4.7 Lattice Gauge theory, cont'd

Last time: strong coupling expansion

$$\langle W(C)\rangle = \int \prod_e dU_e \left(\boxed{\;\;}_C \right) e^{\beta(\square + h.c.)} \sim \beta^{\mathrm{Area}(C)} \left(1+\mathcal{O}(\beta^1)\right)$$

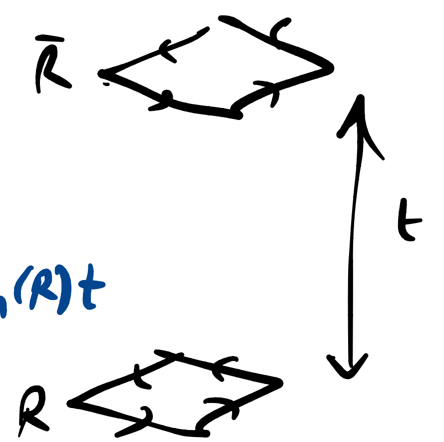
small β

$$\int dU\, U^n = \delta_{n,0} \qquad = e^{-\sigma(\beta)\mathrm{Area}}$$

Area law

$\sigma(\beta) \sim |\ln \beta|$.

How to study the spectrum of hadrons using LGT?



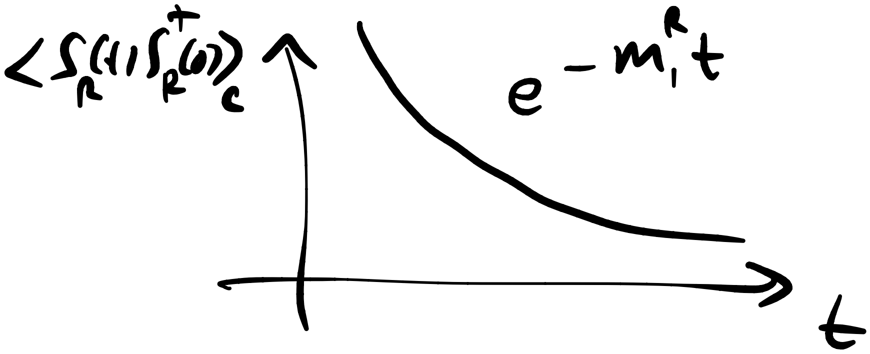
$$\langle S_R(t)\, \Lambda\, S_R^\dagger(0)\rangle_c = \sum_n |c_n^R|^2 e^{-m_n(R)t}$$

$$\mathbb{1} = \sum_n |n\rangle\langle n|$$

$$S_R(t) = e^{Ht} S_R(0) \underline{\underline{e^{-Ht}}}$$

m_n = evals of $\mathcal{H}$

$$c_n^R = \langle n | S_R(0) | \text{vac} \rangle .$$



"glueballs"

(same idea applies $S_R \rightsquigarrow q^\dagger_x U_{xy} q_y$

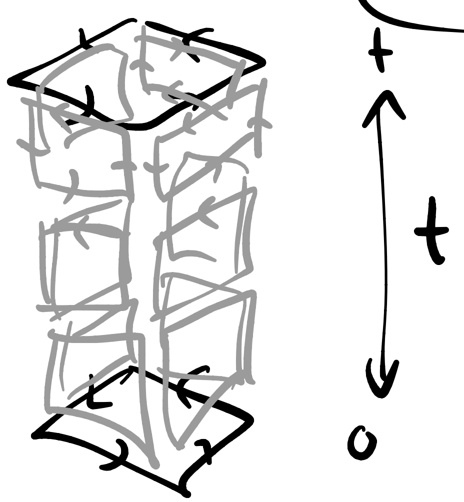
$\longrightarrow$ "mesons"

OR $\rightsquigarrow \sum_{ijk}^{N} q_i \, q_j \, q_k \, \epsilon^{ijk}$

for $N=3$

$\longrightarrow$ "baryons")

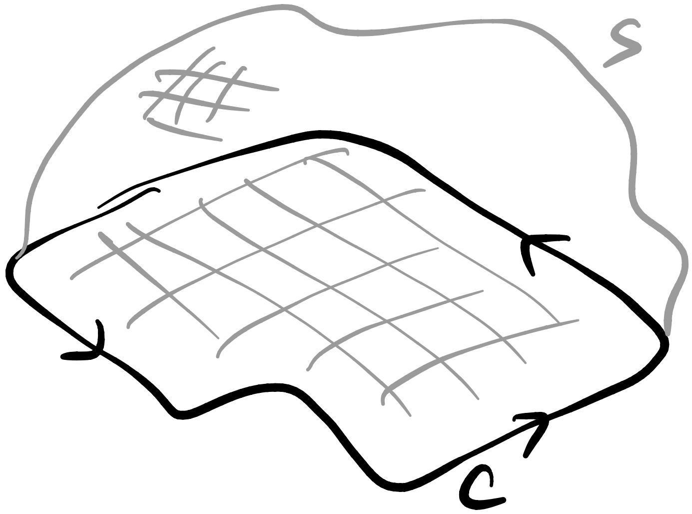
Compute in strong coupling expansion



$$\langle S(t) S(0) \rangle_c \sim \beta^{\text{area of tube}} \left(\dots \right)$$

$$= \beta^{4t} \sim e^{-m_1 t}$$

$$m_1 \sim 4 |\ln \beta| \sim \sigma(\beta).$$



$$\langle W \rangle = \sum_{\substack{\text{surfaces } S \\ \text{w/ } \partial S = C}} \beta^{\text{Area}(S)}$$

has a phase trans at β_c

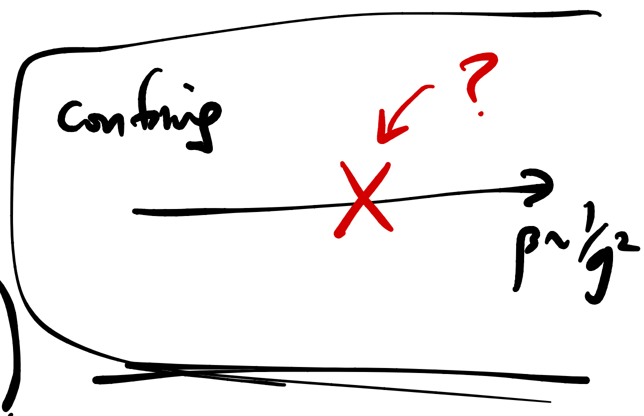
$\beta < \beta_c$: minimal area dominates.

for $\underline{\beta > \beta_c}$: big floppy surfaces dominate

$\beta = \beta_c$

"roughening trans."

<u>weird:</u> not a thermodynamic transition of full YM theory ($\ln Z$ is smooth)



<u>Hint 2</u> : Monopole condensation & dual Meissner effect.

e.m. duality : $F_{\mu\nu} \leftrightarrow \tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$ ie $F \leftrightarrow *F$.

$j_e \leftrightarrow j_m$.

[$dF = *j_m$ $\quad d*F = *j_e$. inv't.]

Solve $dF = *j_m$: (abelian)

$$F_{\mu\nu}(x) = \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\int d^4y\, j_m^\sigma(y) f^\rho(x-y)$$

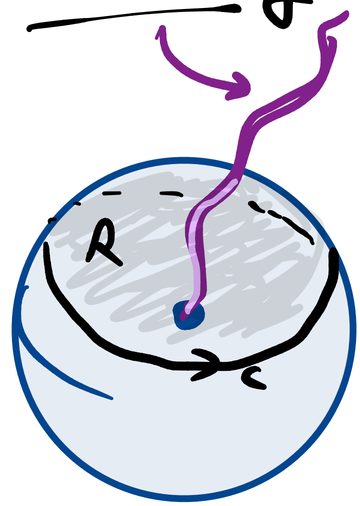
$$\text{w} \quad \partial_\rho f^\rho = \delta^4(x).$$

The support of $f^\rho(x) \equiv$ Dirac string

charged particle along C:

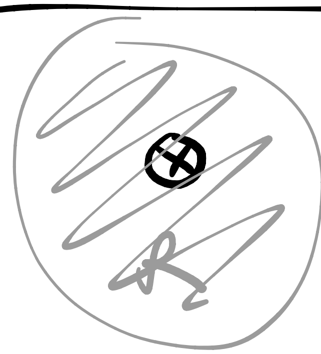
$$e^{ie\oint_{C=\partial R} A} = e^{ie\int_R F}$$

$$= e^{ieg}$$



undetectable if $eg \in 2\pi\mathbb{Z}$. (Dirac quantization condition.)

view from above:



$$\int_R B = g \neq 0$$

$$= \oint_{\partial R} A$$

$\langle \Phi_e \rangle = v \neq 0$

$\Rightarrow$ A is massive

monopoles confined

duality $\updownarrow$

$\langle \Phi_m \rangle = V \neq 0$

$\Rightarrow$ e confined.

5 Non-abelian gauge fields in perturbation theory

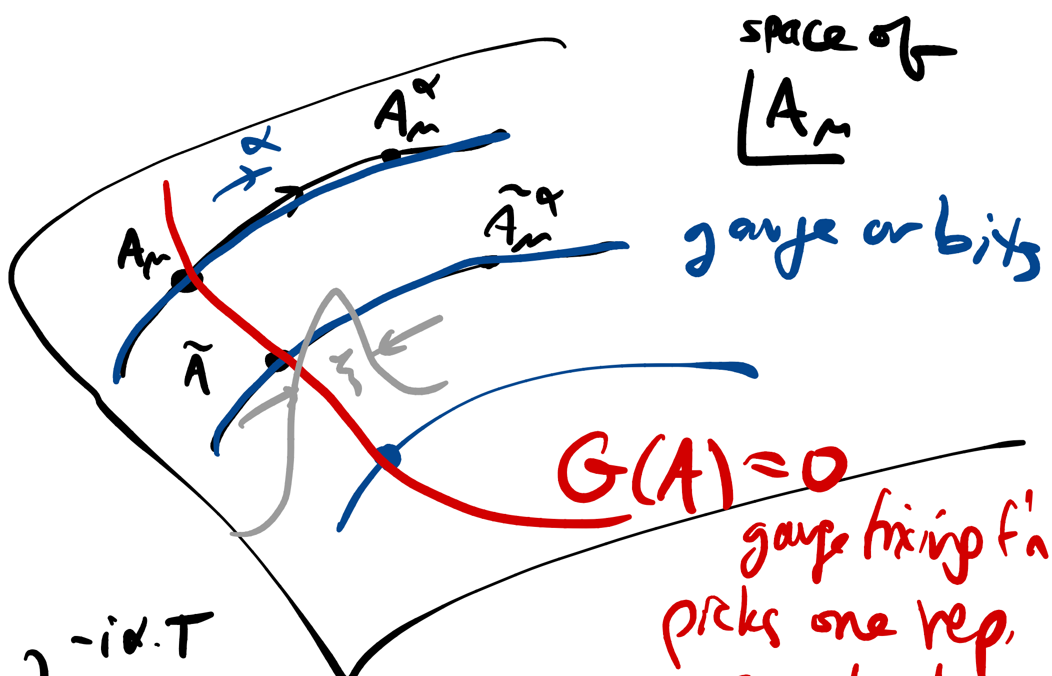
5.1 Gauge-Fixing & Feynman Rules

$$Z_{\substack{\text{pure}\\ \text{YM}}} \equiv \frac{1}{\mathrm{vol}(\mathcal{G})} \int [DA]\, e^{iS[A]}$$

← all gauge field configs

← " $\mathcal{G} = \prod_{x \in \text{spacetime}} G$ "

$$= \frac{\infty}{\infty}.$$



$A_\mu^\alpha \equiv$ image of A_μ under gauge transf α

$$= (A_\mu^\alpha)^a T^a$$

$$= e^{i\alpha\cdot T}\left(A_\mu + \frac{i}{g}\partial_\mu\right)e^{-i\alpha\cdot T}$$

$$\overset{\alpha\text{ small}}{=} \Big(A_\mu^a + \underbrace{\frac{1}{g}\partial_\mu\alpha^a + f^{abc}A_\mu^b\alpha^c}_{\equiv \frac{1}{g}D_\mu\alpha^a} \Big)T^a + \mathcal{O}(\alpha^2) = \left(A_\mu^a + \frac{1}{g}D_\mu\alpha^a\right)T^a + \ldots$$

Note: $G(A(x))$ is not gauge inv't !!

like $\partial^\mu A_\mu(x) = G(A(x))$ local.

$$1 = \Delta[A] \int [D\alpha]\, \delta[\, G(\underline{\underline{A^\alpha}})\,]$$

(def of $\Delta \equiv$ FP determinant)

Haar measure on $\mathcal{G}$

calculus $\Rightarrow$ $\Delta[A] = \det\left(\dfrac{\delta G(A^\alpha)}{\delta \alpha}\right)$

Δ is gauge inv't :

$$\Delta[A^{\alpha'}]^{-1} \overset{\text{def}}{=} \int [D\alpha]\, \delta\left[G(A^{\alpha+\alpha'})\right]$$

$$\overset{\text{Haar}}{=} \int [D(\alpha+\alpha')]\, \delta[\, G(A^{\alpha+\alpha'})]$$

$$= \Delta[A]^{-1}.$$

$$Z = \frac{1}{\mathrm{vol}\,\mathcal{G}} \int [D\alpha] \int [DA]\, \delta[G(A^\alpha)]\, \Delta[A]\, e^{iS[A]}$$

$\int[DA] = \int[DA^\alpha]$

$\Delta[A] = \Delta[A^\alpha]$

$S[A] = S[A]$

$$= \frac{1}{\mathrm{vol}\,\mathcal{G}} \int [D\alpha] \int [DA^\alpha]\, \delta[G(A^\alpha)]\, \Delta[A^\alpha]\, e^{iS[A^\alpha]}$$

$$\tilde{A} \equiv A^\alpha \quad = \cancelto{1}{\frac{\int [D\alpha]}{\mathrm{vol}\,\mathcal{G}}} \int [D\tilde{A}]\, \delta[G(\tilde{A})]\, \Delta[\tilde{A}]\, e^{iS[\tilde{A}]}$$

$$= \int [DA]\, \delta[G(A)]\, \underline{\underline{\Delta(A)}}\, e^{iS[A]}.$$

$\underline{\text{Let's choose}}$: $G(A) = \partial^\mu A^a_\mu(x) - \underline{\underline{\omega^a(x)}}$

$$1 = N(\xi) \int [D\omega]\, e^{-i \int d^Dx \, \frac{\omega^2(x)}{2\xi}}$$

$$\Rightarrow Z = N(\xi) \int [DA]\, \overbrace{\int [D\omega]\, \delta[\partial\cdot A - \omega]}^{=1}\, e^{-i\int \frac{\omega^2}{2\xi}}\, \Delta[A]\, e^{iS[A]}$$

$$= N(\xi) \int [DA]\, \Delta[A]\, e^{i\left(S[A] - \int \frac{(\partial\cdot A)^2}{2\xi}\right)}$$

$$\Delta(A) = \det\left(\frac{\delta G[A^\alpha]}{\delta\alpha}\right)$$

Cov. deriv in adjoint rep

$$\frac{\delta G[A^\alpha]}{\delta\alpha} = \frac{1}{g}\partial^\mu \underline{D_\mu}$$

Recall: $A^\alpha = \left(A^a_\mu + \frac{1}{g} D_\mu \alpha^a\right)T^a + \mathcal{O}(\alpha^2)$

$$G[A] = \partial^\mu A^a_\mu - \omega^a$$

$$D_\mu \alpha^a = \left(\delta^{ac}\partial_\mu + \underbrace{g f^{abc} A^b}_{=0 \text{ in abelian case}}\right)\alpha^c$$

Abelian case: $\frac{\delta G}{\delta\alpha} = \frac{1}{g}\partial^\mu\partial_\mu \longrightarrow \partial_\mu\alpha^a$

indep of A

$$\Longrightarrow \Delta[A] = \text{const.}$$

$$\Delta[A] = \det\left(\frac{1}{g}\partial^\mu D_\mu\right) = \int [Dc\, D\bar{c}]\, e^{i\int d^Dx\, \bar{c}(-\partial^\mu D_\mu)c}$$

Grassman vars.

$c \neq \bar{c}$ is a complex **SCALAR** field
$\;\;= c^\dagger$

in the adjoint rep of G

$c^a T^a$.

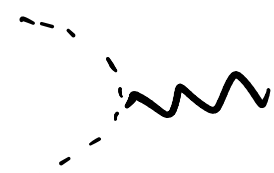
$\Longrightarrow$ it's a **ghost**.

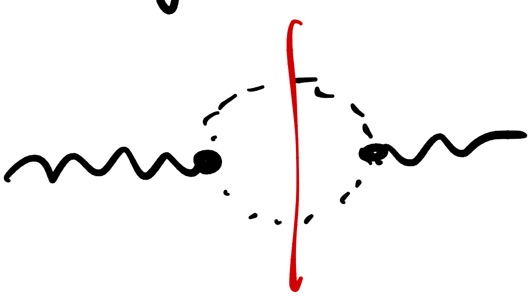
- it's a **fermion**!

violates spin-statistics rule ($\Longrightarrow$ ~~Lorentz~~ OR ~~unitarity~~)

hope: S-matrices w/o external ghosts can be unitary.

(and we'll never make them.)

need ghosts in loops $\xrightarrow{\text{optical thm}}$ $P \sim \sum_f \Big|$  $\Big|^2 \leq 0$



final state w/ ghosts?

ghosts carry negative probability
to cancel unphysical longitudinal pols.
of A_μ.

$$Z = \int [\underline{DA\, Dc\, D\bar{c}}]\; e^{i\left(S[A] - \int \frac{(\partial\cdot A)^2}{2\xi} + \int \bar{c}(-\partial D)c\right)}$$

eg:

for $g=0$

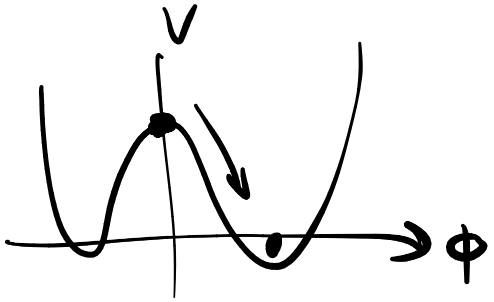
$$Z(g=0) = \left(\underbrace{\det(-\partial^2)^{-D}}_{\text{from } A_\mu} \; \underbrace{\det(-\partial^2)^{+2}}_{\text{from } c,\bar{c}} \right)^{\frac{\dim G}{2}}$$

$$= \left(\det(-\partial^2)^{2-D} \right)^{\frac{\dim G}{2}}$$

"negative dofs".

A_μ	$c,\bar{c}$	$\partial_\mu\varphi$	
4	-2	$+1$	$= 3$

massive vector.



"hope": big machinery to prove it:

BRST ~~symmetry~~ invariance

of $Z = \int [DA\, Dc\, D\bar{c}] \cdots$

$\Longrightarrow$ ① don't make ghosts

$\Rightarrow S$ is unitary

② we don't generate

$\delta S = \int \bar{c} c$.

Feynman Rules: pert thy in g.

$$\mathcal{L}_{ghost} = \bar{c}^a \big(\underbrace{-\partial^2 \delta^{ab}}_{\text{kin.}} + \underbrace{g \overleftarrow{\partial}^\mu f^{abc} A^c_\mu}_{\text{int.}} \big) c^b$$

$$\langle c^a(x) \bar{c}^b(y) \rangle = \delta^{ab} \int d^4k \, e^{-ik(x-y)} \frac{i}{k^2 + i\epsilon}$$

$$S[A] = \int \mathcal{L}_{YM}$$

$$\mathcal{L}_{YM} = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu.$$

$$\xrightarrow[+\frac{(\partial\cdot A)^2}{2\xi}]{\text{quadratic bit of } \mathcal{L}_{YM}}$$

$$\langle A^a_\mu(x)\, A^b_\nu(y)\rangle = \int d^4k\, e^{-ik(x-y)} \cdot \frac{-i}{k^2+i\epsilon}\, \delta^{ab} \left(\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)$$

Physics ~~Nothing~~ can't depend on ξ.