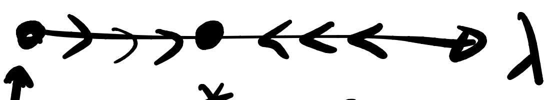
6.2 Renormalization of composite operators & Callan-Symanzik eqn.

Last time:

In $D = 4 - \epsilon < 4$ dims:



What is "ϕ^2" as a particular of WF fixed pt?

$$\mathcal{O}(k) = \text{"}\phi^2\text{"} \quad \text{def'd by}$$

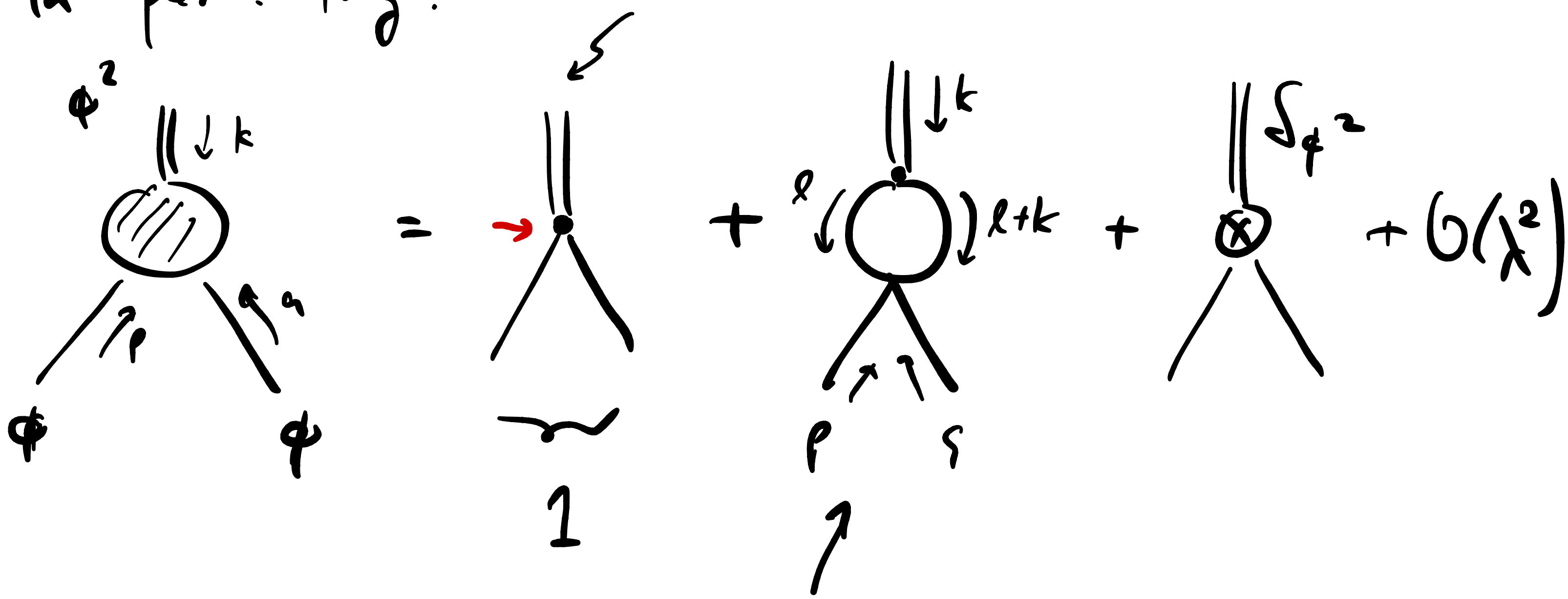
Renormalization condition:

$$\langle \mathcal{O}_\Lambda(k)\, \phi(p)\phi(q) \rangle_{\text{amputated}} \overset{!}{=} 1 \quad \text{at } p^2 = q^2 = k^2 = \underline{\underline{-\Lambda^2}}.$$

$$\mathcal{O}_\Lambda \equiv Z_G^{-1}(\Lambda)\, \mathcal{O}_0$$

$\uparrow$ bare ϕ^2.

In pert. thy:



$$= \frac{1}{2}(-\lambda)\int_0^\infty d^D\ell \; \frac{1}{\ell^2}\,\frac{1}{(k+\ell)^2}$$

$$= -\frac{c\,\lambda}{(k^2)^{\frac{4-D}{2}}} \qquad\qquad c = \frac{\Gamma(2-D/2)}{2\;16\pi^2}\left(1+O(\epsilon)\right)$$

$$Z_G^{-1}(\Lambda) - 1 = \delta_{\phi^2} \overset{R.C.}{=} \frac{\lambda c}{\Lambda^{4-D}}$$

$$G^{(2,1)} \equiv \langle \phi^2_\Lambda(k)\,\phi(p)\,\phi(q) \rangle$$

<u>Callan-Symanzik eqn</u>: physics (G_n) is indep. of our choice of Λ!!

$$G^{2,1} = \underbrace{Z_{\phi^2}^{-1}\sqrt{Z_\phi}\sqrt{Z_\phi}}_{\text{depend on }\Lambda} \langle \underline{\underline{(\phi^2)_0}}(k) \; \underset{\sim\sim\sim}{\phi_0(p)\,\phi_0(q)} \rangle$$

λ depends on Λ

$$0 \overset{!}{=} \Lambda \frac{d}{d\Lambda} G^{n,1} = \left(\Lambda \frac{\partial}{\partial\Lambda} + \beta_\lambda(\lambda)\frac{\partial}{\partial\lambda} + \frac{n}{2}\gamma_\phi - \gamma_G\right) \times G^{n,1}$$

w/
$$\begin{cases} \beta_\lambda(\lambda) = \Lambda \frac{\partial\lambda}{\partial\Lambda} \\ \gamma_G \equiv \Lambda \frac{\partial}{\partial\Lambda} \log Z_G(\Lambda) \\ \gamma_\phi \equiv \Lambda \frac{\partial}{\partial\Lambda} \log Z_\phi(\Lambda) \end{cases}$$

"anomalous dimension"

what is γ_{ϕ^2}?

$$G^{2,1} = \left(1 - \delta_{\phi^2} + 2\cdot\frac{\delta_\phi}{2} + \dots\right)\left(1 - \frac{c\lambda}{(k^2)^{\frac{4-D}{2}}} + \frac{c\lambda}{\Lambda^{(4-D)}} + \dots\right)$$

$$\gamma_{\phi^2} = \Lambda\frac{\partial}{\partial\Lambda}\log Z_{\phi^2} = -\Lambda\frac{\partial}{\partial\Lambda}\log(1+\delta_{\phi^2}) = -\Lambda\frac{\partial}{\partial\Lambda}(\delta_{\phi^2}+\dots)$$

$$= \Lambda\frac{\partial}{\partial\Lambda}\left(\frac{c\lambda}{\Lambda^{4-D}}\right) = (4-D)\,c\lambda\,\frac{1}{\Lambda^{4-D}}$$

$$\Longrightarrow \gamma_{\phi^2} = (4-D)\frac{\Gamma(2-\frac{D}{2})}{2\cdot 16\pi^2} \overset{D\to 4}{=} \frac{\lambda}{16\pi^2}.$$

"Anomalous dim." $+\mathcal{O}(\lambda^2)$

$$\mathcal{L} = \underbrace{\mathcal{L}_0}_{\text{fixed pt action}} + \Lambda^{d_g - D} g\,\mathcal{O}_\Lambda$$

$[g]=0.$

d_g = engineering dim of $\mathcal{O}$.

Eg: $d_g = D-2$ for $\mathcal{O}=\phi^2$

compute $G^n = \langle \phi_1 \cdots \phi_n \rangle_{\mathcal{L}}$

$$0 = \Big(\Lambda\partial_\Lambda + \beta_\lambda(\lambda)\partial_\lambda + \frac{n}{2}\gamma_\phi(\lambda)$$

$$+ \underbrace{(-\gamma_{\mathcal{O}} + d_g - 4)\, g}\underbrace{\partial_g}_{\#\text{ of powers of }\mathcal{O}}\Big)\, G^n$$

$$\equiv \beta_g = g(d_g - 4 - \gamma_{\mathcal{O}})$$

$$= \Big(\Lambda\partial_\Lambda + \sum_{I} \beta_I\, g_I + \frac{n}{2}\gamma_\phi(\lambda, g)\Big)\, G^n$$

I ↑ all couplings

Operator Mixing :

$$\mathcal{O}^I_\Lambda = (Z^{-1}(\Lambda))_{IJ}\ \mathcal{O}^J_0$$

if $\langle \mathcal{O}^I \mathcal{O}^J \rangle \neq 0$.

$$\gamma_{IJ} = -\Lambda \frac{\partial}{\partial \Lambda} \log (Z^{-1}(\Lambda))_{IJ}$$

Resolution : work in a basis of $\{\mathcal{O}^I\}$ w/ γ_{IJ} is diagonal.

Solution to Callan-Symanzik eqn :

is an ODE in G as a fn of Λ .

free theory $S[\phi] = \int \left(\frac{(\partial\phi)^2}{2} - \frac{m^2}{2}\phi^2 \right)$

$$G_n(\{sx_i\}, m^2) \equiv \langle \phi(sx_1) \cdots \phi(sx_n) \rangle_S$$

$$\overset{\text{dim'l analysis}}{=} s^{n\left(\frac{2-D}{2}\right)} \langle \phi'(x_1) \cdots \phi'(x_n) \rangle_{S'}$$

$x \equiv s x'$, $\phi(x) = s^{\frac{2-D}{2}} \phi'(x')$.

S' is the action w/ $m \to s m$.

$s > 1$ (toward IR) makes m more important.

w/ interactions, the Renormalized G satisfies

$$G^n(\{s x_i\}, \{g_I\}, \Lambda)$$

$$= s^{n\left(\frac{2-D}{2}\right)} G_n(\{x_i\}, \{s^{4-D_I} g_I\}, s\Lambda)$$

$$\{g_I\} = \{m^2, \lambda_4, \lambda_6 \dots\}$$

$$\{s^{4-D_I} g_I\} = \{s^2 m^2, \lambda_4, s^{-2}\lambda_6 \dots\}.$$

C-S eqn : $(\Lambda \partial_\Lambda + \beta_I \partial_I + n\gamma_\phi) G_n = 0$

In terms of the running coupling

$$\Lambda \partial_\Lambda g_I(\Lambda) = \beta_I(g_I(\Lambda)),$$

The sol'n of C-S eqn is:

$$G_n(\{x\},\{g_I(\Lambda_1)\},\Lambda_1) =$$

$$e^{-n\int_{\Lambda_1}^{\Lambda_2}\gamma_\phi(\Lambda')\,d\log\Lambda'}\, G_n(\{x\},\{g_I(\Lambda_2)\},\Lambda_2)$$

Combine w/ D.A.:

$\Lambda = \Lambda_1$

$$\Rightarrow G_n(\{sx\},\{g_I(\Lambda)\},\Lambda)$$

$$\overset{CS}{=} e^{-n\int_{\Lambda}^{\Lambda_2}\gamma_\phi(\Lambda')\,d\log\Lambda'}\, G_n(\{sx\},\{g_I(\Lambda_2)\},\Lambda_2)$$

$$\overset{D.A.}{=} s^{n\left[\left(\frac{2-D}{2}\right)-\int_{\Lambda}^{\Lambda/s}\gamma_\phi(\Lambda')\,d\log\Lambda'\right]} \times$$

$$G_n(\{x\},g_I(\Lambda/s),\Lambda)$$

(set $\Lambda_2 = \Lambda/s$.)

$$G_n(\{sx\}, \{g_I(\Lambda)\}, \Lambda) =$$

$$s^{n\left[\left(\frac{2-D}{2}\right) - \int_{\Lambda}^{\Lambda/s} \gamma_\phi(\Lambda')\, d\log\Lambda'\right]} \times$$

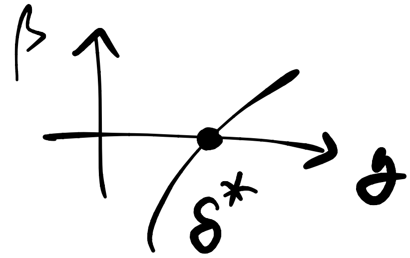
$$G_n(\{x\}, g_I(\Lambda/s), \Lambda)$$

Result of scaling $x \to sx$ is

1. rescaling by engineering dims
2. " " anomalous dims
3. running of the couplings

$$\Lambda \frac{\partial}{\partial\Lambda} g_I(\Lambda) = \beta_I(g_I(\Lambda))$$

eg:



sit at a fixed pt
$g = g^*$
all other $g_I = 0$.

$$e^{-\int_{\Lambda_1}^{\Lambda_2} \gamma_\phi(\Lambda')\, d\log\Lambda'} = \left(\frac{\Lambda_2}{\Lambda_1}\right)^{-\gamma^*}$$

$$\gamma^* = \gamma_\phi(g^*)$$

$$\Longrightarrow G_n(\{s x_i\}, g^*, \Lambda) = s^{n\left(\frac{2-D}{2} + \gamma^*\right)} \times G_n(\{x_i\}, g^*, \Lambda)$$

7 Effective field Theory (EFT)

Wilson's perspective: include all terms consistent w/ symmetries

+ let the RG decide which are relevant.

- explanation for 'renormalizability' of SM & QED
- an EFT comes w/ a cutoff $\overset{E_{new}}{\Lambda}$ which
  - says its regime of validity
  - provides an expansion parameter $\frac{E}{E_{new}}$.

  eg: parable w/ 2 oscillators : $\underline{\underline{\frac{\omega}{\Omega}}}$.
  $\omega_0, \underset{\substack{\uparrow \\ E_{new}}}{\Omega}$

a useful EFT ('renormalizable EFT')
- needs a finite # of counterterms at each order in $\frac{E}{E_{new}}$.

precision $\Delta = \left(\frac{E}{E_{new}}\right)^n$

Notions of EFT: ① we know the UV.
but it's hard.

$\boxed{UV}$		$\boxed{IR}$
• electroweak gauge theory	$\longrightarrow$	QED OR 4-Fermi theory
• QCD	$\longrightarrow$	chiral Lagrangian of pions OR HQET OR SCET or hydro of QGP.
• electrons in a solid w/ Coulomb interactions	$\longrightarrow$	Landau Fermi Liquid Thy or Hubbard Model OR topological field theory OR ...
• water molecules	$\longrightarrow$	hydrodynamics OR thy of phonons in ice.

↑ COARSE-GRAINING.

(2) sometimes we don't know the UV theory.

eg: • (Beyond the) SM physics

• quantum gravity.

WORKSHEET for doing EFT:

* (1) what are the dofs? ______

(2) what are the symmetries? ______

(3) what is the cutoff, Λ,
or its validity? ______

Then write down all interactions between the dof that preserve the symmetries in an expansion in derivatives,
w/ higher-order terms suppressed by $\frac{1}{\Lambda^{\#}}$.

Comments:

- EFT vs doing the integrals

  $\xleftarrow{\text{taylor expand in } \frac{1}{\Lambda}}$ to get a local theory

- writing all the terms : Landau-Ginzburg-Wilson.

- G_{UV} = symmetries of the UV theory.

  vs

  G_{IR} = " " " IR theory.

  — can be emergent symms. = ops that are charged are irrelevant.

  $G_{IR} > G_{UV}$.

  — maybe $H < G_{UV}$ doesn't act on IR dofs.

  eg: no dofs.

7.2 example 1: 4-Fermi theory.

$$\mathcal{L}_{EW} \ni -\frac{1}{2}(\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+)(\partial^\mu W^{-\nu} - \partial^\nu W^{-\mu})$$

$(W^+) = (W^-)^\dagger$

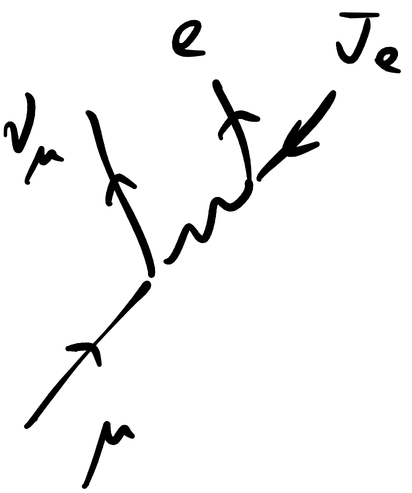
electric charge

$$+ M_W^2 W_\mu^+ W^{-\mu}$$

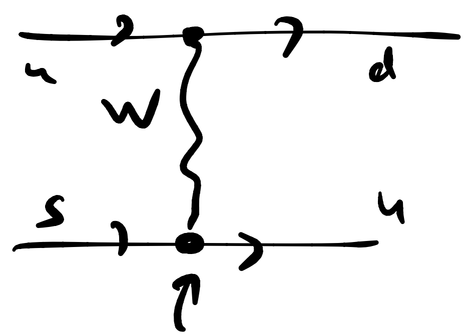
$$-\frac{ig}{\sqrt{2}} \bar{\Psi}_i \gamma^\mu P_L \Psi_i W_\mu^+ \underline{\underline{V_{ij}}} + h.c.$$

+ terms w Z bosons.

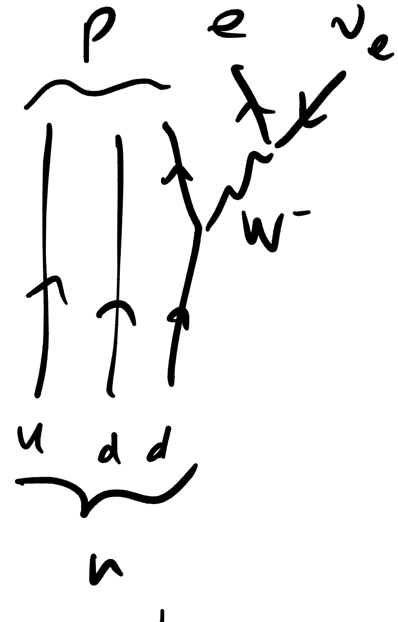
$$P_L = \frac{1}{2}(1-\gamma^5)$$



muon decay



V = CKM matrix



neutron decay.

If we want $E < M_W \sim 10^2\ \mathrm{GeV}$
$\equiv \Lambda$

$$\delta S_{eff} \sim \left(\frac{ig}{\sqrt{2}}\right)^2 V_{ij} V^*_{kl} \int d^D p \, \frac{-i g_{\mu\nu}}{p^2 - M_W^2} \times (\bar{\psi}_i \gamma^\mu P_L \psi_j)(p) \, (\bar{\psi}_k \gamma^\nu P_L \psi_l)(-p)$$

$$\frac{1}{p^2 - M_W^2} \overset{p^2 \ll M_W^2}{\simeq} -\frac{1}{M_W^2}\left(1 + \frac{p^2}{M_W^2} + \frac{p^4}{M_W^4} + \dots\right)$$

derivatives on ψ's.

$$S_F = -\frac{4 G_F}{\sqrt{2}} V_{ij} V^*_{kl} \int d^4 x \, (\bar{\psi}_i \gamma^\mu P_L \psi_j)(x) \, (\bar{\psi}_k \gamma_\mu P_L \psi_l)(x) + \mathcal{O}\left(\frac{1}{M_W^2}\right) + \text{kinetic terms for } \psi$$

$$V \propto T^a_{ij}\, \bar{T}^a_{\bar{k}\bar{l}}$$

$q \qquad \bar{q}$

$\in \quad 3 \otimes \bar{3} = 1 \oplus 8$

$(M)^{i\bar{k}}_{j\bar{l}}$

$$= \left(T^1\right)_{ijkl} + \left(T^8\right)_{ijkl}$$