University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 213 Winter 2023
Assignment 9

Due 11:00am Tuesday March 14, 2023

1. **Brainwarmers.**

   (a) [optional] Is it true that $0 \leq S(A|C) + S(B|C)$? Prove or give a counterexample.

   (b) Show that the von Neumann entropy is the special case $S(\boldsymbol{\rho}) = \lim_{\alpha\to 1} S_\alpha(\boldsymbol{\rho})$ of the Renyi entropies:

   $$S_\alpha(\boldsymbol{\rho}) \equiv \frac{\text{sgn}(\alpha)}{1-\alpha} \log \text{tr}\boldsymbol{\rho}^\alpha = \frac{\text{sgn}(\alpha)}{1-\alpha} \log \sum_a p_a^\alpha \ .$$

2. **Work and the Holevo bound.** [optional]

   (a) Show that the Holevo quantity $\chi(p_a, \rho_a) \equiv S(\rho_{av}) - \sum_a p_a S(\rho_a)$ (with $\rho_{av} \equiv \sum_a p_a \rho_a$) can be written as $\chi(p_a, \rho_a) = \sum_a p_a D(\rho_a||\rho_{av})$.

   (b) Show that

   $$\sum_a p_a D(\rho_a||\sigma) = \chi(p_a, \rho_a) + D(\rho_{av}||\sigma).$$

   (c) Suppose A labors in contact with a heat bath at temperature T, and is governed by hamiltonian H. Convince yourself that in order to create the signal state ρ_a, the required work A must do is

   $$W_a \geq F_T[\rho_a] - F_T[\rho_T] = (k_B T \ln 2)\, D(\rho_a||\rho_T),$$

   where $F_T[\rho] \equiv \text{tr}\rho H - T S_{vN}[\rho]$ is the free energy functional.

   (d) Show that the average work $\bar{W} \equiv \sum_a p_a W_a$ satisfies

   $$\bar{W} \geq (k_B T \ln 2)\, \chi(p_a, \rho_a).$$

   (hint: $D(\rho||\sigma) \geq 0$).

   (e) Apply the Holevo bound to conclude

   $$\bar{W} \geq (k_B T \ln 2)\, I(A:B),$$

   so that that every bit of information A can convey to B requires average work at least $k_B T \ln 2$. Yay, Landauer.

(f) [optional] Estimate the amount of work done per bit sent to your cellular telephone.

3. **Holevo quantity and channel capacity.** [optional] Consider a collection of mutually-commuting density matrices $\{\rho_a\}$. Show that in this case, the Holevo quantity

$$\chi(p_a, \rho_a) \equiv S(\rho_{av}) - \sum_a p_a S(\rho_a) = \sum_a p_a D(\rho_a || \rho_{av}), \quad \rho_{av} \equiv \sum_a p_a \rho_a$$

is the mutual information $I(A:B)$, where the random variable B is the variable b labelling the mutual eigenvectors of the ρ_a: $\rho_a = \sum_b \lambda_a^b |b\rangle\langle b|$.

This suggests that a good definition of the capacity of a quantum channel for sending classical information (let's call it classical capacity) is determined by the Holevo quantity as

$$C = \chi(p_a, \rho_a)/\mathcal{T}$$

(where $\mathcal{T}$ is how long the information takes to go down the channel). And indeed, recall the Holevo bound, which says that $I(A:B) \leq \chi(p_a, \rho_a)$ where B is the outcomes of *any* measurement done on $\sum_a p_a \rho_a$.

4. **Channel capacity of the radiation field.** [optional but highly encouraged]

Suppose (crazy idea) we wanted to send signals using the electromagnetic field.

The radiation field is a collection of quantum harmonic oscillators labelled by frequency, ω. For simplicity, let's consider a one-dimensional field with only one polarization, so there is one oscillator for each value of ω. In the first part of the problem, we'll put the system in a box, so that the allowed frequencies are integer multiples of some fundamental frequency, and the energy of a state with n_j photons in mode j is $E(\{n\}) = \sum_j j n_j h \equiv Nh$ for some constant h.

The signal information could be stored for example in the number of photons $\bar{n}(\omega)$ with a given frequency. As in other examples, to send message a, A puts the field in the state ρ_a. And the message can be extracted by measurements on the resulting radiation field, for example by counting photons.

For practical reasons, we will fix the power P of the signal. There are several ways to implement this constraint; we'll consider two below.

At first we ignore the presence of noise.

(a) Show that the Holevo quantity χ (and hence the channel capacity, no matter what measurement we do) is bounded by the entropy of the average signal $\sum_a p_a \rho_a$.

(b) What is the ρ_{av} that maximizes the entropy, subject to the constraint of fixed energy $E(\{n\}) = P\mathcal{T}$ (where $\mathcal{T}$ is the duration of the signal)?

(c) As a useful intermediate step, show that the entropy for a single harmonic oscillator in thermal equilibrium can be written in terms of the average occupation number $\bar{n}$ as $S_B(\bar{n})$ where

$$S_B(n) \equiv (n+1)\log(n+1) - n\log n.$$

(d) Using the definition of classical capacity in the previous problem, determine the classical capacity of the channel in part 4b at large $\mathcal{T}$.

You may use the Hardy-Ramanujan formula, which counts partitions of N at large N:

$$\mathcal{N}(N) = \frac{1}{4\sqrt{3}N} e^{\pi\sqrt{\frac{2}{3}N}} + \mathcal{O}\left(e^{\frac{\pi}{2}\sqrt{\frac{2}{3}N}}\right).$$

(e) Alternatively, we may impose the condition of fixed power as a condition on the *average* energy. The state which maximizes entropy at fixed average energy is a thermal state. The temperature is determined by the average energy, which is in turn related to the power carried by the signal. Find the relation between T and P. Find a bound on the channel capacity at fixed average energy. (In this part of the problem you may take the infinite-volume limit.)

Inevitably there will be noise, represented by an additional number of photons $\bar{n}(\omega)$ at each frequency which are out of our control. Assume the noise is thermal, in equilibrium at temperature T_N. Suppose the power of the *signal* P (which is some amount of extra photons on top of the noise) is still fixed.

(f) Convince yourself that the upper bound on the channel capacity is now reduced by the entropy of the noise:

$$C\mathcal{T} \le S(\rho_{T_{S+N}}) - S(\rho_{T_N})$$

where ρ_T is the thermal density matrix with temperature T, T_N is the noise temperature, and T_{S+N} is the temperature at an average energy which includes both the noise and the signal. Find T_{S+N} in terms of T_N and P.

(g) Do the integral over frequency. Study the high- and low-temperature limits of your answer. Confirm Landauer's principle in the former case in the following sense: compute the minimum power required to send a single bit.

5. **Direct application of Lieb's theorem.**

   We only used a very special case of Lieb's theorem to prove monotonicity of the relative entropy. Surely there is more to learn from it.

   Consider an ensemble of states $\boldsymbol{\rho} = \sum_i p_i \boldsymbol{\rho}_i$, and a unitary operator $\mathbf{U}$ (for example, it may be closed-system time evolution).

   Show that the relative entropy between $\boldsymbol{\rho}(t) \equiv \mathbf{U}\boldsymbol{\rho}\mathbf{U}^\dagger$ and $\boldsymbol{\rho}$ is convex in $\boldsymbol{\rho}$:

$$D(\boldsymbol{\rho}(t)||\boldsymbol{\rho}) \le \sum_i p_i D(\boldsymbol{\rho}_i(t)||\boldsymbol{\rho}_i).$$

   Open ended bonus problem: see if you can find a better result by directly applying Lieb's joint concavity theorem to a problem in many body physics.

6. **Random singlets.** [optional]

   Consider qbits arranged on a chain. Suppose that the groundstate is made of random singlets, in the following sense: for a given site i, with probability $f(|i-j|a)$ (a is the lattice spacing), the spins at i and j are in the state $(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$. Every spin is paired with some other spin. Consider in turn the case of short-range singlets where $f(x) \propto e^{-x/\xi}$, and long-range singlets where $f(x) \propto \frac{1}{x^2+\delta^2}$.

   (a) Consider a region A which is an interval $\left[-\frac{R-\epsilon}{2}, \frac{R-\epsilon}{2}\right]$ ($\epsilon \ll R$) and B is what we called $\bar{A}^-$ (nearly the complement), more precisely: $B \equiv [-\infty, -\frac{R}{2}] \cup [\frac{R}{2}, \infty]$. Let $I_\epsilon(R) \equiv I(A:B) = S(A) + S(B) - S(AB)$ be their mutual information.

   Find $\overline{\left\langle \vec{\mathbf{S}}_i \cdot \vec{\mathbf{S}}_j \right\rangle}$ (where $\vec{\mathbf{S}} = \frac{1}{2}(\sigma^x, \sigma^y, \sigma^z)$) and $\overline{I_\epsilon(R)}$. In both cases assume the regions are big enough that you can average over regions and use a continuum approximation ($\xi, \delta \gg$ lattice spacing).

   Check that the answer is consistent with the mutual information bound on correlations.

   (b) Consider instead the case where $B = [-\infty, -\frac{R}{2} - L] \cup [\frac{R}{2} + L, \infty]$, so that A and B are separated by a distance L. Show that: for short-range singlets, (i) all (averaged) correlation functions decay exponentially in L (ii) $I(A:B) \sim e^{-L/\xi}$ for large L (and hence the mutual information satisfies an area law). For long-range singlets (i) (averaged) correlation functions have power law decay (ii) $I(A:B) \sim \log(2R - L)$ for large L, and there is no area law.