University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 213 Winter 2023
Assignment 5

Due 11:00am Monday, February 14, 2023

1. **Error rate per bit.** Estimate the probability of failing to decode a message sent through a binary symmetric channel with error rate q per bit, using the Hamming [7,4] code. Note that there is a distinction between the probability of having an error in the decoded string, and an error in a given bit of the message (it doesn't matter if some of the check bits are misconstrued).

2. **Control-X brainwarmer.**

   Show that the operator control-X can be written variously as

$$\mathsf{CX}_{BA} = |0\rangle\langle 0|_B \otimes 1\!\!1_A + |1\rangle\langle 1|_B \otimes \mathbf{X}_A = \mathbf{X}_A^{\frac{1}{2}(1-\mathbf{Z}_B)} = e^{\frac{\mathbf{i}\pi}{4}(1-\mathbf{Z}_B)(1-\mathbf{X}_A)}.$$

3. **Density matrix exercises.**

   (a) Show that the most general density matrix for a single qbit lies in the Bloch ball, *i.e.* is of the form

$$\boldsymbol{\rho}_v = \frac{1}{2}\left(1\!\!1 + \vec{v}\cdot\vec{\boldsymbol{\sigma}}\right), \quad \sum_i v_i^2 \le 1.$$

   Find the determinant, trace, and von Neumann entropy of $\boldsymbol{\rho}_v$.

   (b) A single qbit state has $\langle \mathbf{X} \rangle = s$. Find the most general forms for the corresponding density operator with the minimum and maximum von Neumann entropy. (Hint: the Bloch ball is your friend.)

   (c) Show that the *purity* of a density matrix $\pi[\boldsymbol{\rho}] \equiv \mathrm{tr}\boldsymbol{\rho}^2$ satisfies $\pi[\boldsymbol{\rho}] \le 1$ with saturation only if $\boldsymbol{\rho}$ is pure.

   (d) Show from the definition that the quantum relative entropy satisfies the following

$$D(\boldsymbol{\rho}_A \otimes \boldsymbol{\rho}_B || \boldsymbol{\sigma}_A \otimes \boldsymbol{\sigma}_B) = D(\boldsymbol{\rho}_A||\boldsymbol{\sigma}_A) + D(\boldsymbol{\rho}_B||\boldsymbol{\sigma}_B). \tag{1}$$

$$\sum_i p_i D\left(\boldsymbol{\sigma}_i||\boldsymbol{\rho}\right) = \sum_i p_i D(\boldsymbol{\sigma}_i||\boldsymbol{\sigma}_{\mathrm{av}}) + D(\boldsymbol{\sigma}_{\mathrm{av}}||\boldsymbol{\rho}) \tag{2}$$

$$D\left(\boldsymbol{\sigma}_{\mathrm{av}}||\boldsymbol{\rho}\right) \le \sum_i p_i D(\boldsymbol{\sigma}_i||\boldsymbol{\rho}) \tag{3}$$

   for any probability distribution $\{p_i\}$ and density matrices $\boldsymbol{\rho}, \boldsymbol{\sigma}_i$, and where $\boldsymbol{\sigma}_{\mathrm{av}} \equiv \sum_i p_i \boldsymbol{\sigma}_i$.

4. **Thermal density matrix.** Suppose given a Hamiltonian H. In lecture we showed that the thermal density matrix $\boldsymbol{\rho}_T \equiv \frac{e^{-\frac{H}{k_B T}}}{Z}$ has the maximum von Neumann entropy S_{vN} of any state with the same expected energy. Show that if instead we are given a fixed temperature T, the thermal density matrix minimizes the free energy functional

$$F_T[\boldsymbol{\rho}] \equiv \mathrm{tr}\boldsymbol{\rho}H - TS_{vN}[\boldsymbol{\rho}].$$

5. **Distinguishability of distributions.** Suppose we sample N times a distribution P on a binary variable with $(p_0, p_1) = (p, 1-p)$. What is the probability that we mistake the distribution for Q with probabilities $(q, 1-q)$?

   Hint: the expected number of zeros $\langle n_0 \rangle_P$ is Np. The probability that we get it wrong is the probability that we get Nq zeros instead. Show that

$$\mathrm{Prob}(n_0 = Nq|P) \simeq 2^{-ND(Q||P)}$$

   where $D(Q||P)$ is the relative entropy, and the approximation is Stirling's.