University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 213/113 Winter 2023
Assignment 1 – Solutions

Due 11:00am Tuesday, January 17, 2023

1. **Too many numbers.**

   Find the number of qbits the dimension of whose Hilbert space is the number of atoms in the Earth. (It's not very many.) Now imagining diagonalizing a Hamiltonian acting on this space.

$$N_{♁} \sim 5 \cdot 10^{24} kg \cdot (10^{27} \text{nucleons per kg}) \left(\sim 50 \text{ nucleons per atom}\right)^{-1} = 10^{50}$$

$$1\text{GeV}/c^2 = 10^9 \text{eV}/c^2 \sim 10^{-27} kg.$$

$$2^N = 10^{50} \implies N \sim 165.$$

2. **Warmup for the next problem.**

   Parametrize the general pure state of a qbit in terms of two real angles. A good way to do this is to find the eigenstates of

$$\boldsymbol{\sigma}^n \equiv \check{n} \cdot \vec{\boldsymbol{\sigma}} \equiv n_x \mathbf{X} + n_y \mathbf{Y} + n_z \mathbf{Z}$$

   where $\check{n}$ is a unit vector.

   Compute the expectation values of $\mathbf{X}$ and $\mathbf{Z}$ in this state, as a function of the angles θ, φ.

   $\check{n} \cdot \vec{\sigma}$ squares to one so has eigenvalues ± 1. It is related by a rotation to any one of the Pauli matrices, say σ^x. So we can find its eigenvectors by acting with a rotation on the eigenvectors of σ^x:

$$|+, \check{n}\rangle = R(\theta, \varphi) |\rightarrow\rangle, \quad \sigma^x |\rightarrow\rangle = |\rightarrow\rangle,$$

   and $R(\theta, \varphi) = e^{\mathbf{i}\varphi \frac{\sigma^x}{2}} e^{\mathbf{i}\theta \frac{\sigma^z}{2}}$. Alternatively, we can just diagonalize the matrix

$$\check{n} \cdot \vec{\sigma} = \begin{pmatrix} n_z & n_x - \mathbf{i} n_y \\ n_x + \mathbf{i} n_y & -n_z \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta e^{\mathbf{i}\varphi} \\ \sin\theta e^{-\mathbf{i}\varphi} & -\cos\theta \end{pmatrix}$$

(where I've chosen the polar axis to be $\check{z}$, so that $\check{n} = \cos\theta\cos\varphi\check{x} + \cos\theta\sin\varphi\check{y} + \sin\theta\check{z}$), which has eigenvectors (in the z-basis)

$$|+\check{n}\rangle = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2e^{-\mathbf{i}\varphi} \end{pmatrix}, |-\check{n}\rangle = \begin{pmatrix} \sin\theta/2 \\ -\cos\theta/2e^{-\mathbf{i}\varphi} \end{pmatrix}.$$

That is,

$$|+\check{n}\rangle = \cos\theta/2\,|\uparrow\rangle + e^{-\mathbf{i}\varphi}\sin\theta/2\,|\downarrow\rangle\,.$$

The expectations are

$$\langle\pm,\check{n}|\sigma^i|\pm,\check{n}\rangle = \pm n^i$$

so $\langle Z\rangle = \sin\theta$, $\langle X\rangle = \cos\theta\cos\varphi$.

3. **Mean field theory is product states.**

Consider a spin system on a lattice. More specifically, consider the transverse field Ising model:

$$\mathbf{H} = -J\left(\sum_{\langle x,y\rangle} Z_x Z_y + g\sum_x X_x\right).$$

Consider the mean field state:

$$|\psi_{\mathrm{MF}}\rangle = \otimes_x |\psi\rangle_x = \otimes_x \left(\sum_{s_x=\pm} \psi_{s_x} |s_x\rangle_x\right), \tag{1}$$

i.e., restrict to a product state where the state ψ of each spin is the same.

Write the variational energy for the mean field state, *i.e.* compute the expectation value of $\mathbf{H}$ in the state $|\psi_{\mathrm{MF}}\rangle$, $E(\theta,\varphi) \equiv \langle\psi_{\mathrm{MF}}|\,\mathbf{H}\,|\psi_{\mathrm{MF}}\rangle$.

Assuming s_x is independent of x, minimize $E(\theta,\varphi)$ for each value of the dimensionless parameter g. Find the groundstate magnetization $\langle\psi|Z_x|\psi\rangle$ in this approximation, as a function of g.

The idea of mean field theory is that we completely ignore entanglement between different sites, and suppose that the state is a product state

$$|MFT\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \cdots |\psi_j\rangle \cdots ...$$

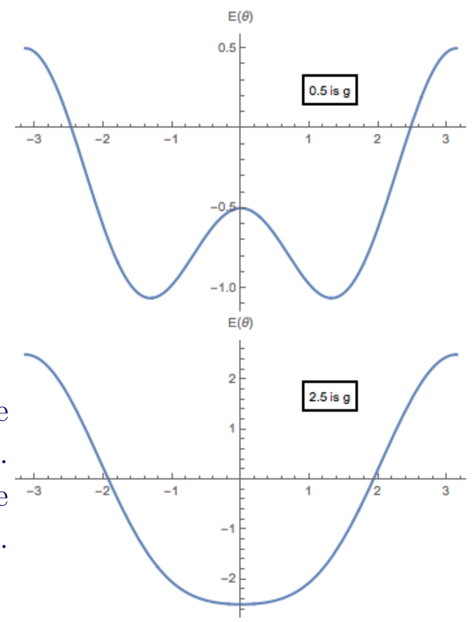
If we further assume translational invariance then the state at every site is the same and we have one bloch sphere to minimize over for each g:

$$|\check{n}\rangle = \otimes_j |\uparrow_{\check{n}}\rangle_j = \otimes_j \left(\cos\frac{\theta}{2}e^{\mathbf{i}\varphi/2}|\rightarrow\rangle + \sin\frac{\theta}{2}e^{-\mathbf{i}\varphi/2}|\leftarrow\rangle\right)_j.$$

Here θ is the angle $\check{n}$ makes with the x axis, and φ is the azimuthal angle in the yz plane, from the z-axis. Perhaps you might have instead chosen the z axis to be the polar axis, as I did in the previous problem. We should get the same answer for the physics. The solution is a little prettier in this basis.

To evaluate the energy expectation in this state, we only need to know single-qbit expectations:

$$\langle \uparrow_{\check{n}} | \, \mathbf{X} \, | \uparrow_{\check{n}} \rangle = \cos\theta, \qquad \langle \uparrow_{\check{n}} | \, \mathbf{Z} \, | \uparrow_{\check{n}} \rangle = \sin\theta\cos\varphi.$$



So the energy expectation is

$$\begin{aligned} E(\theta,\varphi) &\equiv \langle MFT | \, \mathbf{H} \, | MFT \rangle \\ &= -J\left(\sum_{\langle ij \rangle} \langle Z \rangle^2 + g \sum_i \langle X \rangle \right) \\ &= -NJ\left(k \sin^2\theta \cos^2\varphi + g\cos\theta \right) \end{aligned}$$

where N is the total number of sites and k is the number of links incident on each site of the lattice. In $d = 1$, we have $k = 1$, while for the square lattice we have $k = 2$. I've set $k = 1$ in the figures at right.

This is extremized when $\varphi = 0, \pi$ and when

$$0 = \partial_\theta E = NJ\sin\theta\left(2\cos\theta - g\right).$$

Notice that when $\theta = 0$, the two solutions of φ are the same, since the φ coordinate degenerates at the pole. The solutions at $\cos\theta = g/2$ only exist when $g/2 < 1$. In that case they are minima (see the figure) since $\partial_\theta^2 E|_{\cos\theta = g/2} > 0$, while $\partial_\theta^2 E|_{\theta=0} = NJ(g-2)$ is negative for $g < 2$. (Notice that $\varphi = \pi$ can be included by allowing $\theta \in (-\pi, \pi]$, as in the figure.)

So in $d = 1$, mean field theory predicts a phase transition at $g = 2$, from two states where $\langle \mathbf{Z}_j \rangle = \sin\theta = \pm\sqrt{1 - \frac{g^2}{.4}}$ to one where $\langle \mathbf{Z} \rangle = 0$. The actual transition is at $g = 1$, as we know from the solution using Jordan-Wigner (see *e.g.* my Whence QFT? notes); MFT overestimates the range of the ordered phase because it leaves out fluctuations which tend to destroy the order.

Let's study the behavior near the transition, where θ is small. Then the energy can be approximated by its Taylor expansion

$$E(\theta) \simeq NJ\left(-2 + \frac{g-2}{2}\theta^2 + \frac{1}{4}\theta^4\right)$$

(where I have set $g = g_c = 2$ except in the crucial quadratic term). This has minima at

$$\langle \mathbf{Z}_j \rangle = \sin\theta \simeq \theta = \pm\sqrt{g_c - g}\ . \tag{2}$$

The energy behaves like

$$E_{MFT}(g) = \begin{cases} \frac{3}{4}(g_c - g)^2\ , & g < g_c \\ \\ 0\ , & g \geq g_c \end{cases}$$

Notice that $\partial_g E$ is continuous at the transition. (Recall that the groundstate energy of the quantum system is equal to the free energy of the corresponding stat mech system, so $\partial_g E \propto \partial_T F$ continuous is the same criterion for a continuous transition.) So mean field theory (correctly) predicts a continuous quantum phase transition between the ordered phase and the disordered phase. The location of the transition is wrong (mean field theory overestimates the size of the ordered region because it leaves out lots of order-destroying fluctuations), and so are other properties, such as the exponent in (**??**), which should be 1/8 instead of 1/2.

4. **Classical versus quantum circuit sampling.** [This is an optional open-ended problem intended as food for thought.]

   We showed in lecture that the set of states reachable from a given state by polynomial-depth quantum circuits is a small fraction of the whole Hilbert space. This followed by close analogy with the statement that most boolean functions aren't computable using a polynomial number of gates. The closeness of this analogy leads to the following question:

   Let $P_C(s,t)$ be the probability of obtaining bit string s when starting with N uniform iid bits and feeding them through a classical circuit C made of t layers of 2-bit gates.

   Let
   $$P_U(s,t) = \left|\langle s^z = s| U \otimes_{i=1}^N |s^x = 1\rangle\right|^2$$
   where U is a quantum circuit made from t layers of neighboring 2-qbit gates. This is the probability distribution for measurements of σ_i^z on the state resulting from acting a quantum circuit U on a product of σ^x eigenstates.

Show that when $t = 0$ the distributions are the same.

Under some assumptions about the scaling of t with N, can we find a $P_U(s,t)$ that can never be a $P_C(s,t)$?

If we were allowed to measure in the X-basis as well as the Z-basis then it would be easy, because we could for example just design the circuit to produce at time t Bell pairs between spins $2n - 1$ and $2n$, and do exactly the Bell protocol on them.

Warning: I don't know the answer.

I still don't know the answer, but here are some references I know that discuss attempts to distinguish between classical and quantum by sampling problems:

Rather than thinking about qbits realized by spins, we could think about a Hilbert space spanned by states of photons moving through optical fibers. Then gates are made out of beamsplitters and mirrors. This paper by Aaronson and Arkhipov shows that sampling the output of such circuits of linear optical elements should be difficult for a classical computer, based on the known hardness of computing permanents of matrices.

This paper by Bravyi, Gosset and Koenig gives an example of a problem that no classical finite-depth circuit can solve (basically because of Bell inequalities), but which can be solved by a local quantum circuit of the same size. The answer to the computation arises by sampling P_U as defined above. Further recent progress in this direction appears here.

This paper (from around the same time as the boson sampling paper) by Bremner, Jozsa and Shepherd studies circuits where all the gates are diagonal in some basis (say x). The initial state and the measurements are done in the z basis. They show (somehow I haven't managed to extract) that being able to simulate the output of such a thing with a classical circuit would imply Bad Things for complexity theory.

Finally, the recent breakthrough by Google is just such a sampling problem. They create a *random* quantum circuit and sample from it in the z basis. Some of the theory about this is here.