University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 213 Winter 2023
Assignment 8 – Solutions

Due 11:00am, Tuesday March 7, 2023

Note that I've made most of the problems optional so that you have time to think about the final paper.

1. **Brain-warmer.** [optional] Let $Z = \sum_j |j\rangle \langle j| \omega^j$ be the clock operator on a d-dimensional Hilbert space, $\omega \equiv e^{2\pi \mathbf{i}/d}$. Show that $K_j = Z^j/\sqrt{d}$ are Kraus operators for the diagonal-part channel. That is: a density matrix can be scrambled (as defined in lecture) by only d operators.

$$\rho \mapsto \frac{1}{d}\sum_j Z^j \rho Z^{-j}$$

in the Z-basis, has ρ_{jj} on the diagonal and

$$\sum_{j=0}^{d} \omega^j = 0$$

on the off-diagonal.

2. **Equivalent Kraus representations.** [optional] Show that $\{\mathcal{K}\} \simeq \{\tilde{\mathcal{K}}\}$ produce the same channel iff $\mathcal{K}_k = \sum_l u_{kl}\tilde{\mathcal{K}}_l$ where u_{kl} is a unitary matrix in the kl indices.

Open-ended bonus problem: find invariants of $\{\mathcal{K}\}$ which label equivalence classes under the above equivalence relation.

We're going to substitute in $K_k = \sum_l u_{kl}\tilde{K}_l$ and $K_k^\dagger = \sum_{l'} u^\star_{kl'}\tilde{K}^\dagger_{l'}$ into

$$\rho \mapsto \sum_k K_k \rho K_k^\dagger = \sum_{k,l,l'} u_{kl}\tilde{K}_l \rho u^\star_{kl'}\tilde{K}^\dagger_{l'} = \sum_{ll'} \tilde{K}_l \rho K^\dagger_{l'} \underbrace{\sum_k u_{kl}\left(u^\dagger\right)_{l'k}}_{=\delta_{ll'}} = \sum_l \tilde{K}_l \rho \tilde{K}_l^\dagger .$$

3. **The entropy exchange.** [optional] [Petz' book] Many of the quantum channels we considered in previous homeworks have the property that they increase the von Neumann entropy of their victim. An exception is the amplitude damping channel, which makes the state more pure. An implementation of a second law of thermodynamics would require us to identify a total entropy that probably increases.

We showed that any channel can be regarded as unitary evolution on a larger space (followed by partial trace), and unitary evolution doesn't change the entropy (of the whole system) at all. Even without introducing the environment explicitly, the Kraus operators give us a way to keep track of the entropy:

Suppose a CPTP map $\mathcal{E} : \text{End}(A) \to \text{End}(B)$ has Kraus representation $\{K_i\}_{i=1..r}$. For any density matrix on A, let

$$\varsigma_{ij} \equiv \text{tr} K_i \boldsymbol{\rho} K_j^\dagger.$$

(Recall that the index on the Kraus operators comes from a basis on the environment.)

(a) Show that ς_{ij} is positive and has unit trace as a matrix in the ij space. (So it can be regarded as a density matrix on that space.)

Unit trace follows from cyclicity of the trace and $\sum_i K_i^\dagger K_i = \mathbb{1}$. To show that ς is positive consider an arbitrary diagonal matrix element:

$$\begin{aligned} v_i^\star \varsigma_{ij} v_j &= v_i^\star \text{tr} K_i \rho K_j^\dagger v_j & (1) \\ &= \text{tr} \left(\sum_i v_i^\star K_i \right) \rho \left(\sum_j v_j K_j^\dagger \right) & (2) \\ &= \text{tr} X \rho X^\dagger = \sum_a \langle a| X \rho X^\dagger |a\rangle = \sum_{a'} \langle a'| \rho |a'\rangle \geq 0. & (3) \end{aligned}$$

(b) The von Neumann entropy of ς_{ij}, $S(\varsigma) = -\text{tr}\varsigma \log \varsigma$ is called the *entropy exchange*. Show that $S(\varsigma)$ is preserved by the equivalence relation between Kraus representations of $\mathcal{E}$.

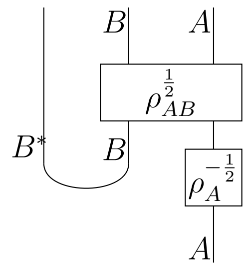
It's independent of the choice of basis on the environment.

(c) Show (by Stinespring dilation) that ς is the reduced density matrix of the *environment* after the action of the channel, and therefore that the entropy exchange is equal to the entropy of the environment after the action of the channel.

Dilation says we can realize the channel as unitary evolution U on a larger space AE, in which case $K_i = {}_E\langle i| U |0\rangle_E$. Therefore

$$\varsigma_{ij} = \text{tr}_A K_i \rho K_j^\dagger = \text{tr}_A \langle i| U \rho \otimes |0\rangle\langle 0|_E U^\dagger |j\rangle = {}_E\langle i| \left(\text{tr}_A U \rho \otimes |0\rangle\langle 0|_E U^\dagger \right) |j\rangle_E \,.$$

4. **Isometries from density matrices.** [optional] Given a positive density matrix (no zero eigenvalues) on a bipartite system ρ_{AB}, show that the following object $V^{\rho}_{A\to BB^{\star}A}$ (time goes up in the picture):

$V^{\rho}_{A\to BB^{\star}A} =$ 

(4)

is an isometry, $V^{\dagger}V = \mathbb{1}_A$.

5. **Strong subadditivity implies weak monotonicity.** [important] Show that for any state $\boldsymbol{\rho}_{ABC}$ on $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, the condition

$$S(A) + S(B) \le S(AC) + S(BC)$$

is equivalent to strong subadditivity in the form

$$S(A'C') + S(A'B') \ge S(A') + S(A'B'C') \qquad \forall A'B'C' \ .$$

Hint: use the Araki-Lieb purification trick.

[CN page 521] Choose a purification $\boldsymbol{\rho}_{ABCR}$ and use the second version in the form

$$S(BC) + S(BR) \ge S(B) + S(RBC)$$

($i.e.$ set $A' = B, B' = R, C' = C$). Then purity of the whole state says $S(RBC) = S(A), S(BR) = S(AC)$ and we have

$$S(BC) + S(AC) \ge S(B) + S(A)$$

which is equivalent to the requested form. This argument is reversible.

6. **Consequences of SSA for mutual information.** [optional] Prove that SSA implies

$$I(A:B) + I(A:C) \le 2S(A) \ .$$

$$I(A:B) + I(A:C) = S_A + S_B - S_{AB} + S_A + S_C - S_{AC} \le 2S_A$$

since SSA says the total of the red terms is ≤ 0.

Is the analogous inequality for Shannon entropies true?

Shannon entropy is a special case of vN entropy for diagonal density matrices.

Alternatively, a more direct proof is the following:

$$-H(A:B)+H(A) = -H(A)-H(B)+H(AB)+H(A) = H(AB)-H(B) = H(A|B)$$

is a conditional entropy, which classically is positive.

Find an example of a state where $I(A:B) > S(A)$.

A Bell pair does the trick, since $I(A:B) = 2 > 1 = S_A$.

7. **Measurement is coarse-graining.** [optional]

   Let $\boldsymbol{\rho}, \boldsymbol{\sigma}$ be two states on $\mathcal{H}$, and let $\{\mathcal{M}_x\}$ be a POVM. Define the classical probability distributions p_x, q_x from the outcomes of a measurement of $\{\mathcal{M}_x\}$ on the states $\boldsymbol{\rho}, \boldsymbol{\sigma}$ respectively (that is, $p_x = \mathrm{tr}\boldsymbol{\rho}\mathcal{M}_x$ etc). Show that

   $$\hat{D}(\boldsymbol{\rho}||\boldsymbol{\sigma}) \geq D(p||q).$$

   Define a quantum channel $\mathcal{M} : \mathrm{End}(\mathcal{H}) \to \mathrm{End}\left(\mathrm{span}\{|x\rangle\}_{\mathrm{ON}}\right)$ by

   $$\boldsymbol{\rho} \mapsto \mathcal{M}(\boldsymbol{\rho}) = \sum_x |x\rangle\langle x| \, \mathrm{tr}_{\mathcal{H}} \mathcal{M}_x \boldsymbol{\rho}.$$

   This is trace-preserving by $\sum_x \mathcal{M}_x = \mathbb{1}$ and completely positive by $\mathcal{M}_x \geq 0$. Then $D(p||q) = \hat{D}(\mathcal{M}(\boldsymbol{\rho})||\mathcal{M}(\boldsymbol{\sigma}))$ and the statement follows from monotonicity of the relative entropy under the action of any quantum channel.

8. **Scramble.** [optional]

   For this problem $\mathcal{H}_A$ has dimension d.

   (a) **Warmup.** The set of linear operators $\mathrm{End}(\mathcal{H}_A)$ is itself a Hilbert space with the Hilbert-Schmidt inner product $\langle \mathbf{A}, \mathbf{B}\rangle = \mathrm{tr}\mathbf{A}^\dagger\mathbf{B}$. Find an orthogonal basis $\{\mathbf{U}_a\}$ for this space (over $\mathbb{C}$) whose elements are themselves unitary operators, $\mathrm{tr}\mathbf{U}_a^\dagger\mathbf{U}_b = d\delta_{ab}$.

   [Hint: consider the algebra generated by the unitaries $\mathbf{X}, \mathbf{Z}$ on the qdit teleportation problem on the previous problem set.]

   One nice set of such operators is

   $$\{\mathbf{U}_{ij} \equiv \mathbf{X}^i\mathbf{Z}^j\}_{ij=1..d}$$

   which are orthogonal since $\mathbf{X}^i\mathbf{Z}^j$ is unitary and traceless unless $i = j = 0$.

   Bonus: For the case of $|A| = 2^k$ find such a basis whose elements square to one.

Alternatively, in the case when the dimension is 2^k, we can just take tensor products of pauli matrices:

$$\underbrace{\mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \cdots}_{1}, \underbrace{\mathbb{1} \otimes \boldsymbol{\sigma}^a \otimes \mathbb{1} \otimes \mathbb{1} \cdots}_{3k}, \underbrace{\mathbb{1} \otimes \boldsymbol{\sigma}^a \otimes \mathbb{1} \otimes \boldsymbol{\sigma}^b \cdots}_{3^{\frac{k(k-1)}{2}}}, \ldots$$

and these are orthogonal because the paulis are traceless, and normalized because the paulis are hermitian and square to one. And there is just the right number of them because

$$\sum_{l=0}^{k} 3^l \binom{k}{l} = (1+3)^k = 4^k = 2^{2k} = |A|^2.$$

(b) Consider a maximally entangled state $|\Phi\rangle \equiv \frac{1}{\sqrt{d}} \sum_i |ii\rangle \in \mathcal{H}_A \otimes \mathcal{H}_A$. Show that the d^2 maximally entangled states

$$|\Phi_a\rangle \equiv \mathbf{U}_a \otimes \mathbb{1} \, |\Phi\rangle$$

form an orthonormal basis of $\mathcal{H}_A \otimes \mathcal{H}_A$.

(c) Check your answers to the previous two parts for the case of qbits $d = 2$. Make a basis of product states from linear combinations of maximally entangled states.

(d) Find $\{p_a, \mathbf{U}_a\}$ with p_a probabilities and $\mathbf{U}_a$ unitary such that the associated channel *scrambles* an arbitrary operator $\mathbf{A} \in \mathrm{End}(A)$, in the sense that

$$\sum_a p_a \mathbf{U}_a \mathbf{A} \mathbf{U}_a^\dagger = \frac{\mathrm{tr}\mathbf{A}}{d} \mathbb{1}.$$

$p_a = 1/d^2$.

(e) Use the previous result and the concavity of the entropy to show that the uniform state $\mathbf{u} = \mathbb{1}/d$ has the maximum von Neumann entropy on A.

For any state $\boldsymbol{\rho}$,

$$S\left(\frac{\mathbb{1}}{d}\right) = S\left(\sum_a p_a \mathbf{U}_a \boldsymbol{\rho} \mathbf{U}_a^\dagger\right) \overset{\text{concavity}}{\geq} \sum_a p_a S\left(\mathbf{U}_a \boldsymbol{\rho} \mathbf{U}_a^\dagger\right) \overset{\text{basis independence}}{=} S(\boldsymbol{\rho}).$$

9. **Random quantum expanders.** [Very optional, somewhat open-ended and numerical]

Consider the family of quantum channels of the form

$$\boldsymbol{\rho} \mapsto \mathcal{E}_\chi(\boldsymbol{\rho}) = \sum_{i=1}^{\chi} p_i \mathbf{U}_i \boldsymbol{\rho} \mathbf{U}_i^\dagger$$

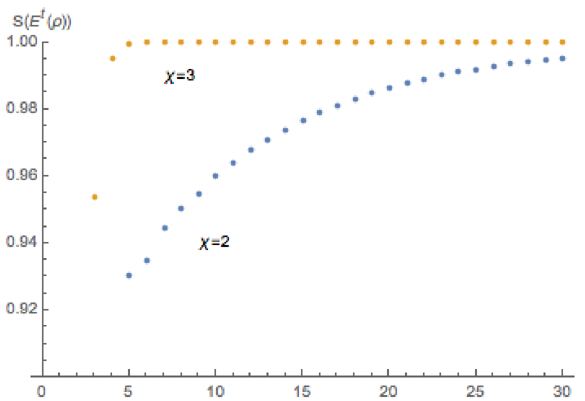
with $\{\mathbf{U}_i\}$ a collection of unitaries. Such a channel is called a *quantum expander*. Show that such a channel is unital.

Sample χ random unitaries from the Haar measure on $\mathsf{U}(d)$ *e.g.* in Mathematica[1]. (You can take $p_i = 1/\chi$ for definiteness if you wish.)

Sample a random initial density matrix[2].

Consider the rate at which repeated action of the channel $\mathcal{E}_\chi$, $\boldsymbol{\rho}_n = \mathcal{E}^n(\boldsymbol{\rho})$ mixes the initial state $\boldsymbol{\rho}$ as a function of χ (and d). We can use the von Neumann entropy as a measure of this mixing. Make some plots and some estimates.

Choosing $\chi = 2$ and $\chi = 3$ random unitaries and applying the resulting channel t times to a random initial density matrix, I find plots that looks like this:



In fact it should be possible to say something analytically about, say, the behavior of the purity. After one application of the channel, $\boldsymbol{\rho}_0 \to \sum_{a=1}^{\chi} \frac{1}{\chi} \mathbf{U}_a \boldsymbol{\rho}_0 \mathbf{U}_a^\dagger = \boldsymbol{\rho}$, the purity becomes

$$\mathrm{tr}\boldsymbol{\rho}^2 = \frac{1}{\chi^2} \sum_{ab}^{\chi} \mathrm{tr} \mathbf{U}_b^\dagger \mathbf{U}_a \boldsymbol{\rho}_0 \mathbf{U}_a^\dagger \mathbf{U}_b \boldsymbol{\rho}$$

[1]Haar measure means the measure which is invariant under the group action. I did this by choosing a $d \times d$ complex matrix X with entries chosen from the gaussian distribution (which is indeed invariant under $\mathsf{U}(d)$) and then taking $Y = X + X^\dagger$ to make it hermitian, and then using the matrix U which diagonalizes Y.

[2]I did this by choosing a complex matrix X with entries chosen from the gaussian distribution, and then taking $Y = X + X^\dagger$ to make it hermitian and then taking $Z = Y^2$ to make it positive and then taking $\boldsymbol{\rho} = Z/\mathrm{tr}Z$ to make it a density matrix. What distribution did I use?

and we need to evaluate Haar averages

$$\int d\Omega(U) U_{ij} U^\dagger_{kl} = \delta_{il}\delta_{jk}$$

$$\int d\Omega(U) U_{ij} U^\dagger_{kl} U_{mn} U^\dagger_{pr} = (\delta_{il}\delta_{mr} + \delta_{ir}\delta_{ml})(\delta_{jk}\delta_{np} + \delta_{jp}\delta_{kn}) \ldots$$

Taking the average, I'm finding

$$\overline{\mathrm{tr}\boldsymbol{\rho}^2} = \frac{1}{\chi^2}\left(\frac{\chi(\chi-1)}{2} + \chi(\ldots)\right)$$

something which rapidly decreases for large χ.

If n is very large, how many terms do I actually need to include in the sum in

$$\mathcal{E}^n(\boldsymbol{\rho}) = \sum_{i_1..i_n} p_{i_n} \cdots p_{i_1} \mathbf{U}_{i_1} \cdots \mathbf{U}_{i_n} \boldsymbol{\rho} \mathbf{U}^\dagger_{i_n} \cdots \mathbf{U}^\dagger_{i_1} \quad ?$$

By Shannon's noiseless channel theorem, the typical subspace has $2^{nS(p)}$ elements.

Consider the eigenstates (eigenoperators) of the (super)operator $\mathcal{E}_\chi$. Can you show that any state orthogonal (in the Hilbert-Schmidt norm) to $\mathbb{1}$ has an eigenvalue less than 1?

In lecture we showed several results beginning with monotonicity of the relative entropy as the starting point. Here we will show, following Ruskai, that SSA is just as good a starting point.

10. **SSA implies concavity of the conditional entropy.**

(a) Show that SSA can be rewritten as

$$D(\boldsymbol{\rho}_{12}||\boldsymbol{\rho}_2) \leq D(\boldsymbol{\rho}_{123}||\boldsymbol{\rho}_{23}) \tag{5}$$

where $\boldsymbol{\rho}_2$ means $\mathbb{1}_1 \otimes \boldsymbol{\rho}_2$ etc. (Note that in this expression the arguments are not density matrices and positivity of the BHS is not guaranteed.)

(b) Consider a bipartite state $\boldsymbol{\rho}_{12}$. Show that

$$D(\boldsymbol{\rho}_{12}||\mathbb{1}/d_1 \otimes \boldsymbol{\rho}_2) = -S(12) + S(2) + \log d_1 = -S(1|2) + \log d_1 \ .$$

Just use the definition and $\log(\mathbb{1}/d_1 \otimes \boldsymbol{\rho}_2) = -\log d_1 + \log \boldsymbol{\rho}_2$. This is a useful relation between the conditional entropy and a relative entropy (minus something, since it's not positive). Notice that the mutual information would have been $D(\boldsymbol{\rho}_{12}||\boldsymbol{\rho}_1 \otimes \boldsymbol{\rho}_2) = I(1:2)$.

(c) Apply SSA in the form (5) to the state

$$\boldsymbol{\rho}_{123} = \sum_i p_i \boldsymbol{\rho}_{12}^i \otimes |i\rangle \langle i|_3 .$$

Conclude the statement in the title of this problem.

For this density matrix,

$$S_{23} = \sum_i p_i S(\rho_2^i) + H(p), \quad S_{123} = \sum_i p_i S(\rho_{12}^i) + H(p).$$

So SSA says

$$S_{123} - S_{23} \leq S_{12} - S_2 = S_\rho(1|2)$$

The LHS is

$$\sum_i p_i \left(\underbrace{S(\rho_{12}^i) - S(\rho_2^i)}_{S_{\rho_i}(1|2)} \right) = \sum_i p_i S_{\rho_i}(1|2)$$

which says that the conditional entropy is concave.

11. **SSA implies monotonicity of the relative entropy.**

(a) Show that for $F(A)$ convex and homogeneous $F(xA) = xF(A)$,

$$\lim_{x\to 0} \frac{F(A+xB) - F(A)}{x} \leq F(B). \tag{6}$$

Convexity says

$$F(pA + (1-p)B) \leq pF(A) + (1-p)F(B) \tag{7}$$

for $p \in [0,1]$. Homogeneity says that this implies

$$pF\left(A + \frac{(1-p)}{p}B\right) \leq pF(A) + (1-p)F(B). \tag{8}$$

$$F\left(A + \frac{(1-p)}{p}B\right) \leq F(A) + \frac{(1-p)}{p}F(B). \tag{9}$$

Then letting $x = \frac{1-p}{p}$ we have

$$F\left(A + xB\right) - F(A) \leq xF(B) \tag{10}$$

for any $x > 0$ and we are done.

(b) Recall from problem (10) that SSA implies concavity of $S(2|1) \equiv S(\boldsymbol{\rho}_{12}) - S(\boldsymbol{\rho}_1)$.

We are going to let $F(\rho) = -S(2|1)_\rho$. By SSA, this function is convex, and it is homogeneous because $S(x\rho) = -\mathrm{tr} x\rho \log x\rho = xS(\rho) - x\log x$ so for any state on 12,

$$F(x\rho) = S(x\rho_1) - S(x\rho_{12}) = x(S(\rho_1) - S(\rho_{12})) = xF(\rho). \tag{11}$$

(c) Combine the first two parts of this problem, setting

$$A \equiv \boldsymbol{\sigma}_{12}, B \equiv \boldsymbol{\rho}_{12}$$

in (6) to show monotonicity of the relative entropy under partial trace.

Applying (6) with the given operators we have

$$\mathrm{tr}\rho_{12}\log\rho_{12} - \mathrm{tr}\rho_1\log\rho_1 \geq \lim_{x\to 0}\frac{1}{x}\left(-S(\sigma_{12}+x\rho_{12}) + S(\sigma_1+x\rho_1) + S(\sigma_{12}) - S(\sigma_1)\right) \tag{12}$$

$$= \mathrm{tr}\rho_{12}\log\sigma_{12} + \mathrm{tr}\rho_{12} - \mathrm{tr}\rho_1\log\sigma_1 - \mathrm{tr}\rho_1 \tag{13}$$

$$= \mathrm{tr}\rho_{12}\log\sigma_{12} + 1 - \mathrm{tr}\rho_1\log\sigma_1 - 1 \tag{14}$$

$$= \mathrm{tr}\rho_{12}\log\sigma_{12} - \mathrm{tr}\rho_1\log\sigma_1 \tag{15}$$

which can be rearranged to say

$$D(\rho_{12}||\sigma_{12}) \geq D(\rho_1||\sigma_1).$$

At step (13), I used $\partial_x|_{x=0}S(\sigma+\rho x) = -\mathrm{tr}\rho\log\sigma - \mathrm{tr}\rho$. In general the Taylor expansion of $\log(\sigma + x\rho)$ in x is a frightening thing (since ρ and σ need not commute), but cyclicity of the trace means that this does not cause trouble here.

12. **SSA implies joint convexity of relative entropy.**

(a) **Monotonicity of the relative entropy implies joint convexity.** Apply monotonicity of the relative entropy to the following block-diagonal bipartite states

$$\boldsymbol{\rho}_{AB} = \sum_i p_i \boldsymbol{\rho}^i_A \otimes |i\rangle\langle i|_B\,, \quad \boldsymbol{\sigma}_{AB} = \sum_i p_i \boldsymbol{\sigma}^i_A \otimes |i\rangle\langle i|_B \quad . \tag{16}$$

Conclude the boldface statement.

$$D(\rho_A||\sigma_A) \leq D(\rho_{AB}||\sigma_{AB}). \tag{17}$$

For the given density matrices, the LHS is

$$D(\rho_A||\sigma_A) = D(\sum_i p_i \rho_A^i || \sum_j p_j \sigma_A^j).$$

The RHS is

$$D(\rho_{AB}||\sigma_{AB}) = \mathrm{tr}\rho_{AB} \log \rho_{AB} - \mathrm{tr}\rho_{AB} \log \sigma_{AB}. \tag{18}$$

Again we use the fact form of the eigenvectors of states of the form (16) (which are sometimes called qc states): if the eigenvectors and eigenvalues of σ_A^i are $|k(i)\rangle\langle k(i)|$ and $\lambda_k^{(i)}$ respectively then those of σ_{AB} are

$$|k(i)\rangle\langle k(i)| \otimes |i\rangle\langle i| \quad \text{and} \quad p_i \lambda_k^{(i)}$$

respectively. Therefore

$$\log \sigma_{AB} = \sum_{i,k} |k(i)\rangle\langle k(i)| \otimes |i\rangle\langle i| \log \left(p_i \lambda_k^{(i)} \right)$$

and the RHS of (17) is

$$-H(p) + \sum_i p_i \mathrm{tr} \rho_A^i \log \rho_A^i + H(p) - \sum_i p_i \mathrm{tr} \rho_A^i \log \sigma_A^i = \sum_i p_i D(\rho_A^i || \sigma_A^i).$$

(b) Conclude from the previous part (12a) and (11) that SSA implies joint convexity of the relative entropy.

(c) Check that there are no loops in the above chains of reasoning.