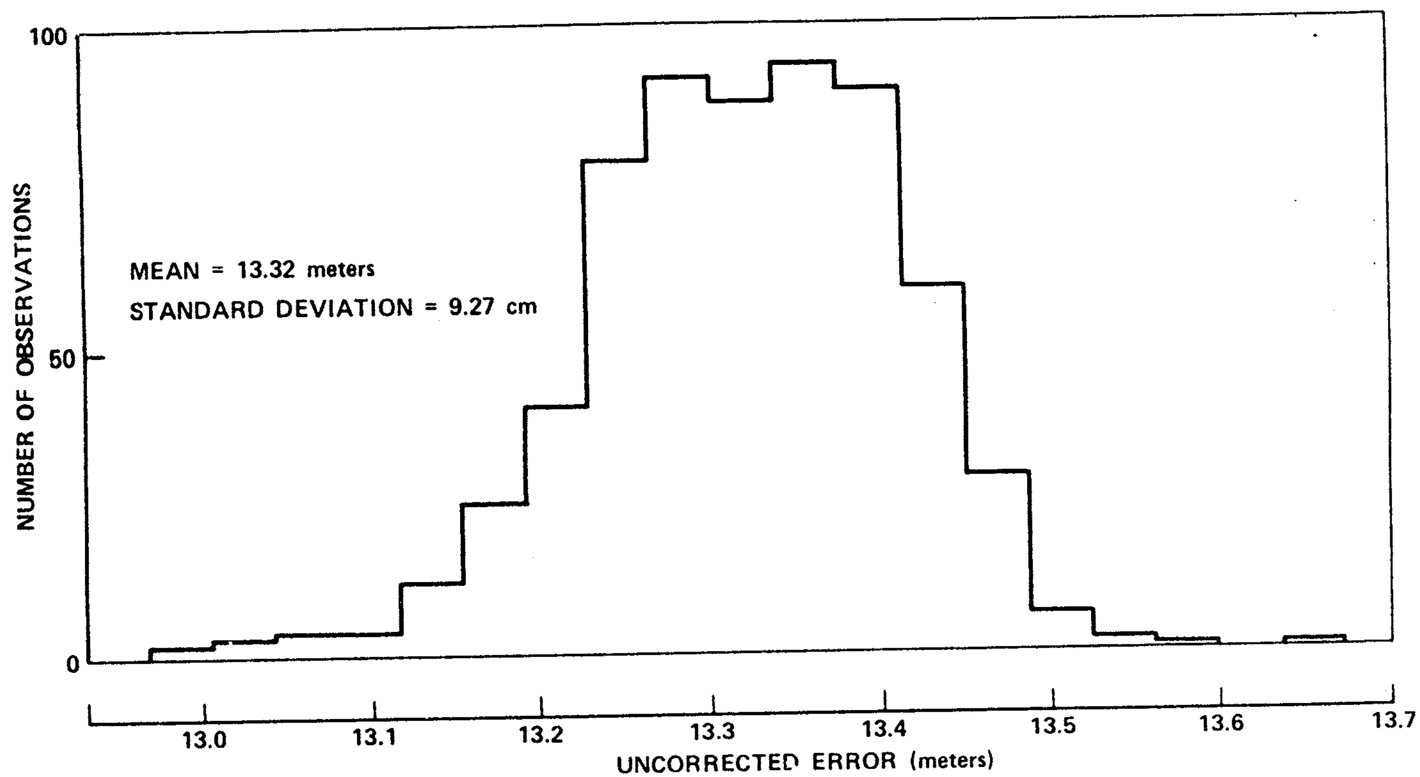

Figure 10. Tropospheric Range Error at About 10 Degrees Elevation for Laser Frequency of 0.6943 Microns

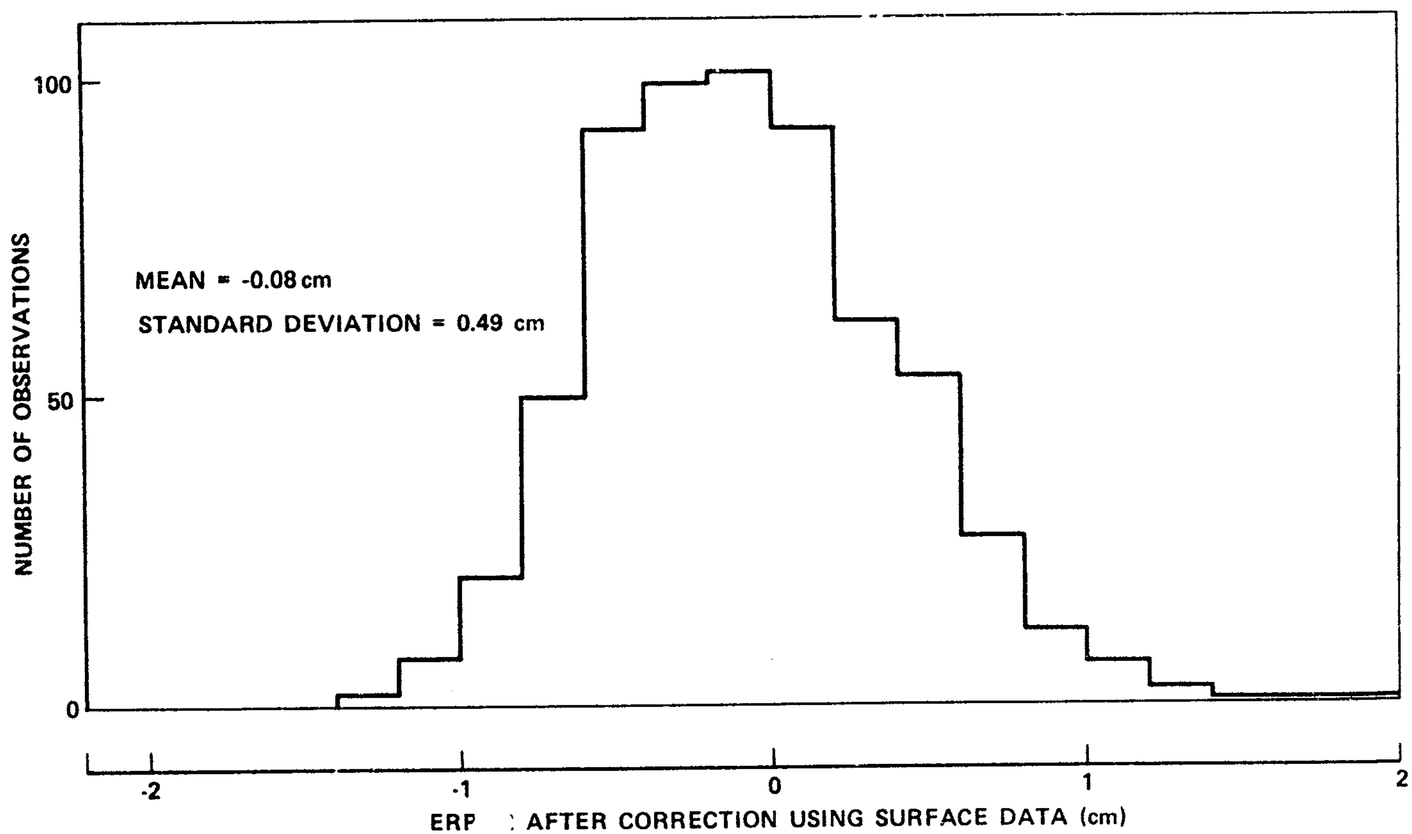

Figure 11. Tropospheric Range Error at About 10 Degrees Elevation for Laser Frequency of 0.6943 Microns

It should be pointed out that only the relative accuracy of the two procedures has been tested, and that errors caused by factors common to both methods are not in evidence. For example, equation (4) for the group refractive index is used both in (18) and in the ray-trace equations, and any error in its magnitude would reflect equally in the corrections. Similarly, the hydrostatic equation used in equation (2-1) and hence (18) is also implicit in the ray-tracing method because the heights that appear in radiosonde profiles are not measured quantities but rather are calculated from the measured pressures, temperatures, and relative humidities using the hydrostatic equation. Also, both methods assume horizontal homogeneity. Saastamoinen [6] has estimated the standard error from such sources to be less than 1 or 2 centimeters at 10° elevation.

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APPENDIX 1

NEGLECT OF SATELLITE RANGE

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The correction

$$\Delta R = \int_{r_0}^{r_1} \frac{n_g}{\sin\theta} dr - R \tag{1-1}$$

can be written as

$$\Delta R = 10^{-6} \int_{r_0}^{r_1} \frac{N_g \, dr}{\sin\theta} + \left[\int_{r_0}^{r_1} \frac{1}{\sin\theta} dr - R \right] \tag{1-2}$$

The expansion of the first term in (1-2), using suitable approximations [3] , gives

$$\begin{aligned} 10^{-6} \int_{r_0}^{r_1} \frac{N_g \, dr}{\sin\theta} &= \frac{10^{-6}}{\sin\theta_0} \int_0^{\infty} N_g \, dh \\ &- \frac{1}{\sin^3\theta_0} \left[\frac{10^{-6}}{r_0} \int h N_g \, dh \right. \\ &\left. -10^{-12} \int N_g (N_0 - N) \, dh \right] \\ &+ \circ \circ \circ \end{aligned} \tag{1-3}$$

The expansion of the bracketed second term in (1-2), which represents the difference ΔR_g between the geometrical lengths of the phase and the straight-line paths between the satellite and the tracking station, can be obtained by expanding equation (A5) of reference [3] in inverse powers of $\sin\theta_0$ giving

$$\begin{aligned} \Delta R, &= \frac{1}{\sin^3\theta_0} \cdot \frac{1}{2} 10^{-12} \int N^2 \, dh \\ &- \frac{1}{2} 10^{-12} \frac{(\int N \, dh)^2}{R} \cdot \frac{\cos^2\theta_0}{\sin\theta_0{}^4} \end{aligned} \tag{1-4}$$

The relative error incurred in neglecting the last term in (1-4) is estimated by dividing it by the (dominant) first term in (1-3), ignoring the small difference between the magnitudes of N and N_g

$$\text{relative error} = \frac{1}{2} \frac{10^{-6} \int N\, dh}{R} \cdot \frac{\cos^2 \theta_0}{\sin^3 \theta_0} \tag{1-5}$$

The satellite height h_1 is roughly approximated by approximated by $R \sin\theta_0$, and the zenith integral is about 2 meters:

$$\text{relative error} \approx \frac{1}{h_1} \tan^2 \theta_0 \tag{1-6}$$

where h_1 is the satellite height in meters. Taking $h_1 \geqq 70$ km and $\theta_0 \geqslant 10°$, the error calculated from (1-6) is less than 0.05% which can be neglected.

APPENDIX 2

EVALUATION OF INTEGRALS

From the perfect-gas law, the law of partial pressures, and the hydrostatic equation

$$-\frac{dP}{dh} = \frac{Mg(P-e)}{RT} + \frac{Mwge}{RT}$$

$$= -\frac{MgP}{RT} - \frac{0.378\,Mge}{RT} \qquad (2\text{-}1)$$

where [15]

M = 28.966 = Molecular weight of dry air
Mw = 18.016 = Molecular weight of water vapor
R = 8314.36 Joules $(^\circ K)^{-1}$ $(Kg - Mole)^{-1}$
= Universal gas constant
g = acceleration of gravity (m/s)
h = height (m)

Combining (2-1) and (4)

$$\int N_g\, dh = -80.343\, f(\lambda)\frac{R}{M}\int \frac{1}{g} dP$$

$$+ [30.5\, f(\lambda) - 11.3]\int \frac{e}{T} dh \qquad (2\text{-}2)$$

The first integral on the right side of (2-2) above can be evaluated using the approximation [15].

$$g = 9.806\,[1 - 0.0026 \cos 2\varphi - 0.00031\,(H + h)] \qquad (2\text{-}3)$$

$$\frac{1}{g} \overset{\circ}{=} \frac{1}{9.806}[1 + 0.0026 \cos 2\varphi + 0.00031\,(H + h)] \qquad (2\text{-}4)$$

from which, integrating the last term by parts,

$$- \int \frac{1}{g} dP = P_0 \frac{1}{9.806} [1 + 0.0026 \cos 2\varphi + 0.00031 (H + \frac{1}{P_0} \int Pdh)] \qquad (2\text{-}5)$$

$$= P_0 / \bar{g}$$

where $\bar{g}$ is the value of g at the height

$$\bar{h} = \frac{1}{P_0} \int P \, dh \qquad (2\text{-}6)$$

above the tracking station or $H + \bar{h}$ above sea level. Saastamoinen uses a gravitational constant evaluated at*

$$H + \bar{h} = 7.3 + 0.9 \, H \text{ km} \qquad (2\text{-}7)$$

From (2-7) and (2-3)

$$\bar{g} = 9.784 (1 - 0.0026 \cos 2\varphi - 0.00028 \, H) \qquad (2\text{-}8)$$

$$\equiv 9.784 \, f(\varphi, H)$$

where H is the station elevation in kilometers. Saastamoinen has also evaluated the integral

$$\int \frac{e}{T} dh \overset{\circ}{=} \frac{R}{4M\bar{g}} e_0 \qquad (2\text{-}9)$$

where the $\bar{g}$ appearing in (2-9) is set equal to $\bar{g}$ in (2-8) as a convenient approximation. The expression for the zenith integral becomes

$$\int N dh = 80.343 \, f(\lambda) \frac{R}{M\bar{g}} P_0$$

$$+ [30.5 \, f(\lambda) - 11.3] \frac{R}{4M\bar{g}} e_0 \qquad (2\text{-}10)$$

*An equivalent result can be obtained by numerically estimating $\bar{h}$ using (2-17) with T_0 set equal to $T_e + \beta H$ where T_e is the sea level temperature.

$$10^{-6} \int N\,dh = \frac{f(\lambda)}{f(\varphi,H)} \left[0.002357\,P_0 + \frac{(30.5 - 11.3/f(\lambda))}{19.2}\;0.000141\,e_0\right] \qquad (2\text{-}11)$$

Neglecting small errors in the second term of (2-11), equation (13) results.

SECOND INTEGRAL

In equation (12), the magnitude of the coefficient of 1/sin E is about 2.4 meters, while the coefficient of $1/\sin^3 E$, is only about $\frac{1}{4}$ centimeters. At E = 10°, the magnitude of the first term is about 12 meters, while the second is about half a meter. Consequently the second term need not be as accurately evaluated as the first, and it is sufficient to use the approximation

$$\frac{10^{-6}}{r_0} \int h N_g\,dh \overset{\circ}{=} \frac{10^{-6}}{r_e} \int \frac{80.343\, f(\lambda)\, P}{T}\, h\,dh \qquad (2\text{-}12)$$

where r_e is a nominal earth radius (6378 km) and the air is assumed to be dry. It is also sufficient to treat g as a constant throughout.

From (2-1), and integrating by parts

$$\int \frac{P}{T} h\,dh = \frac{R}{Mg} \int P\,dh \qquad (2\text{-}13)$$

The pressure P in (2-13) is obtained by integrating (2-1)

$$P = P_0 \exp\left[\frac{Mg}{R} \int \frac{1}{T}\,dh\right] \qquad (2\text{-}14)$$

The temperature T is assumed to have a linear slope

$$T = T_0 + \beta h \qquad (2\text{-}15)$$

and the integration in (2-14) is carried out giving

$$P = P_0 \left(\frac{T}{T_0}\right)^{-Mg/R\beta} \qquad (2\text{-}16)$$

The integration in (2-13) may now be performed

$$\int P\,dh = P_0 \cdot \frac{R\,T_0}{Mg} \cdot \frac{1}{1 - \frac{R\beta}{Mg}} \tag{2-17}$$

From (2-12), (2-13), and (2-17)

$$\frac{10^{-6}}{r_0} \int h\,N_g\,dh \overset{\circ}{=} f(\lambda)\,\frac{10^{-6}\,(80.343)R^2}{r_e\,M^2\,g^2}\,P_0\,T_0\,K \tag{2-18}$$

$$= f(\lambda)\,(1.084 \times 10^{-8})\,P_0\,T_0\,K$$

where g has been set equal to 9.784 and the factor

$$K \equiv \frac{1}{1 - \frac{R\beta}{Mg}} \tag{2-19}$$

is equal to unity in an isothermal atmosphere ($\beta = 0$) and is equal to about 0.8 in an atmosphere in which the temperature lapse rate is a constant 6°/km ($\beta = -6°/km$).

Rather than use the theoretical value for K given by (2-19), which is based on a constant lapse rate, the value of K used in the corrections equations is taken to be an empirical constant which was determined by solving (2-18) for k and calculating its value by numerically integrating through the atmospheres of the U.S. Standard Atmosphere Supplements, 1966. Using linear regression on the values so obtained, the formula

$$K = 1.163 - 0.00968 \cos 2\varphi - 0.00104\,T_0 + 0.00001435\,P_0 \tag{2-20}$$

resulted. Here φ is the latitude of the tracking station.

THIRD INTEGRAL

The contribution from the third integral in (12) is only marginally significant, and the term can be approximated by

$$\frac{1}{2} 10^{-12} \int N_g N\,dh = \frac{1}{2} 10^{-12}\,(80.343)^2\,f(\lambda) \int \frac{P^2}{T^2}\,dh \tag{2-21}$$

Assuming a constant temperature gradient, and using (2-16)

$$10^{-12} \int \left(N\,Ng - \frac{1}{2} N^2 \right) dh \doteq \frac{10^{-12}}{4} (80.343)^2 f(\lambda) \frac{R}{Mg} \frac{P_0{}^2}{T_0} \cdot \frac{1}{1 + \frac{RP}{2mg}} \qquad (2\text{-}22)$$

The last factor in (2-22) can be expressed in terms of K using (2-19), giving (15).

APPENDIX 3

PROGRAM FOR CALCULATING REFRACTIVITY PROFILES FROM RADIOSONDE DATA

APPENDIX 3

PROGRAM FOR CALCULATING REFRACTIVITY PROFILES FROM RADIOSONDE DATA*

RADIOSONDE DATA

Radiosonde observations are measurements of pressure, temperature, and humidity taken from the surface up to the point where the balloon that carries the sensors bursts [1]. The values of temperature, pressure and relative humidity measured at certain standard and significant levels during each balloon ascent from numerous weather stations is available from the National Climatic Center. This data can be used to construct continuous refractivity profiles from the surface up to the point of highest measurement. Above the latter point, the refractivity profile can be extended by assuming a suitable temperature profile.

GEOPOTENTIAL ALTITUDE

The equations used to calculate the refractivity profiles employ the geopotential altitude H [2, p. 217], which is given by

$$H = \frac{1}{G} \int_0^Z g \, dZ \qquad (1)$$

where Z is the geometric altitude, and the lower limit of integration is from sea level (Z = 0). H is in geopotential meters when G equals 9.8 m/sec^2. The local acceleration of gravity is calculated from the latitude ϕ by [2, p. 488]

$$g_0 = 9.780356 \, (1 + 0.0052885 \sin^2 \phi - 5.9 \times 10^{-6} \sin^2 2\phi) \qquad (2)$$

and [2, p. 217]

$$g = \frac{g_0 \, r_0^2}{(r_0 + Z)^2} \; (\mathrm{m/sec^2}) \qquad (3)$$

Here r_0 is an effective earth radius given by [2, p. 218]

$$r_0 = \frac{2 g_0}{3.085462 \times 10^{-6} + 2.27 \times 10^{-9} \cos 2\phi - 2 \times 10^{-12} \cos 4\phi} \; (\mathrm{m}) \qquad (4)$$

*This appendix is self-contained. It has separate references, and the notation used differs from that in the rest of the report.

From (1) and (3) the conversion between geopotential and geometric altitude is given by [2, p. 218]

$$H = \frac{g_0}{G} \left(\frac{r_0 Z}{r_0 + Z} \right) \tag{5}$$

and

$$Z = \frac{r_0 H}{\frac{g_0 r_0}{G} - H} \tag{6}$$

VIRTUAL TEMPERATURE

The calculations also make use of the virtual temperature T_v [3] which is related to the ordinary temperature T (°K) by

$$T_v = \frac{T}{1 - 0.379 \frac{e}{P}} \tag{7}$$

where e is the partial pressure of the water vapor in the air, and is given by [4, p. 343]

$$e = \left(\frac{R_h}{100}\right) (6.11)\ 10^{\frac{7.5\,(T - 273.15)}{237.3 + (T - 273.15)}} \text{ (mbar)} \tag{8}$$

R_h being the relative humidity in percent.*

CALCULATION OF GEOPOTENTIAL ALTITUDES

The first step in the calculation of refractivity profiles from the radiosonde measurements of pressure, temperature, and relative humidity is to establish a

*If the dewpoint temperature T_d (°K) is given instead of the relative humidity, e can be calculated from (8) by setting R_h = 100 and T = T_d.

table of pressure, temperature, and virtual temperature versus geopotential altitude. The virtual temperatures at the given points are calculated from the measured values of P, T, and R_h using (8) and (7).

To calculate the geopotential altitudes, it is necessary to assume hydrostatic equilibrium [3]

$$dP = -\rho g dZ \tag{9}$$

The density ρ is given with sufficient accuracy by [3]

$$\rho = \frac{MP}{RT_v} \tag{10}$$

The apparent molecular weight of dry air is taken to be [2, p. 289]

$$M = 28.966 \tag{11}$$

and the universal gas constant [2, p. 289]

$$R = 8314.36 \text{ Joules } (°K)^{-1} (\text{Kg-mole})^{-1} \tag{12}$$

Using the assumption that the virtual temperature is a linear function of geopotential height between any two adjacent measured points H_1 and H_2, (9) may be integrated with the use of (1) and (10) to give

$$\frac{P_2}{P_1} = \left(\frac{T_{v1}}{T_{v2}}\right)^{\frac{GM(H_2 - H_1)}{R(T_{v2} - T_{v1})}} \tag{13}$$

which may be written as

$$H_2 = H_1 + \left(\frac{RT_{v1}}{GM}\right) \frac{\chi \ln (P_2/P_1)}{\ln (1 + \chi)}$$

$$= H_1 + \left(\frac{RT_{v1}}{GM}\right) \ln \left(\frac{P_2}{P_1}\right) \left(1 - \frac{\chi}{2} + \frac{\chi^2}{3} \ldots\right)^{-1} \tag{14}$$

where

$$\chi \equiv (T_{v2} - T_{v1})/T_{v1} \tag{15}$$

Equation (14) can be used stepwise starting at the known geopotential elevation of the radiosonde station to compute the geopotential altitudes.* In this way the required table of pressure, temperature, and virtual temperature versus height is established.

CALCULATION OF REFRACTIVITY PROFILES

The radio refractivity N is given by the formula† [5, p. 7]

$$N = 77.6 \frac{P}{T} + 3.73 \times 10^5 \frac{e}{T^2} \tag{16}$$

with P and e expressed in millibars and T in degrees Kelvin.

To calculate N at a given height, i.e., to obtain a point of a refractivity profile, it is necessary to know the values of P, T and e at that height. These are obtained as follows:

The height is converted to a geopotential altitude by adding it to the geometric station elevation to obtain the geometric altitude Z, and applying (5). Using the geopotential altitude so calculated, the temperature and the virtual temperature at the given height are obtained from the table of P, T, and T_v vs. H by linear interpolation. The pressure at the given height is calculated using (13) with P_2, T_{v2} and H_2 replaced by the values associated with the given height. Finally the vapor pressure e is calculated from (7). Substitution into (16) then gives the required refractivity.

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*The geopotential altitudes are computed at the radio-sonde stations and are included in the data stored at the National Climatic Center. The altitudes are recomputed both as a check of the self-consistency of the data and also to generate geopotential altitudes consistent with the values of the fundamental constants (R and M, for example) adopted.

†At optical frequencies (2) and (4) of the main text are used.

$$N = 77.6 \frac{P}{T} \left(1 + \frac{7.52 \times 10^{-3}}{\lambda^2}\right)$$

where the wavelength λ is in microns.

A listing of the FORTRANH program with a sample profile calculated from meteorological data taken at Dulles airport on 1 January 1967 is shown in Appendix 4.

Also shown are the surface measurements of temperature, pressure and relative humidity, the tropospheric range error obtained from ray-trace (RANGE ERROR), the tropospheric elevation angle error, the tropospheric range error approximation (RANGE ERROR APPROX) obtained from using equation (18) of the main text, and the difference between the ray-trace and the approximation (RANGE DIFF) for arrival angles of 10°, 15°, 20°, 40° and 80°.

REFERENCES FOR APPENDIX 3

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