University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 213/113 Winter 2023
Assignment 0.5 – Solutions

Due 12:30pm Tuesday, January 17, 2023

Here are some bonus problems for the benefit of those of you with limited prior experience with quantum mechanics. These problems are strictly optional, unless you find them difficult, in which case they are compulsory. If any of the notation is not clear please ask. You may find these notes helpful.

1. **Pauli spin matrix gymnastics.**

   (This problem is long but each part is pretty simple.) Recall the definition of the Pauli spin matrices:

$$\boldsymbol{\sigma}^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \boldsymbol{\sigma}^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \boldsymbol{\sigma}^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

   (Occasionally I will write $\boldsymbol{\sigma}^x \equiv \boldsymbol{\sigma}^1 \equiv \boldsymbol{X}, \boldsymbol{\sigma}^y \equiv \boldsymbol{\sigma}^2 \equiv \boldsymbol{Y}, \boldsymbol{\sigma}^z \equiv \boldsymbol{\sigma}^3 \equiv \boldsymbol{Z}$.)

   (a) Show that the $\boldsymbol{\sigma}^i$ are Hermitian.

   X and Z are real and symmetric and thus hermitian, while Y is imaginary and antisymmetric.

   (b) Check that the $\boldsymbol{\sigma}^i$ all square to the identity operator, $(\boldsymbol{\sigma}^i)^2 = \mathbb{1}, \forall i$.

   (c) Check that the $\boldsymbol{\sigma}$s are all traceless, $\text{tr}\boldsymbol{\sigma}^i = 0, \forall i$, where the trace operation is defined as $\text{tr}\,(M) \equiv \sum_a M_{aa}$.

   (d) Find their eigenvalues and eigenvectors.

   $Z \;:\; |0\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with eigenvalues ± 1 respectively.

   $X \;:\; |\pm\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ with eigenvalues ± 1 respectively.

   $Y \;:\; |y^\pm\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle \pm \mathbf{i}\,|1\rangle)$ eigenvalues ± 1 respectively.

   (e) There are only so many two-by-two matrices. A product of sigmas can be written in terms of sigmas. Show that

$$\boldsymbol{\sigma}^i \boldsymbol{\sigma}^j = i\epsilon^{ijk}\boldsymbol{\sigma}^k + \delta^{ij}\mathbb{1} \tag{1}$$

   where ϵ^{ijk} is the completely antisymmetric object with $\epsilon^{123} = 1$ (that is: $\epsilon^{ijk} = 0$ if any of ijk are the same, $= 1$ if ijk is a cyclic permutation of 123 and $= -1$ if ijk is an odd permutation of 123, like 132). You may prefer to do parts 1g and 1h of the problem first.

First note that $(\sigma^i)^2 = \mathbb{1}$ for any i. This is the source of the $\delta^{ij}\mathbb{1}$ term. The ϵ^{ijk} term will vanish if $i = j$ because by definition it is 0 on repeated indices. The $\mathbf{i}\epsilon^{ijk}\sigma^k$ piece one can find by exhaustively multiplying pairs, shown not here. You always get the third Pauli matrix with a factor of $\mathbf{i}$ and sign which depends on whether the permutation (the order of the product) is even or odd.

(f) There are only so many two-by-two hermitian matrices. Convince yourself that an arbitrary hermitian operator $\mathbf{A}$ acting on a two-dimensional Hilbert space can be decomposed as

$$\mathbf{A} = a_0\mathbb{1} + a_1\mathbf{X} + a_2\mathbf{Y} + a_3\mathbf{Z} \equiv a_\mu \boldsymbol{\sigma}^\mu$$

where $\boldsymbol{\sigma}^0 \equiv \mathbb{1}$ is the identity operator (which does nothing to everyone). Furthermore, show that the coefficients a_μ can be extracted by taking traces:

$$a^\mu = c\,\mathrm{tr}\left(\mathbf{A}\sigma^\mu\right)$$

for some constant c. Find c.

(g) Convince yourself that (1) implies

$$[\boldsymbol{\sigma}^i, \boldsymbol{\sigma}^j] = 2i\epsilon^{ijk}\boldsymbol{\sigma}^k,$$

(where $[A, B] \equiv AB - BA$ is the *commutator*) and that therefore $\mathbf{J}^i_{\frac{1}{2}} \equiv \frac{1}{2}\boldsymbol{\sigma}^i$ satisfy

$$[\mathbf{J}^i_{\frac{1}{2}}, \mathbf{J}^j_{\frac{1}{2}}] = i\epsilon^{ijk}\mathbf{J}^k_{\frac{1}{2}},$$

the same algebra as the rotation generators on the 3-state system in §1.5 of these lecture notes. [Cultural note: this is the Lie algebra called $\mathbf{su}(2)$.]

Use the above: $[\sigma^i, \sigma^j] = \sigma^i\sigma^j - \sigma^j\sigma^i = \mathbf{i}\epsilon^{ijk}\sigma^k - \mathbf{i}\epsilon^{jik}\sigma^k = \mathbf{i}\epsilon^{ijk}\sigma^k + \mathbf{i}\epsilon^{ijk}\sigma^k$

(h) Convince yourself that (1) implies

$$\{\boldsymbol{\sigma}^i, \boldsymbol{\sigma}^j\} = 2\delta^{ij}$$

where $\{A, B\} \equiv AB + BA$ is called the *anti-commutator*.

[Cultural note: this is called the Dirac algebra or Clifford algebra.]

It may be useful to note that $\{A, B\} + [A, B] = 2AB$.

$\{\sigma^i, \sigma^j\} = 2\sigma^i\sigma^j - [\sigma^i, \sigma^j] = (2\mathbf{i}\epsilon^{ijk}\sigma^k + 2\delta^{ij}) - 2\mathbf{i}\epsilon^{ijk}\sigma^k = 2\delta^{ij}$

(i) Convince yourself that (1) is the same as

$$(\boldsymbol{\vec{\sigma}} \cdot \vec{a})(\boldsymbol{\vec{\sigma}} \cdot \vec{b}) = \vec{a} \cdot \vec{b} + i\boldsymbol{\vec{\sigma}} \cdot \left(\vec{a} \times \vec{b}\right).$$

In particular, check that $(\boldsymbol{\vec{\sigma}} \cdot \hat{n})^2 = \mathbb{1}$ if $\hat{n}$ is a unit vector.

$(\vec{\sigma} \cdot \vec{a})(\vec{\sigma} \cdot \vec{b}) = \sum_{ij} a_i b_j \sigma^i \sigma^j$

$= \sum_{ij} a_i b_j (\mathbf{i}\epsilon^{ijk}\sigma^k + \delta^{ij}\mathbb{1}) = \sum_{ij} a_i b_j \delta^{ij} + \mathbf{i}\sum_{ij} \epsilon^{ijk} a_i b_j \sigma^k$

$= \vec{a} \cdot \vec{b} + \mathbf{i}\vec{\sigma} \cdot (\vec{a} \times \vec{b})$ as $(\vec{a} \times \vec{b})^k = \sum_{ij} \epsilon^{ijk} a_i b_j$

$(\vec{\sigma} \cdot \hat{n})^2 = \mathbb{1}$ is then simply a special case.

(j) Show that

$$e^{i\frac{\theta}{2}\boldsymbol{\vec{\sigma}}\cdot\hat{n}} = \mathbb{1}\cos\frac{\theta}{2} + i\boldsymbol{\vec{\sigma}} \cdot \hat{n}\sin\frac{\theta}{2}$$

where $\hat{n}$ is a unit vector.

[Hint: use the Taylor expansion of the LHS $e^A = 1 + A + A^2/2! + ...$ and the previous results for $(\boldsymbol{\vec{\sigma}} \cdot \hat{n})^2$.]

$e^{\mathbf{i}\frac{\theta}{2}\vec{\sigma}\cdot\hat{n}} = 1 + (\mathbf{i}\frac{\theta}{2}\vec{\sigma} \cdot \hat{n}) + \frac{1}{2}(\mathbf{i}\frac{\theta}{2}\vec{\sigma} \cdot \hat{n})^2 + \frac{1}{6}(\mathbf{i}\frac{\theta}{2}\vec{\sigma} \cdot \hat{n})^3 + \frac{1}{4!}(\mathbf{i}\frac{\theta}{2}\vec{\sigma} \cdot \hat{n})^4 \cdots$

$= (1 - \frac{1}{2}(\frac{\theta}{2})^2 + \frac{1}{4!}(\frac{\theta}{2})^4 + \cdots)\mathbb{1} + \mathbf{i}\vec{\sigma} \cdot \hat{n}(1 - \frac{1}{6}(\frac{\theta}{2})^3 + \cdots)$

$= \cos\frac{\theta}{2}\mathbb{1} + \mathbf{i}\vec{\sigma} \cdot \hat{n}\sin\frac{\theta}{2}$

2. **Dirac notation exercises.**

Dirac's notation for state vectors is extremely useful and we will use it all the time. The following problems are intended to test your understanding of the discussion on pages 1-6 – 1-10 of these notes.

(a) Consider some operators acting on a Hilbert space with a resolution of the identity of the form

$$\mathbb{1} = \sum_n |n\rangle \langle n| \ .$$

Recall that the matrix representation of an operator in this basis is $A_{nm} = \langle n| \hat{A} |m\rangle$. Using Dirac notation, show that the matrix representation of a product of operators $\left(\hat{A}\hat{B}\right)_{nr}$ is given by the matrix product of the associated matrices $\sum_m A_{nm}B_{mr}$.

$\left(\hat{A}\hat{B}\right)_{nr} = \langle n| AB |r\rangle = \langle n| A\mathbb{1}B |r\rangle = \sum_m \langle n| A |m\rangle \langle m| B |r\rangle = \sum_m A_{nm}B_{mr}$

(b) For a normalized state $|a\rangle$ (normalized means $\langle a|a\rangle = 1$), show that the operator

$$P_a \equiv |a\rangle\langle a|$$

is a *projector*, in the sense that $P_a^2 = P_a$ (doing it twice is the same as doing it once), and $P_a = P_a^\dagger$.

3. **Time evolution.** Suppose we have a two state system with Hilbert space spanned by $|0\rangle, |1\rangle$, the eigenbasis of $\mathbf{Z} \equiv |0\rangle\langle 0| - |1\rangle\langle 1|$. At time $t = 0$ we measure the observable $\mathbf{Z}$ and find the answer 1. The system evolves under the Hamiltonian

$$\mathbf{H} = g\mathbf{X}, \qquad \mathbf{X} \equiv |0\rangle\langle 1| + |1\rangle\langle 0|.$$

After time t, what is the probability that we will get 1 if we measure $\mathbf{Z}$ again?

By the axioms of QM, the initial state is $|\psi(0)\rangle = |0\rangle$, and the state at time t is

$$|\psi(t)\rangle = e^{-\mathbf{i}\mathbf{H}t}|0\rangle .$$

The eigenvectors of $\mathbf{X}$ are $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$. Expanding the initial state in this basis, we have

$$|\psi(t)\rangle = e^{-\mathbf{i}\mathbf{H}t}(|+\rangle + |-\rangle)\frac{1}{\sqrt{2}} = \left(e^{-\mathbf{i}gt}|+\rangle + e^{\mathbf{i}gt}|-\rangle\right)\frac{1}{\sqrt{2}}.$$

The amplitude to measure $\mathbf{Z} = +1$ in this state is

$$\langle 0|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-\mathbf{i}gt} + e^{\mathbf{i}gt}\right)$$

and so the probability is

$$P(t) = \cos gt.$$

This kind of oscillation is generally named after Rabi.

4. **Normal matrices.**

An operator (or matrix) $\hat{A}$ is *normal* if it satisfies the condition $[\hat{A}, \hat{A}^\dagger] = 0$.

(a) Show that real symmetric, hermitian, real orthogonal and unitary operators are normal.

Real symmetric is a special case of hermitian.

Let H be hermitian. $[H, H^\dagger] = [H, H] = 0$

Real orthogonal is a special case of unitary.

Let U be unitary. $[U, U^\dagger] = UU^\dagger - U^\dagger U = 1\!\!1 - 1\!\!1 = 0$

(b) Show that any operator can be written as $\hat{A} = \hat{H} + \mathbf{i}\hat{G}$ where $\hat{H}, \hat{G}$ are Hermitian. [Hint: consider the combinations $\hat{A} + \hat{A}^\dagger, \hat{A} - \hat{A}^\dagger$.] Show that $\hat{A}$ is normal if and only if $[\hat{H}, \hat{G}] = 0$.

Let $H = \frac{1}{2}(A + A^\dagger)$ and $G = \frac{1}{2i}(A - A^\dagger)$. By inspection H and G are hermitian.

The combination $H + iG = \frac{1}{2}(A + A^\dagger) + \frac{1}{2}(A - A^\dagger) = A$

$[A, A^\dagger] = [H + iG, H - iG] = [H, -iG] + [iG, H] = 2i[G, H]$ which is 0 iff $[H, G] = 0$

(c) Show that a normal operator $\hat{A}$ admits a spectral representation

$$\hat{A} = \sum_{i=1}^{N} \lambda_i \hat{P}_i$$

for a set of projectors $\hat{P}_i$, and complex numbers λ_i.

By the above if A is normal then $[H, G] = 0$ which allows us to simultaneously diagonalize them with the same set of projectors $\{P_j\}$. Denote their respective eigenvalues h_j and g_j.

$A = \sum_j (h_j + ig_j) P_j$

5. **Clock and shift operators.**

Consider an N-dimensional Hilbert space, with orthonormal basis $\{|n\rangle, n = 0, \ldots, N-1\}$. Consider operators $\mathbf{T}$ and $\mathbf{U}$ which act on this N-state system by

$$\mathbf{T}|n\rangle = |n+1\rangle, \qquad \mathbf{U}|n\rangle = e^{\frac{2\pi \mathbf{i} n}{N}}|n\rangle \ .$$

In the definition of $\mathbf{T}$, the label on the ket should be understood as its value modulo N, so $N + n \equiv n$ (like a clock).

(a) Find the matrix representations of $\mathbf{T}$ and $\mathbf{U}$ in the basis $\{|n\rangle\}$.

Define $\omega = e^{\frac{2\pi i}{N}}$. $\mathbf{T} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$ and $\mathbf{U} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \omega & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega^{N-1} \end{pmatrix}$

(b) What are the eigenvalues of $\mathbf{U}$? What are the eigenvalues of its adjoint, $\mathbf{U}^\dagger$?

$e^{\frac{2\pi \mathbf{i} n}{N}}$ and $e^{\frac{-2\pi \mathbf{i} n}{N}}$ respectively for $n \in \{0, \cdots, N-1\}$

(c) Show (using Dirac notation, not matrices) that

$$\mathbf{UT} = e^{\frac{2\pi \mathbf{i}}{N}} \mathbf{TU}.$$

$\mathbf{UT}|n\rangle = \mathbf{U}|n+1\rangle = e^{\frac{2\pi i(n+1)}{N}}|n+1\rangle$

$\mathbf{TU}|n\rangle = \mathbf{T}e^{\frac{2\pi i n}{N}}|n\rangle = e^{\frac{2\pi i n}{N}}|n+1\rangle$

Comparing the coefficients yields the result above.

(d) From the definition of adjoint, how does $\mathbf{T}^\dagger$ act, *i.e.*

$$\mathbf{T}^\dagger|n\rangle = ?$$

$T^\dagger|n\rangle = |n-1\rangle$

(e) Show that the 'clock operator' $\mathbf{T}$ is normal – that is, commutes with its adjoint – and therefore can be diagonalized by a unitary basis rotation.

Consider $[\mathbf{T}, \mathbf{T}^\dagger] |n\rangle = \mathbf{T}\mathbf{T}^\dagger |n\rangle - \mathbf{T}^\dagger\mathbf{T} |n\rangle = \mathbf{T} |n-1\rangle - \mathbf{T}^\dagger |n+1\rangle = 0$

(f) Find the eigenvalues and eigenvectors of $\mathbf{T}$.

[Hint: consider states of the form $|\theta\rangle \equiv \sum_n e^{\mathbf{i}n\theta} |n\rangle$.]

Consider $\mathbf{T} |\theta\rangle = \mathbf{T} |0\rangle + \mathbf{T}e^{i\theta} |1\rangle + \cdots + \mathbf{T}e^{i(N-1)\theta} |N-1\rangle$

$= |1\rangle + e^{i\theta} |2\rangle + \cdots + e^{i(N-1)\theta} |0\rangle = e^{-i\theta} |\theta\rangle$ where θ must be such that $e^{iN\theta} = 1$

The most general solution to $e^{iN\theta} = 1$ is for $\theta = \frac{2\pi j}{N}$ for $j \in \{0, \cdots, N-1\}$

This defines a basis of $|\omega^j\rangle \equiv \sum_n \omega^{j*n} |n\rangle$ where j runs from 0 to $N-1$.