EUROPHYSICS LETTERS 1 November 2001

Europhys. Lett., **56** (3), pp. 354–360 (2001)

Noise-mediated dynamics in a two-dimensional oscillator: Exact solutions and numerical results

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(received 1 June 2001; accepted in final form 1 August 2001)

PACS. 05.45.-a – Nonlinear dynamics and nonlinear dynamical systems.
PACS. 05.40.Ca – Noise.
PACS. 85.25.Dq – Superconducting quantum interference devices (SQUIDs).

Abstract. – We derive a Fokker-Planck equation (FPE) to analyze the oscillator equations describing a nonlinear amplifier, exemplified by a two-junction Superconducting Quantum Interference Device (SQUID), in the presence of thermal noise. We show that the FPE admits a unique stationary solution and obtain analytical results for several parameters ranges. To solve the FPE numerically, we develop an efficient spectral method which exploits the periodicity of the probability density. The numerical method, combined with the exact solutions, allow us to rapidly explore the noise-mediated dynamics as a function of the control parameters.

The study of nonlinear dynamical behavior in systems that undergo bifurcations via changing a control parameter is of considerable interest. Tuning these systems to the onset of the bifurcation can lead to very large changes in the output in response to a very small external perturbation, resulting in a high gain and great sensitivity to a "target" input or forcing signal. Recent work has focused on the onset of spontaneous oscillations, via a saddle-node connection, in one realization out of many sharing such dynamics: the two-junction (or "dc") SQUID [1]. The dc SQUID is characterized [2] by a two-dimensional (2D) set of dynamical equations for the junction Schrödinger phase differences. Our interest in the SQUID stems from its relevance as the most sensitive detector of magnetic fields. Experimental and numerical results [3, 4] have shown that the optimal response of the SQUID to an input signal occurs just beyond the bifurcation point. Noise, present from a variety of sources, can however change the dynamical response of the system and understanding the effect of noise on the SQUID is of obvious importance.

In this letter, we examine the dynamics of a dc SQUID in the presence of thermal Johnson noise generated in the resistive shunts; this manifests itself as white voltage noise sources in the junctions. Previous calculations have been mostly numerical (*e.g.*, [5]); instead, in this letter we use a Fokker-Planck equation (FPE) approach which yields the possibility of extracting analytic information. Our analytical results are compared against very efficient numerical simulations of the FPE.

The SQUID dynamics are described by equations for the time-derivatives of the Schrödinger phase differences δ_i across the (assumed identical) Josephson junctions [2, 4]:

$$\tau\dot{\delta}_i = I_{\mathrm{b}}/2 + (-1)^i I_{\mathrm{s}} - I_0 \sin\delta_i, \quad i = 1, 2. \tag{1}$$

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The circulating current I_s, induced in the loop by an external magnetic flux, is the experimental observable of interest and can be written in the form $\beta I_\mathrm{s}/I_0 = \delta_1 - \delta_2 - 2\pi\Phi_e/\Phi_0$. Here, $\tau = \hbar/(2eR)$ is a characteristic time constant (R being the normal-state resistance of the junctions), $\beta \equiv 2\pi L I_0/\Phi_0$ the nonlinearity parameter, L the loop inductance, I_0 the junction critical current and $\Phi_0 \equiv h/2e$ the flux quantum. The two natural experimental control parameters are the applied dc magnetic flux Φ_e and the dc bias current I_b, which we take to be symmetrically applied to the loop. It is convenient to rescale time by τ and introduce a scaled flux $\Phi_\mathrm{ex} \equiv \Phi_e/\Phi_0$ and bias current $J \equiv I_\mathrm{b}/(2I_0)$. Inserting the Langevin noise sources in the dynamics yields the system

$$\begin{aligned}\dot{\delta}_1 &= J - \frac{1}{\beta}(\delta_1 - \delta_2 - 2\pi\Phi_\mathrm{ex}) - \sin\delta_1 + \xi_1(t),\\ \dot{\delta}_2 &= J + \frac{1}{\beta}(\delta_1 - \delta_2 - 2\pi\Phi_\mathrm{ex}) - \sin\delta_2 + \xi_2(t),\end{aligned} \tag{2}$$

where ξ_i is white Gaussian noise having zero mean and correlation function $\langle \xi_i(t)\xi_j(t')\rangle = 2D\delta_{ij}\delta(t-t')$.

As mentioned above, this system exhibits two (in the absence of noise, distinct) regimes of operation [4, 6]. For a fixed Φ_ex, a saddle-node connection takes place when the bias current J exceeds a critical value J_c. For $J < J_\mathrm{c}$, the noiseless system has two fixed points, one stable (a node) and one unstable (a saddle) [7]. This is the "superconducting regime" with the potential energy function admitting of stable minima corresponding to a current conservation $2J = \sin\delta_1 + \sin\delta_2$. For $J > J_\mathrm{c}$ the fixed points disappear and we obtain oscillatory solutions whose frequency obeys the characteristic square-root scaling law [6]. This latter regime is the so-called "running regime." The properties of the solutions near the bifurcation have recently been studied [6].

The coupled Langevin equations (2) lead to a Fokker-Planck equation for the probability density $\rho(\delta_1, \delta_2, t)$:

$$\frac{\partial\rho}{\partial t} = D\left[\frac{\partial^2\rho}{\partial\delta_1^2} + \frac{\partial^2\rho}{\partial\delta_2^2}\right] - \frac{\partial}{\partial\delta_1}(v_1\,\rho) - \frac{\partial}{\partial\delta_2}(v_2\,\rho), \tag{3}$$

and

$$\begin{aligned}v_1(\delta_1,\delta_2,t) &= J - \frac{1}{\beta}(\delta_1 - \delta_2 - 2\pi n - 2\pi\Phi_\mathrm{ex}) - \sin\delta_1,\\ v_2(\delta_1,\delta_2,t) &= J + \frac{1}{\beta}(\delta_1 - \delta_2 - 2\pi n - 2\pi\Phi_\mathrm{ex}) - \sin\delta_2,\end{aligned} \tag{4}$$

where $v_{1,2}$ are the drift terms (with 2π-periodicity in δ_1 and δ_2), and the density function is normalized, $\int_0^{2\pi}\int_0^{2\pi}\rho(\delta_1,\delta_2,t)\,\mathrm{d}\delta_1\,\mathrm{d}\delta_2 = 1$, with n chosen in order to obtain a periodic continuation of the coefficients.

We are interested in finding solutions of the FPE for large time. This search is greatly facilitated by the fact that the FPE has a unique stationary solution. This can be seen by noting that the functional $H(t) = \int \rho\ln(\rho/\rho_0)\,\mathrm{d}\delta_1\,\mathrm{d}\delta_2$, where ρ_0 is the stationary solution, is a Lyapunov function (see [8] and references therein). It then follows that such a stationary solution is unique and globally stable.

For $J = 0$, we have found an exact (stationary) solution:

$$\rho_0(\delta_1,\delta_2) = \alpha e^{-\frac{1}{2\beta D}(\delta_1-\delta_2-2\pi n-2\pi\Phi_\mathrm{ex})^2}\, e^{\frac{1}{D}\cos\delta_1}\, e^{\frac{1}{D}\cos\delta_2}, \tag{5}$$

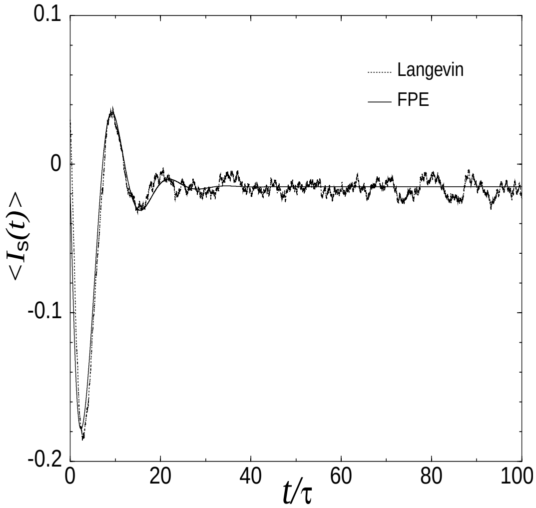


Fig. 1 – Comparison between the numerical solution of the Langevin equations (averaged over 10000 realizations) and the solution of the Fokker-Planck equation by the spectral method with $N = M = 20$ moments. Parameters are $D = 0.1$, $J = 0.9$, $\beta = 1$, and $\Phi_{\rm ex} = 0.2$.

where

$$\alpha = \left[\int_0^{2\pi} \int_0^{2\pi} \mathrm{d}\delta_1 \, \mathrm{d}\delta_2 e^{-\frac{1}{2\beta D}(\delta_1 - \delta_2 - 2\pi n - 2\pi \Phi_{\rm ex})^2} e^{\frac{1}{D}\cos\delta_1} e^{\frac{1}{D}\cos\delta_2} \right]^{-1}. \tag{6}$$

In the general case of $J \neq 0$, information may be extracted when β is large. Making use of the *ansatz*

$$\rho_0(\delta_1, \delta_2) = e^{-\frac{1}{2\beta D}(\delta_1 - \delta_2 - 2\pi n - 2\pi \Phi_{\rm ex})^2} \Psi(\delta_1, \delta_2), \tag{7}$$

and expanding in powers of $1/\beta$,

$$\Psi(\delta_1, \delta_2) = \Psi_0 + \frac{1}{\beta}\Psi_1 + O(1/\beta^2), \tag{8}$$

the zeroth-order term can be obtained:

$$\Psi_0(\delta_1, \delta_2) = \alpha \, e^{\frac{1}{D}\cos\delta_1} e^{\frac{1}{D}\cos\delta_2} f(\delta_1) f(\delta_2), \tag{9}$$

with

$$f(\delta_{1,2}) = \int_0^{2\pi} e^{-\frac{1}{D}[J\eta + \cos(\delta_{1,2}+\eta)]} \, \mathrm{d}\eta, \tag{10}$$

with α chosen to satisfy the normalization condition

$$\int_0^{2\pi} \int_0^{2\pi} \mathrm{d}\delta_1 \, \mathrm{d}\delta_2 e^{-\frac{1}{2\beta D}(\delta_1 - \delta_2 - 2\pi n - 2\pi \Phi_{\rm ex})^2} \Psi_0 = 1. \tag{11}$$

For parameters values for which analytical progress is difficult to achieve one has to resort to numerics. Direct simulation of the Langevin equations (2), as has been commonly done in

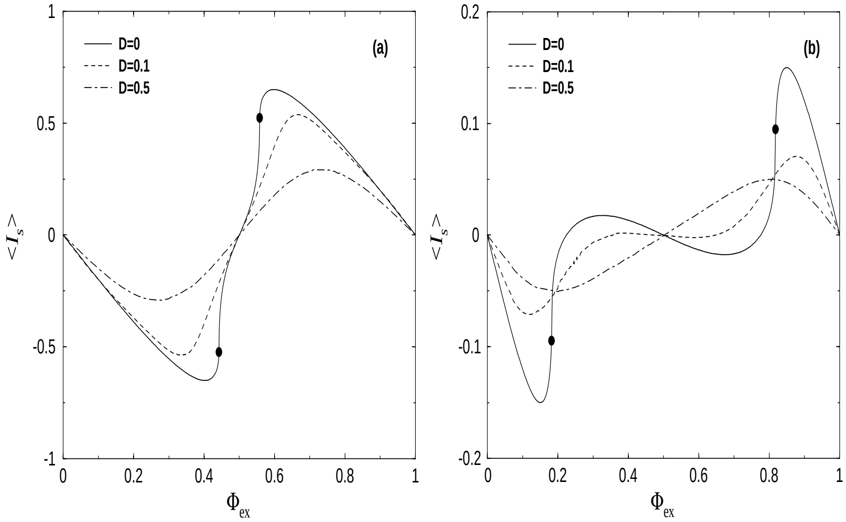


Fig. 2 – Transfer characteristics for different noise values for two different values of the bias current J ($J = 0.35$ in (a) and $J = 0.85$ in (b)). For the noiseless case, running solutions are present for parameter values that lie between the two solid circles [4].

the dc SQUID repertoire (see, *e.g.*, [5]), can be computationally intensive. For reasonably accurate results one typically has to average over many realizations. This is particularly the case for systems close to a bifurcation point where one has to distinguish between different stable solutions. Numerical solutions of the FPE, on the other hand, can be obtained much faster. Rather than using a finite-difference scheme we have used an efficient spectral method [9].

In this method, we expand ρ in a Fourier series,

$$\rho(\delta_1, \delta_2, t) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} r_n^m(t) e^{i\, n\delta_1} e^{i\, m\delta_2}, \tag{12}$$

exploiting the 2π-periodicity in δ_1, and δ_2. Introducing this expansion into the Fokker-Planck equation (3), we obtain the following hierarchy of ordinary differential equations for the moments r_n^m:

$$\begin{aligned} \dot{r}_n^m &= -D\big(n^2 + m^2\big) r_n^m - i\, J(n+m) r_n^m + \frac{n}{2}\big(r_{n-1}^m - r_{n+1}^m\big) + \frac{m}{2}\big(r_n^{m-1} - r_n^{m+1}\big) + \\ &\quad + \frac{n-m}{\beta} \sum_{l=1}^{\infty} \frac{(-1)^l}{l} \big[\cos(2\pi\, l\Phi_{\rm ex})\big(r_{n-l}^{m+l} - r_{n+l}^{m-l}\big) - i\, \sin(2\pi\, l\Phi_{\rm ex})\big(r_{n-l}^{m+l} + r_{n+l}^{m-l}\big)\big], \\ n &= -\infty, \dots, \infty, \quad m = -\infty, \dots, \infty. \end{aligned} \tag{13}$$

In practice, we numerically solve (13) for $n = -N, \dots, N$, and $m = -M, \dots, M$, setting $r_{N+1}^{M+1} = r_{-N-1}^{-M-1} = 0$. The number of modes is chosen such that the absolute error in ρ drops below a given tolerance, typically 10^{-12}. This number depends on the noise strength D and increases when D decreases.

Once the moments are obtained, we can calculate the various quantities of interest. The input-output transfer characteristic (TC) is a convenient descriptor of the system response in terms of experimentally controllable or measurable quantities. The TC is a plot of the average screening current (averaged over the phases) $\langle I_{\rm s}(t)\rangle$ *vs.* the external flux $\Phi_{\rm ex}$, with

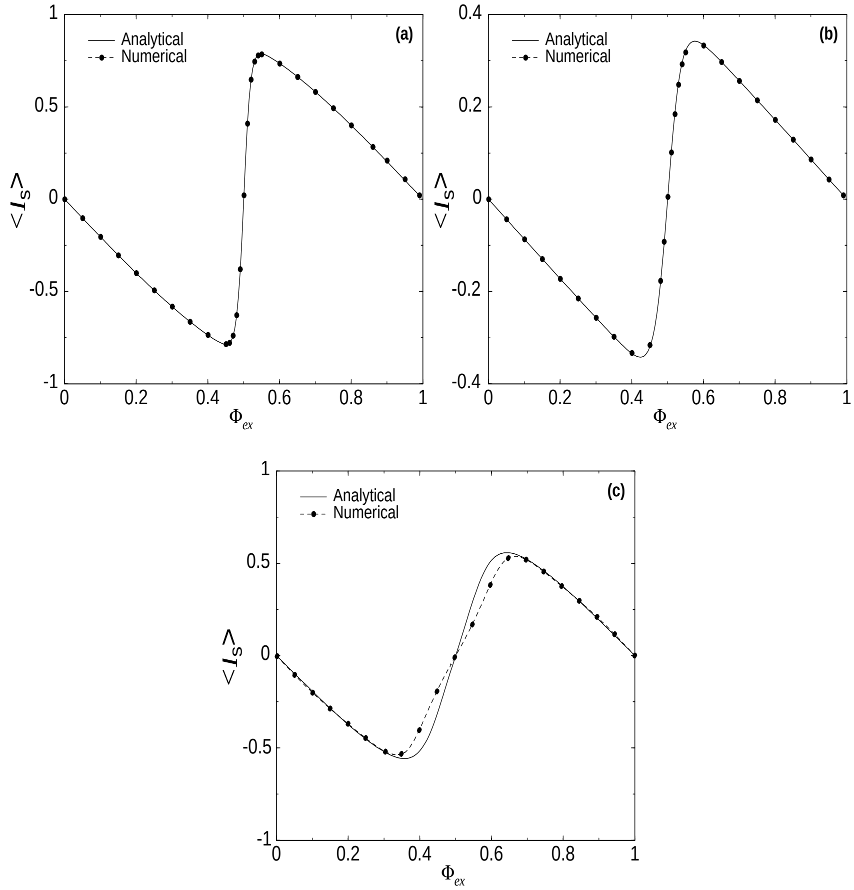


Fig. 3 – Comparison between the analytical and numerical solution. Parameters are $D = 0.1$; (a) $J = 0$, $\beta = 1$, (b) $J = 0.35$, $\beta = 5$, and (c) $J = 0.35$, $\beta = 1$.

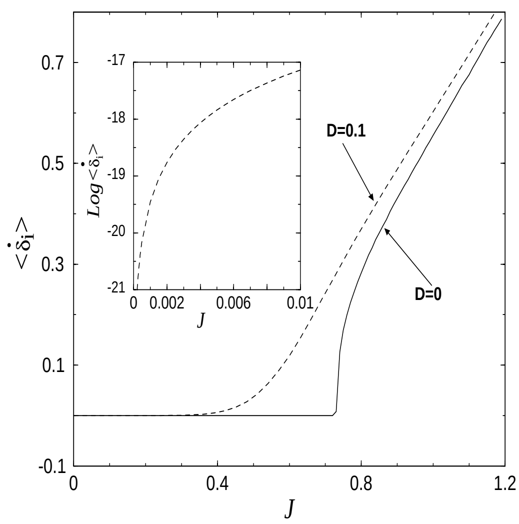
$\langle I_\mathrm{s}(t)\rangle$ computed as

$$\langle I_\mathrm{s}(t)\rangle = \frac{I_o}{\beta}\int_0^{2\pi}\int_0^{2\pi} \mathrm{d}\delta_1\,\mathrm{d}\delta_2 I_\mathrm{s}(\delta_1,\delta_2)\rho(\delta_1,\delta_2,t). \tag{14}$$

In terms of the moments in our Fourier expansion, this becomes

$$\langle I_\mathrm{s}(t)\rangle = \frac{I_0}{\beta}\left[-8\pi^2\sum_{l=1}^{\infty}\frac{(-1)^l}{l}\,\mathrm{Im}\left(r_l^{-l}e^{-i\,2\pi\,l\Phi_\mathrm{ex}}\right)\right]. \tag{15}$$

In fig. 1, we plot the average screening current $\langle I_\mathrm{s}\rangle$ as a function of time, obtained numerically by solving the Langevin equations (2) and by solving the FPE using the above-described spectral method. The spectral method (with $N = M = 20$ moments) is seen to provide excellent agreement with the more conventional and time-consuming technique based on numerically integrating the coupled stochastic differential equations (2). We have also verified our spectral method by solving the FPE using a finite-difference algorithm. To achieve an absolute error below 10^{-3} our spectral method required $N = M = 10$ moments while direct simulations of the Langevin equations required $N = 10^6$ realizations (the error in this method



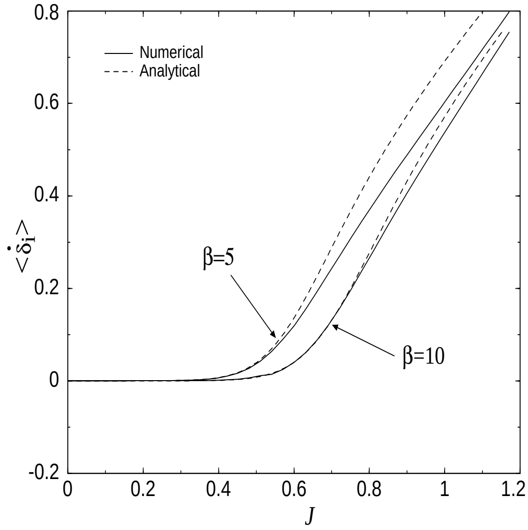


Fig. 4 – Numerically obtained average frequency for $D = 0$ (solid line) and $D = 0.1$ (dashed line). Parameters are $\Phi_{\rm ex} = 0.4$, and $\beta = 5$. The inset shows a blow-up of the $D = 0.1$ curve near $J = 0$ on a log-linear scale and illustrates the nonzero average frequency for small J.

Fig. 5 – Comparison between the average frequency obtained numerically and analytically. Note that the agreement improves when β increases. Parameters are $\Phi_{\rm ex} = 0.4$ and $D = 0.1$.

scales like $\sqrt{(1/N)}$). A speed comparison between the two methods revealed that our spectral method was more than three orders of magnitude faster.

In fig. 2 we show the effect of the noise intensity on the TCs for two different values of the bias current. In the noiseless case, the TC was obtained by solving eq. (2) (setting $D = 0$) while in the noisy case the TC was obtained by direct numerical simulation of the FPE. Figure 3 shows a comparison between the analytical and numerical solutions of the FPE for different J and β. Figure 3(a) corresponds to $J = 0$ for which the analytical solution (5) is exact. Indeed, the analytical and numerical solution are indistinguishable. Figure 3(b) shows the TC for a large value of β ($\beta = 5$) and demonstrates that the zeroth-order term of our expansion (9) already produces excellent results. In (c), for $\beta = 1$, we see that the analytical solution starts to deviate from the numerical one.

Let us investigate the dynamics of the noisy system near the bifurcation point, where the system is highly sensitive to input signals. We are particularly interested in the average frequency of each junction, which is of importance to know the device operation regime. This frequency, $\dot{\delta}_i$, can be readily calculated by noting that $\langle\dot{\delta}_i\rangle = \langle v_i\rangle$, since the white noise has zero mean. In the FPE, the average drift is calculated as

$$\langle v_i\rangle = \int_0^{2\pi}\int_0^{2\pi} \mathrm{d}\delta_1\,\mathrm{d}\delta_2 v_i(\delta_1,\delta_2)\rho_0(\delta_1,\delta_2) \tag{16}$$

which, in terms of the moments in our Fourier expansion, becomes

$$\langle v_i\rangle = J - (-1)^i\langle I_{\rm s}\rangle + 4\pi^2\,\mathrm{Re}(r_1^0). \tag{17}$$

In fig. 4, we show the average frequency $\langle\dot{\delta}_i\rangle$ as a function of J for the noisy ($D = 0.1$) and noiseless case. In the absence of noise, there exists a sharp transition, which corresponds to the bifurcation point that separates the superconducting and running regimes. In the noisy case, on the other hand, thermal fluctuations lead to a nonzero average frequency for arbitrary small bias currents J. This is illustrated in the inset in fig. 4 which shows $\langle \dot{\delta}_i \rangle$ near $J = 0$. Finally, in fig. 5 we show a comparison between the analytical and numerical solution for fixed noise level. As expected from the perturbation analysis, the agreement improves as β is increased.

To summarize, we have analyzed the dynamics of the dc SQUID in the presence of thermal Johnson noise. Our approach consists of using the 2D Fokker-Planck equation associated with the stochastic nonlinear Langevin equation. The advantage of this formulation is that it gives a simpler picture of the dynamics, now described in terms of a probability density. This density has been shown to evolve towards a stationary solution. For certain parameter values, we have been able to find exact solutions while for other regimes in parameter space we have provided an expansion. Finally, we have constructed a very efficient way of numerically solving the FPE. It should be emphasized that the techniques described here are not limited to SQUIDs. In fact, it can be used to describe the dynamics of a large class of nonlinear dynamical systems with noisy input. We apply it to a prototype system (the dc SQUID) which is of considerable interest as the most sensitive device to measure magnetic fields in a variety of physics and biomedical applications. Furthermore, it should be stressed that the present study can be extended to account for an external injection signal, arrays of globally coupled SQUIDs or even modified potentials used to study ratchet effects in dc SQUIDs [10]. In the latter cases, nonlinearities in the 2D Fokker-Planck can lead to the appearance of bifurcations, which might be analytically tractable. These extensions are currently under investigation.

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This work has been supported by the Office of Naval Research (Code 331). We also thank the National Partnership for Advanced Computational Infrastructure at the San Diego Supercomputer Center for computing resources.

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