University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 213 Fall 2023
Assignment 6 – Solutions

Due 11:00am Tuesday, February 21, 2023

1. **Copying classical information is OK.** Construct a *linear* operator $\mathcal{O}$ on a system of two qbits that acts as follows on the computational basis states of the first qbit:

$$\mathcal{O}\left|0\right\rangle \otimes \left|a\right\rangle = \left|0\right\rangle \otimes \left|0\right\rangle, \quad \mathcal{O}\left|1\right\rangle \otimes \left|a\right\rangle = \left|1\right\rangle \otimes \left|1\right\rangle,$$

for any computational-basis state $|a\rangle$ of the second qbit.

What is $\mathcal{O}\left(\cos\theta\left|0\right\rangle + \sin\theta\left|1\right\rangle\right) \otimes \left|a\right\rangle$?

$\mathcal{O}$ is uniquely specified to be

$$\mathcal{O} = \left|0\right\rangle\left\langle 0\right| \otimes \left|0\right\rangle\left(\left\langle 0\right| + \left\langle 1\right|\right) + \left|1\right\rangle\left\langle 1\right| \otimes \left|1\right\rangle\left(\left\langle 0\right| + \left\langle 1\right|\right).$$

$$\mathcal{O}\left(\cos\theta\left|0\right\rangle + \sin\theta\left|1\right\rangle\right) \otimes \left|a\right\rangle = \cos\theta\left|00\right\rangle + \sin\theta\left|11\right\rangle.$$

Can this operator $\mathcal{O}$ be unitary? Find a *unitary* operator $\mathcal{U}$ that acts as follows:

$$\mathcal{U}\left|0\right\rangle \otimes \left|0\right\rangle = \left|0\right\rangle \otimes \left|0\right\rangle, \quad \mathcal{U}\left|1\right\rangle \otimes \left|0\right\rangle = \left|1\right\rangle \otimes \left|1\right\rangle,$$

but acts in some other way when the second register is in the state $|1\rangle$.

[We can think of the operator U as copying *classical* information (onto a known register): if we are forced (*e.g.* by decoherence) to remain in the computational basis, the information in the first qbit is just a classical bit; the operator $\mathcal{O}$ copies this classical bit into the second register. This shows that the quantum no-cloning theorem does not forbid the cloning of classical information.]

$\mathcal{O}$ is not unitary, since it takes the unnormalized state state $|0\rangle \otimes |0\rangle + |1\rangle$ to the normalized state $|00\rangle$. If we only demand that the operator acts as a copier when the target bit is initialized to zero, we can make it unitary. For example:

$$\mathcal{U} = |00\rangle\langle 00| + |01\rangle\langle 11| + |01\rangle\langle 01| + |10\rangle\langle 11|. \tag{1}$$

More generally, the action on the subspace with target bit $= 1$ is completely unspecified, so we could do instead:

$$\mathcal{U}' = |00\rangle\langle 00| + |11\rangle\langle 01| + \cos\theta|01\rangle\langle 01| - \sin\theta|01\rangle\langle 11| + \sin\theta|10\rangle\langle 01| + \cos\theta|10\rangle\langle 11|. \tag{2}$$

As a matrix, this looks like:

	1	0	0	0
	0	Cos[θ]	0	-Sin[θ]
	0	Sin[θ]	0	Cos[θ]
	0	0	1	0

One nice example is the control-X gate, $\mathsf{CX} = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes X$.

Extra credit: show that it is not possible to construct such a linear operator that acts as above for *arbitrary* states $|a\rangle$ of the second qbit.

Since a linear operator is determined by its action on a basis, this would completely specify the operator to be the operator $\mathcal{O}$ above, which is not unitary.

2. **Brain-warmer: Entanglement cannot be created locally.** Consider a bipartite hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Define a *local unitary* to be an operator of the form $\mathbf{U}_A \otimes \mathbf{U}_B$ where $\mathbf{U}_{A,B}$ acts only on $\mathcal{H}_{A,B}$. These are the operations that can be done by actors with access only to A *or* B. Show that by acting on a state of $\mathcal{H}$ with a local unitary we cannot change the Schmidt number or the entanglement entropy of either factor. Consider both the case of a pure state of $\mathcal{H}$ and a mixed state of $\mathcal{H}$; note that the action of a unitary $\mathbf{U}$ on a density matrix $\boldsymbol{\rho}$ is
$$\boldsymbol{\rho} \to \mathbf{U}\boldsymbol{\rho}\mathbf{U}^\dagger.$$
[We conclude from this that to create an entangled state from an unentangled state, we must bring the two subsystems together and let them interact, resulting in a more general unitary evolution than a local unitary.]

The vN entropy $S(\rho_A)$ is invariant under changes of basis on A. The partial trace $\rho_A = \mathrm{tr}_B \rho_{AB}$ is invariant under changes of basis on B.

3. **Partial transpose is not completely positive.** Find the spectrum of the operator
$$\sum_{i,j=1}^{d} |ji\rangle\langle ij| \tag{3}$$
acting on two d-state systems. (This operator is the image of the maximally entangled state $\sum_{ij=1}^{d} |ii\rangle\langle jj|$ under the partial transpose operation $T \otimes \mathbb{1}_B$.)

For each of the $n(n-1)/2$ values of (i,j) with $i \neq j$ we get an eigenvector of the form

$$|ij\rangle - |ji\rangle\,, \quad \text{with eigenvalue } -1$$

and an eigenvector of the form

$$|ij\rangle + |ji\rangle\,, \quad \text{with eigenvalue } +1.$$

And for each $i = 1..d$ we get an eigenvector $|ii\rangle$ with eigenvalue $+1$. Altogether, $n(n-1)/2$ times -1 and the other $n(n-1)/2 + n = n^2 - n(n-1)/2$ $+1$.

The operator in (3) is the SWAP operator that interchanges the states of the two factors. Its eigenspaces are just the $d(d-1)/2$-dimensional antisymmetric subspace and the $d(d+1)/2$-dimensional symmetric subspace.

4. **Brainwarmers on Kraus operators.**

   (a) Check that the Kraus operators

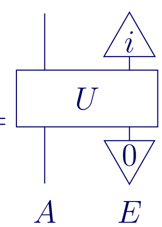
$$\mathcal{K}_i = \langle i|U|0\rangle$$

   (where U is a unitary on $A \otimes \bar{A}$, $\{|i\rangle\}$ is an ON basis of $\bar{A}$, and $|0\rangle$ is a reference state in $\bar{A}$) satisfy the condition

$$\sum_i \mathcal{K}_i^\dagger \mathcal{K}_i = \mathbb{1}_A \tag{4}$$

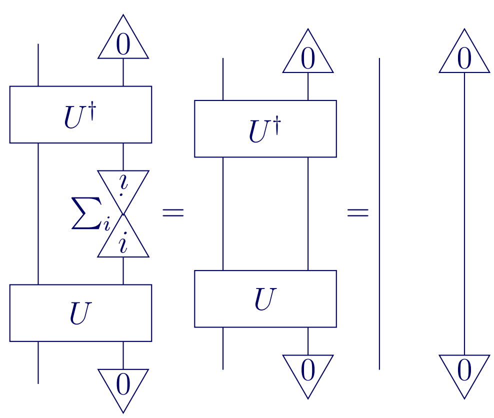
   A useful way to visualize the Kraus operators is with the following diagram:

$$\mathcal{K}_i = \tag{5}$$



   where the triangles indicate the states $|0\rangle$ and $|i\rangle$ on the $E = \bar{A}$ Hilbert space. This expression is very general and different channels just involve different choices of U. In this notation, the desired equation is:

$$\sum_i \mathcal{K}_i^\dagger \mathcal{K}_i =$$



$$= \mathbb{1}_A. \tag{6}$$

(b) Check that the condition (4) implies that the Kraus evolution $\boldsymbol{\rho} \to \sum_i \mathcal{K}_i \boldsymbol{\rho} \mathcal{K}_i^\dagger$ preserves the trace.

(c) Find a set of Kraus operators for the erasure (or reset) channel that takes $\boldsymbol{\rho} \mapsto |0\rangle\langle 0|$ for every $\boldsymbol{\rho}$. Check that they satisfy (4).

Choose an ON basis $\{|i\rangle\}$ of $\mathcal{H}$. Then $\mathcal{K}_i = |0\rangle\langle i|$ will work. $\sum_i \mathcal{K}_i^\dagger \mathcal{K}_i = \sum_i |i\rangle \langle 0|0\rangle \langle i| = 1\!\!1$. This is an extreme example where $\sum_i \mathcal{K}_i \mathcal{K}_i^\dagger = |0\rangle\langle 0|$ is very different.

(d) **Stationary states of unital channels.** Check that the unital condition $\sum_i \mathcal{K}_i \mathcal{K}_i^\dagger = 1\!\!1$ implies that the uniform density matrix $\mathbf{u} \equiv 1\!\!1 \frac{1}{|\mathcal{H}|}$ is a fixed point of the associated quantum channel, $\mathcal{E}$, *i.e.*

$$\mathcal{E}(\mathbf{u}) = \mathbf{u}.$$

$$\mathcal{E}(\mathbf{u}) = \sum_i \mathcal{K}_i \mathbf{u} \mathcal{K}_i^\dagger = \sum_i \mathcal{K}_i \mathcal{K}_i^\dagger \frac{1}{|\mathcal{H}|} = \frac{1\!\!1}{|\mathcal{H}|} = \mathbf{u}.$$

5. **Phase-damping channel.**

(a) Show that the Kraus operators given in lecture indeed reproduce the action of the phase-damping channel.

(b) Show that the Kraus operators given in lecture indeed result from the given unitary action U_{AE} on the combined system and environment.

6. **Amplitude-damping channel.** [from Preskill 3.4.3, Le Bellac §15.2.4] This is a very simple model for a two-level atom, coupled to an environment in the form of a (crude rendering of a) radiation field.

The atom has a groundstate $|0\rangle_A$; if it starts in this state, it stays in this state, and the radiation field stays in its groundstate $|0\rangle_E$ (zero photons). If the atom starts in the excited state $|1\rangle_A$, it has some probability p per time dt to return to the groundstate and emit a photon, exciting the environment into the state $|1\rangle_E$ (one photon). This is described by the time evolution

$$\mathbf{U}_{AE} |0\rangle_A \otimes |0\rangle_E = |0\rangle_A \otimes |0\rangle_E$$

$$\mathbf{U}_{AE} |1\rangle_A \otimes |0\rangle_E = \sqrt{1-p}\, |1\rangle_A \otimes |0\rangle_E + \sqrt{p}\, |0\rangle_A \otimes |1\rangle_E .$$

(a) Show that the evolution of the atom's density matrix can be written in terms of two Kraus operators $\mathcal{K}_i$, find those operators and show that they satisfy $\sum_i \mathcal{K}_i^\dagger \mathcal{K}_i = 1\!\!1_{\text{atom}}$.

$$\mathbf{M}_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \mathbf{M}_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}.$$

Probability is conserved because

$$\mathbf{M}_0^\dagger \mathbf{M}_0 + \mathbf{M}_1^\dagger \mathbf{M}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1-p \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix} = 1\!\!1.$$

So the density matrix evolves according to

$$\begin{aligned} \boldsymbol{\rho} \to \mathcal{K}(\boldsymbol{\rho}) &= \mathbf{M}_0 \boldsymbol{\rho} \mathbf{M}_0^\dagger + \mathbf{M}_1 \boldsymbol{\rho} \mathbf{M}_1^\dagger \\ &= \begin{pmatrix} \rho_{00} & \sqrt{1-p}\rho_{01} \\ \sqrt{1-p}\rho_{10} & (1-p)\rho_{11} \end{pmatrix} + \begin{pmatrix} p\rho_{11} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \rho_{00} + p\rho_{11} & \sqrt{1-p}\rho_{01} \\ \sqrt{1-p}\rho_{10} & (1-p)\rho_{11} \end{pmatrix} \end{aligned}$$

(b) Assuming that the environment is forgetful and resets to $|0\rangle_E$ after each time step dt, find the fate of the density matrix after time $t = ndt$ for late times $n \gg 1$, *i.e.* upon repeated application of the channel.

After n steps (in time $t = n \cdot dt$), the 11 matrix element has undergone $\rho_{11} \to (1-p)^n \rho_{11} = e^{-\gamma t}$, again exponential decay with rate $-\log(1-p)/dt \sim p/dt$ (for small p). Using $\rho_{00} + \rho_{11} = 1$, the whole matrix is:

$$\mathcal{K}^n(\boldsymbol{\rho}) = \begin{pmatrix} 1 + (1-p)^n \rho_{11} & (1-p)^{n/2}\rho_{01} \\ (1-p)^{n/2}\rho_{10} & (1-p)^n \rho_{11} \end{pmatrix} .$$

If you wait long enough, the atom ends up in its groundstate:

$$\lim_{n\to\infty} \mathcal{K}^n(\boldsymbol{\rho}) = \begin{pmatrix} \rho_{00} + \rho_{11} & 0 \\ 0 & 0 \end{pmatrix} = |0\rangle_A \langle 0|_A .$$

This example of open-system evolution takes a mixed initial state (say some incoherent sum of ground and excited state) to a (particular) pure final state. (Note that the off-diagonal elements (the 'coherences') decay at half the rate of ρ_{11} (the population of the excited state).

(c) Evaluate the *purity* $\text{tr}\boldsymbol{\rho}_n^2$ of the nth iterate. (Recall that the purity is 1 IFF the state is pure.)

7. **Phase-flipping decoherence channel.** [from Schumacher] Consider the following model of decoherence on an N-state Hilbert space, with basis $\{|k\rangle, k = 1..N\}$.

   Define the unitary operator

$$\mathbf{U}_\alpha \equiv \sum_k \alpha_k |k\rangle \langle k|$$

   where α_k is an N-component vector of signs, ± 1 – it flips the signs of some of the basis states. There are 2^N distinct such operators.

   Imagine that interactions with the environment act on any state of the system with the operator $\mathbf{U}_\alpha$, for some α, chosen randomly (with uniform probability from the 2^N choices).

   [Hint: If you wish, set $N = 2$.]

   (a) Warmup question: If the initial state is $|\psi\rangle$, what is the probability that the resulting output state is $\mathbf{U}_\alpha |\psi\rangle$?
   $\frac{1}{2^N}$.

   (b) Write an expression for the resulting density matrix, $\mathcal{D}(\boldsymbol{\rho})$, in terms of $\boldsymbol{\rho}$.
   The coupling to the environment takes any density matrix and replaces it with

$$\mathbf{U}_\alpha^\dagger \boldsymbol{\rho} \mathbf{U}_\alpha$$

   with probability $1/2^N$. Therefore, our 'superoperator' acts by

$$\boldsymbol{\rho} \to \mathcal{D}(\boldsymbol{\rho}) = 2^{-N} \sum_\alpha \mathbf{U}_\alpha^\dagger \boldsymbol{\rho} \mathbf{U}_\alpha.$$

   (c) Think of $\mathcal{D}$ as a superoperator, an operator on density matrices. How does $\mathcal{D}$ act on a density matrix which is diagonal in the given basis,

$$\boldsymbol{\rho}_{\text{diagonal}} = \sum_k p_k |k\rangle \langle k| \quad ?$$

   It leaves them alone, since $|k\rangle \langle k| \mapsto (\pm 1)^2 |k\rangle \langle k|$ no matter which $\mathbf{U}_\alpha$ we choose.

   (d) The most general initial density matrix is not diagonal in the k-basis:

$$\boldsymbol{\rho}_{\text{general}} = \sum_{kl} \rho_{kl} |k\rangle \langle l| \quad .$$

what does $\mathcal{D}$ do to the off-diagonal elements of the density matrix?

Averaging over α sets them to zero.

8. **Turtles all the way down.** [optional, open-ended]

   A question you may have about our discussion of polarization-damping as a model of decoherence is: why does the environment reset to the reference state $|0\rangle_E$?

   We can postpone the question a bit by coupling the environment to its own environment, according to an amplitude damping channel. On the previous problem set, you saw that the result of the repeated action of such a channel can set $\boldsymbol{\rho}_E = |0\rangle\langle 0|$. This statement in turn assumes a forgetful meta-environment. A thermodynamic limit is required to postpone the question indefinitely. Construct such a thermodynamic limit.