4.7 Lattice Gauge Theory

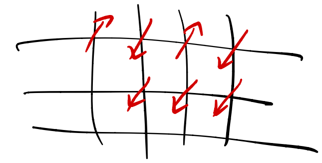
QFT can emerge from lattice models

eg: scalar field theory ⟵ 

OR

spin system

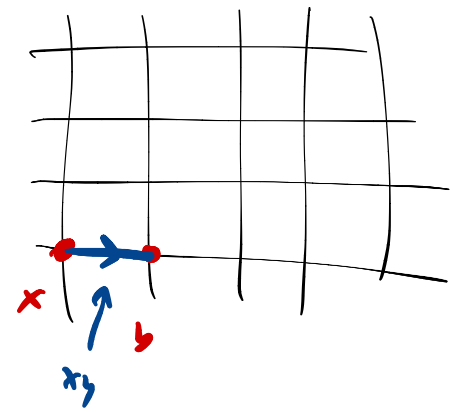
$$Z = \underset{\{\downarrow\uparrow\}}{\mathrm{tr}}\, e^{-\beta H}$$



This is a non-perturbative, completely unitary regulator which moreover can be simulated efficiently classically.

Further virtue: allows strong-coupling expansion. (small β)

① Discretize euclidean spacetime.



② Place on each link xy of the lattice a G-valued matrix $U_{xy}^{\alpha\beta}$

$\alpha,\beta = 1 \ldots \dim R$ ← a unitary rep of G.

③ define $U_{yx} \equiv U_{xy}^{-1}$.

examples: ① $G = U(N)$

U is $N\times N$ matrix w/ $U^\dagger U = \mathbb{1}$.

$\alpha, \beta = 1 \ldots N$

② $G = U(1)$

$$U_{xy} = e^{i\theta_{xy}} \qquad \theta_{xy} \in [0, 2\pi)$$

$$\theta_{yx} = -\theta_{xy}.$$

③ $G = \mathbb{Z}_n$

$$U = e^{2\pi i \ell/n} \qquad \ell = 1 \ldots n$$

a phase w/ $U^n = \mathbb{1}$.

eg for $n=2$ this is just a classical on each link.

Think of $U_{xy} = $ " $\mathcal{P} e^{i\int_x^y A_\mu(r) dr^\mu}$ ".

Impose gauge redundancy:

$$U_{xy} \longmapsto g_x^\dagger U_{xy} g_y .$$

$g_{x,y} \in G$ for each x, y.

- choose $S[U]$ which is gauge invariant
- integrate over U with an invariant measure.

$$Z = \int \prod_\ell \underline{\underline{dU_\ell}} \, e^{-S[U]}$$

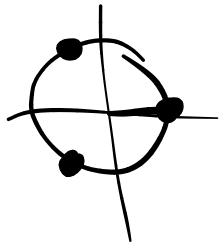
For each link

$\int_G dU$ is the Haar (G-inv't) measure on G.

<u>def'd by:</u> • $1 \overset{!}{=} \int_G dU$ • $\int_G dU f(U) = \int_G dU f(VU) = \int_G dU f(UV)$ $\forall V \in G$.

In the examples: for $G = \mathbb{Z}_n$:

$$\text{"}\int_{\mathbb{Z}_n} dU\text{"} = \frac{1}{n} \sum_{l=1}^{n}$$



for $G = U(1)$:

$$\text{"}\int_{U(1)} dU\text{"} = \oint d\theta$$

for $G = SU(2) \simeq S^3$

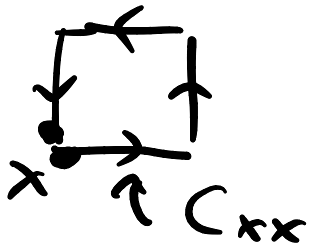
$$\int_{SU(2)} dU = \int (\text{round measure on } S^3)$$

Beautiful feature of $\int_G dU = 1$:

$$\int [dU]\, e^{-S}\Big|_{S=0} = 1.$$

$\Longrightarrow$ no gauge-fixing is required.

what should we use for $S[U]$?



$$W(C_{xx}) = \text{"} P\, e^{-i\oint_{C_{xx}} A} \text{"}$$

is still a matrix

$$W(C_{xx}) \longrightarrow g_x^{-1}\, W(C_{xx})\, g_x$$

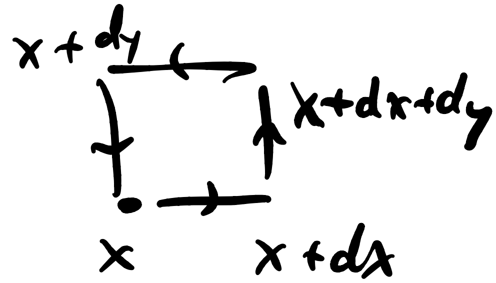
A gauge <u>IN</u>variant object is

$$W(C) \equiv \mathrm{tr}\, W(C_{xx}) = \text{"} \mathrm{tr}\, P\, e^{-i\oint_C A} \text{"}$$

$$W(C) \longrightarrow \mathrm{tr}\, g_x^{-1}\, W(C_{xx})\, g_x \overset{\text{c.o.t}}{=} \mathrm{tr}\, W(C_{xx}) = W(C).$$

<u>Action should be local.</u>

smallest $C = \partial\Box$



$$\underset{\text{Wilson}}{S}[U] = \frac{1}{2f^2}\sum_{\Box} \mathrm{Re}\, S_\Box$$

$$\underline{S_\Box = W(\partial\Box)} = \mathrm{tr}\prod_{\ell\in\partial\Box} U_\ell = \mathrm{tr}\big(U_{x,x+dx}\, U_{x+dx,x+dx+dy}\, U_{x+dx+dy,x+dy}\, U_{x+dy,x}\big).$$

focus on nonabelian continuous case eg $G = SU(N)$

$$\mathcal{U}_{x,x+dx} = e^{-i \cancel{\int_x^{x+dx}} A_\mu dx^\mu}$$

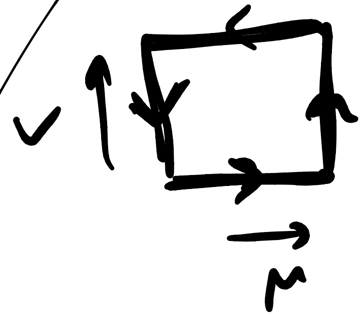
$A_\mu \in \mathfrak{g}$ the Lie algebra $su(N)$

$s \propto a$, lattice spacing

$$e^{sA} e^{sB} \overset{CBH}{=} e^{s(A+B) + \frac{s^2}{2}[A,B] + O(s^3)}$$

$$\mathrm{Re}\, S_{\square_{\mu\nu}} = \frac{1}{2f^2} \mathrm{Re}\,\mathrm{tr}\left(e^{-ia^2 F_{\mu\nu}} \cancel{+ O(a^3)}\right)$$

no sum on $\mu\nu$.



$$F_{\mu\nu} = [D_\mu, D_\nu]$$

pure imaginary

$$= \frac{1}{2f^2} \mathrm{Re}\,\mathrm{tr}\left(\mathbb{1} - \cancel{i a^2 F_{\mu\nu}} - \frac{1}{2} a^4 F_{\mu\nu} F^{\mu\nu} + O(a^5)\right)$$

(no sum on $\mu\nu$)

$$= \frac{1}{2f^2}\left(\mathrm{tr}\, \mathbb{1} - \frac{a^4}{2} \mathrm{tr}\, F_{\mu\nu} F^{\mu\nu} + \dots\right)$$

$$= \mathcal{L}_{YM}(\square_{\mu\nu}) + \text{const} + O(a^5)$$

this works if a is small compared to ξ (the correlation length).

this is a dynamical question.

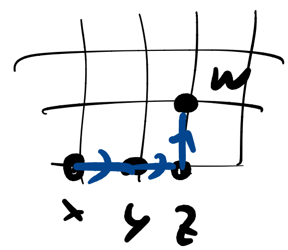
[Add charged matter (quarks)

④ Place fundamentals V^{q_x} on the sites of the lattice (some rep R_q of G)

for fermionic quarks q's are grassmann #s

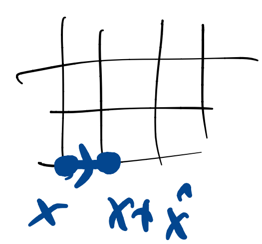
$$\begin{cases} q_x \longmapsto g_x q_x . \\ q^\dagger_x \longmapsto q^\dagger_x g_x \end{cases}$$

$q_x^\dagger U_{xy} U_{yz} U_{zw} q_w$



is gauge invariant.

$$S_{kin}[q] = \frac{1}{a^\#} \sum_{x,\hat{e}} q_x^\dagger U_{x,x+\hat{e}} q_{x+\hat{e}}$$



fermions

$$\simeq \int d^D x\, \bar{q}(x) (\not{D} - m) q(x)$$

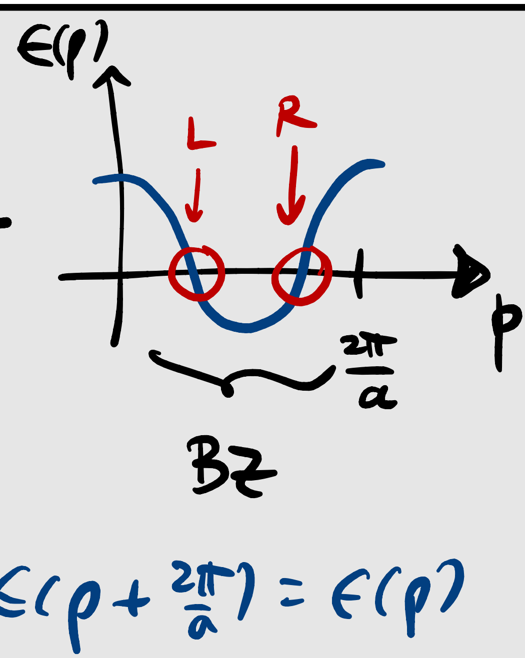
$\Rightarrow D_\mu = \partial_\mu - i A_\mu$ is covariant deriv.

$$\Delta x^\mu D_\mu q \equiv U_{x,x+\Delta x}\, q_{x+\Delta x} - q_x .$$

Note: It's impossible to get a chiral spectrum of fermion from a gaussian lattice action.

chiral $\equiv$ L fermion in R
R fermion in R'
$\Rightarrow R' \neq R^*$.

Pf:



[Nielsen & Ninomiya fermion-doubling theorem.]

Q: How to learn physics from $\int [dU][dq]\, e^{-S[U,q]}$?

Elitzur's Thm:

$$\int dU\, e^{-S[U]}\, \Upsilon[U] = 0$$

if Υ is not gauge inv't.

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gauge-inv't observables in pure YM thy (no quarks):

$$W(C) = \mathrm{tr} \prod_{\ell \in C} U_\ell \simeq \mathrm{tr}\, P e^{i \oint_C A}$$

path in lattice.

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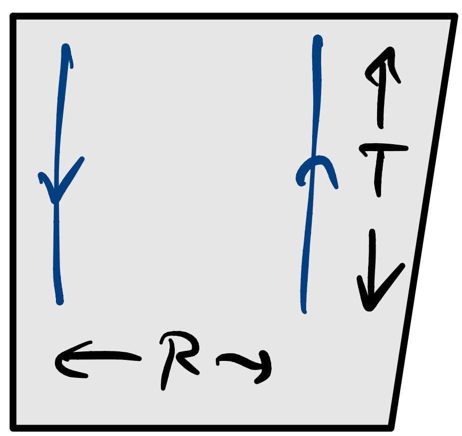
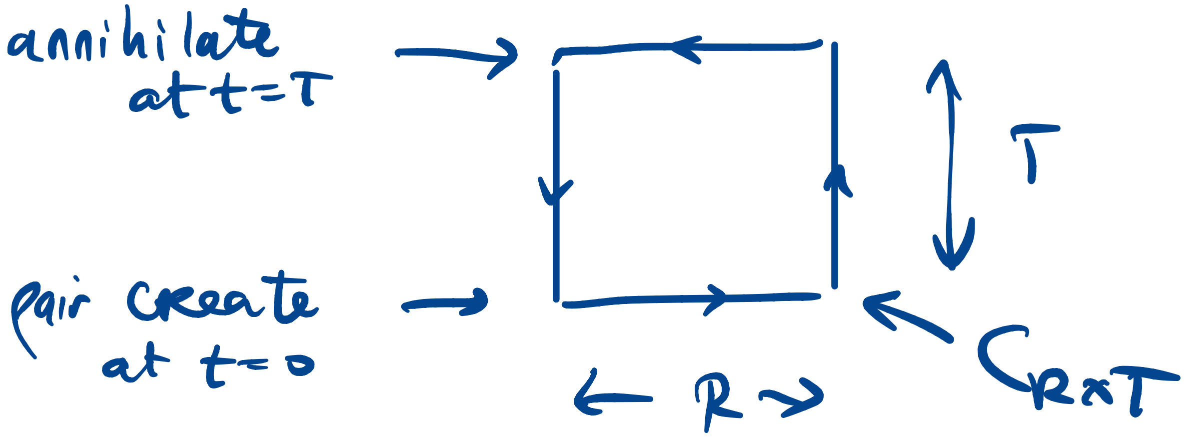
add static sources in QED:

$$\lim_{T\to\infty} Z^{-1} \int [DA]\, e^{-S_{\text{maxwell}}[A] - \int A_\mu J^\mu} = e^{-V(R)T}$$



$$J^\mu(x) = \eta^{\mu 0}\left(\delta^d(\vec{x}) - \delta^d(\vec{x} - (R,0,0))\right)$$

$V(R)$ = Coulomb potential.



$$\langle W(C_{R\times T})\rangle = Z^{-1}\int \prod dU\, e^{-S[U]}\, W(C_{R\times T})$$

$$\overset{T\gg R}{\simeq} e^{-V(R)\,T}$$

piece extensive in T.

measure the potential between static sources!

massive vector of mass m

For S_{Max} in continuum: $\int DA$ is gaussian.

$$\log \langle e^{i\oint_{C_{R\times T}} A}\rangle \simeq -E(R)T - f(T)R$$

$$E(R) \sim \frac{1}{R}e^{-mR}$$

$\sim$ perimeter of the loop $\sim 2T+2R$

In a confining phase:

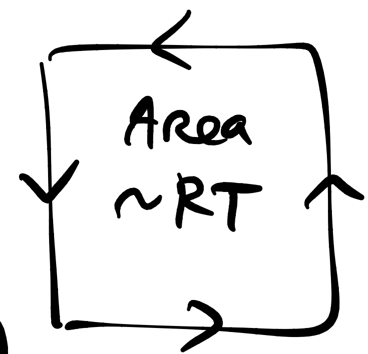
$$\langle W(C_{R\times T})\rangle \overset{T\gg R}{\simeq} e^{-V(R)T}$$ AREA LAW

w/ $V(R) = \sigma R$

$$\Rightarrow \quad F = -\frac{\partial V}{\partial R} = -\sigma$$

$\sigma \equiv$ string tension

$$\log \langle W(C_{R\times T})\rangle \sim -\,\mathrm{Area}(C_{R\times T})$$



Warning: area law $\Rightarrow$ confinement

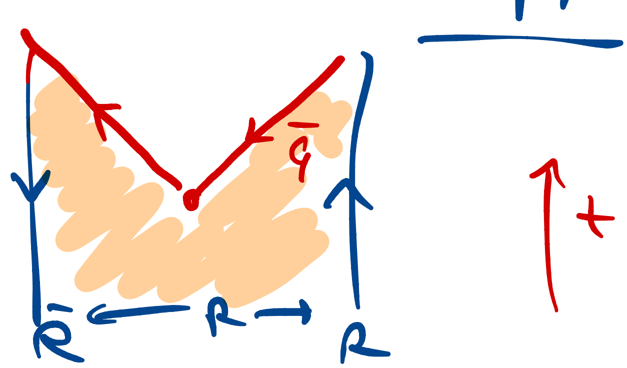
w/ dynamical quarks $\longrightarrow$ $\not\Leftarrow$

$$W(C,R) = \mathrm{tr}_R\, P\, e^{i\oint_C A^A T^A_R}$$

in rep R.

if $\sigma R \gtrsim 2M_q$

can pair produce dynamical charges



Whence area law?

Hint #1: Monte Carlo simulations.

Hint #2: Strong coupling expansion

$$\int \prod dU \; e^{\beta \sum_{\Box} S_{\Box}} W(C)$$ like $\mathrm{tr}\, e^{-\beta H}$.

wants to be expanded in small β.

(high-temp expansion) $\beta \sim \frac{1}{T}$

Prop: Strong-coupling expansion has finite radius of convergence (unlike weak coupling!).

"Pf": $\mathrm{tr}\, e^{-\beta h} = \sum_c e^{-\beta h(c)}$

$e^{-\beta h(c)}$ is analytic at $\beta = 0$.

A singularity would have to come from $\sum_c$. = a continuous phase trans at $T = \infty$.

But $\xi \to 0$ at $T \to \infty$.

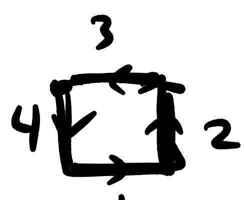
Eg $G = U(1)$. $\qquad U_\ell = e^{i\theta_\ell} \in U(1)$

$$S_{\square_{\mu\nu}}[U] = -(1 - \cos\theta_{\mu\nu})$$

$$\theta_{\mu\nu}(x) = \theta_\mu(x+\nu) - \theta_\mu(x) - \theta_\nu(x+\mu) - \theta_\nu(x)$$
$$= \Delta_\nu \theta_\mu - \Delta_\mu \theta_\nu(x).$$

lattice version of $F_{\mu\nu}$

<u>one-plaquette model</u>:



$$\langle W(\square)\rangle = \int \prod_{l=1}^{4} dU_1 dU_2 dU_3 dU_4 \, U_1U_2U_3U_4 \, e^{\overbrace{\beta(S_\square + S_{-\square})}^{\text{wilson action}}}$$

$$\left[\begin{aligned} S_\square &= U_1U_2U_3U_4 \\ S_{-\square} &= U_4^\dagger U_3^\dagger U_2^\dagger U_1^\dagger . \end{aligned}\right]$$

$$\Rightarrow \Big(\cancelto{0}{1} + \beta(S_\square + \underline{S_{-\square}}) + \cancelto{0}{\frac{\beta^2}{2}(S_\square + S_{-\square})^2}$$
$$+ \frac{\beta^3}{3!}\underbrace{(S_\square + S_{-\square})^3}_{3 S_\square S_{-\square}^2} + \dots\Big)$$

[key fact: $\displaystyle\int_0^{2\pi} d\theta \, e^{in\theta} = \delta_{n,0}$]

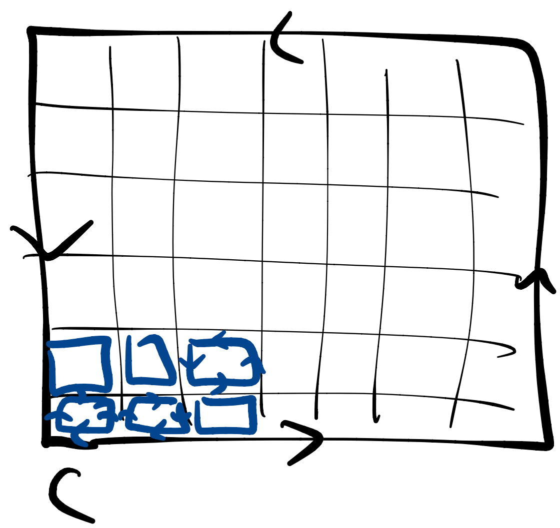
$$\langle W[\square]\rangle = \beta \langle \underbrace{S_\square S_{-\square}}_{1} \rangle + \frac{\beta^3}{2}\langle \underbrace{S_\square^2 S_{-\square}^2}_{\#} \rangle \Big] + O(\beta^5)$$

$$= \beta^{A(\square)}\left(1 + O(\beta^2)\right)$$

$$= e^{-\sigma(\beta)\,\mathrm{Area}}$$

tension $= \sigma(\beta) = |\ln\beta|$.

$$\langle W[C]\rangle = \int \prod dU \prod_{l\in C} U_l \cdot e^{\beta(S_\square + S_\square)}$$



requires one factor

$\sim \beta S_{-\square}$

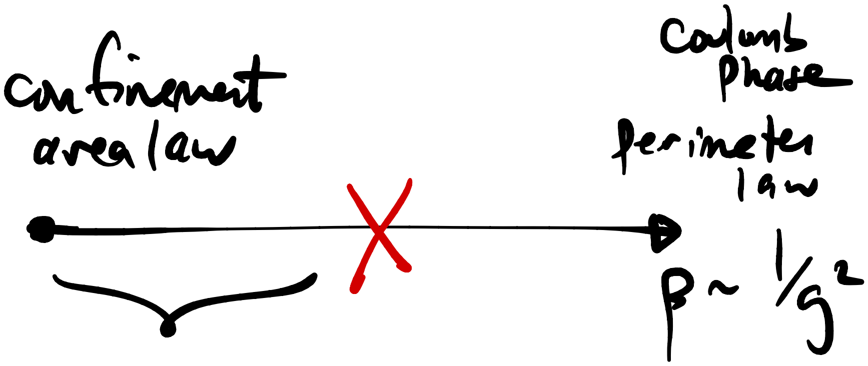
for each plaquette inside the loop

multiple wrappings.

$$= e^{-\sigma(\beta)\,\mathrm{Area}(C)}\left(1 + O(\beta^2)\right)$$

area law! $\sigma(\beta) \sim |\log \beta|$.

$\Longrightarrow$ confinement of ~~color~~ charge !



Conjecture: no such transition
in the non-abelian case.

$G = SU(2)$: $A = 1..3$

$(T^A)_{BC} = -i\,\epsilon_{ABC}$ spin 1 rotation generators.

In any rep R of the Lie algebra

$(T_R^A)_{ab}$ satisfy $a = 1..\dim R$

$$[T_R^A, T_R^B] = i f^{ABC} T_R^C$$

these satisfy Jacobi

$$0 = [T_R^A, [T_R^B, T_R^C]] + \text{perms}$$

$$= f^{BCD} f^{ADE} T_R^E + \text{perms}$$

Choose T_R^E to satisfy $\mathrm{tr}_R T_R^E T_R^F = \delta^{EF}$

$$\omega_p \wedge \omega_q = (-1)^{pq}\, \omega_q \wedge \omega_p$$

$$\underline{\underline{A = A^A T^A}} \quad \text{is a matrix}$$

then $\quad A \wedge A \neq 0$

$$F = F^A T^A$$

$$F = dA + A \wedge A$$

$$F_{\mu\nu} = [D_\mu, D_\nu] = [(\partial_\mu + A_\mu), (\partial_\nu + A_\nu)]$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu + \underline{\underline{[A_\mu, A_\nu]}}$$

$$[T^A, T^B] = i f^{ABC} T^C$$

$$F^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu + i \underline{\underline{f}}^{ABC} A_\mu{}^B A_\nu{}^C$$

$$F = \frac{1}{2} F_{\mu\nu}\, dx^\mu dx^\nu$$

$$= \frac{1}{2} dx^\mu dx^\nu \left(\partial_\mu A_\nu - \partial_\nu A_\mu - ie [A_\mu, A_\nu] \right)$$

$$[A_\mu, A_\nu] = i f_{ABC} T^C A_\mu^B A_\nu^C$$

$$F_{\mu\nu} = \frac{i}{e} [D_\mu, D_\nu]$$

$$D_\mu = \partial_\mu - i A_\mu$$

$$\boxed{(A \wedge B)_{m_1 \ldots m_{p+q}} = \frac{1}{(p+q)!} A_{m_1 \ldots m_p} B_{m_{p+1}, \ldots m_q}}$$