4. GAUGE THEORY

4.1 Anderson-Higgs mechanism (and superconductors)

massive vector fields as gauge fields.

$$\mathcal{L}_B = -\frac{1}{4e^2}(dB)_{\mu\nu}(dB)^{\mu\nu} \underline{+\frac{1}{2}m^2 B_\mu B^\mu}$$

$$(dB)_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \qquad B^\mu B_\mu = B^0 B_0 - B^i B_i$$

is not invariant under $B_\mu \to B_\mu + \partial_\mu \theta$.

<u>Stueckelberg trick:</u> pretend the gauge parameter θ is a field.

Let $B_\mu \equiv A_\mu - \partial_\mu \theta$

added 1 dof
added 1 redundancy
$*$

is inv't under
$$* \begin{cases} A_\mu(x) \to A_\mu + \partial_\mu \lambda \\ \theta(x) \to \theta(x) + \lambda(x) \end{cases}.$$

<u>nice things</u>: • $(dB)_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

if θ is smooth $\Uparrow$ $= F_{\mu\nu}$.

$\partial_\mu \partial_\nu \theta - \partial_\nu \partial_\mu \theta = 0$

• $B_\mu B^\mu = (A_\mu - \partial_\mu \theta) \underbrace{(A^\mu - \partial^\mu \theta)}$

a kinetic term for θ !!

• coupling conserved current j_μ , $\underline{\partial^\mu j_\mu = 0}$
to B :

$$\int d^D x \, j_\mu B^\mu \overset{\text{IBP}}{=} \int d^D x \, j_\mu A^\mu .$$

<u>option 1</u> : choose a gauge where $\theta = 0$.
(unitary gauge) $\rightsquigarrow \mathcal{L}_B$.

<u>option 2</u> : who is θ?

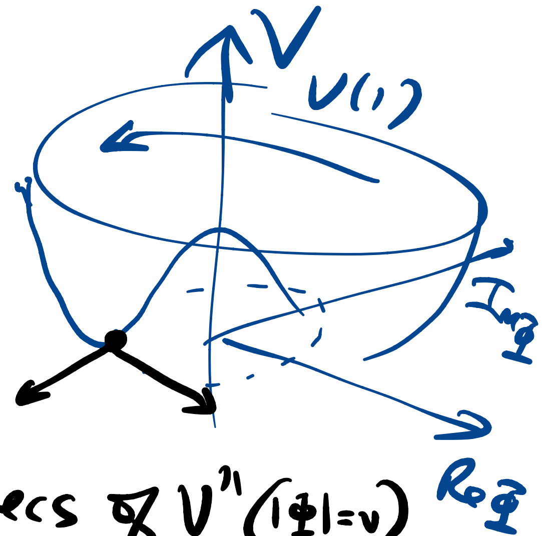
consider:

$$\mathcal{L}_{global} \equiv \frac{1}{2}|\partial\Phi|^2 - V(|\Phi|)$$

$$V(|\Phi|) = \kappa(|\Phi|^2 - v^2)^2$$

$$m_0^2 \equiv \partial_\Phi^2 V\Big|_{\Phi=0} < 0.$$

$v^2 > 0$:



evecs of $V''(|\Phi|=v)$

w/ evals: 0, $8\kappa v^2 > 0$

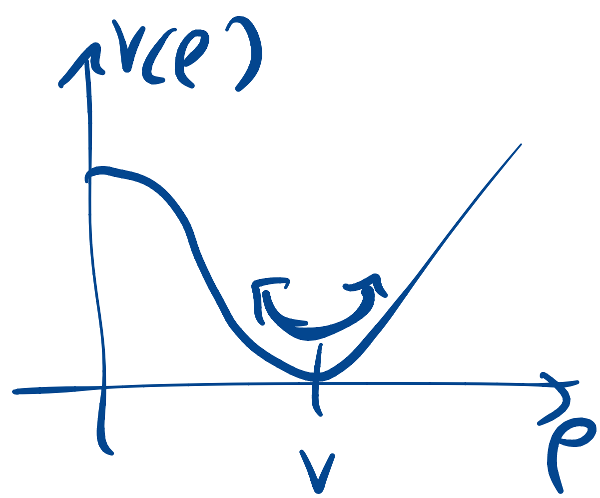
Polar coords on field space:

$$\Phi = \rho e^{i\theta}$$

$$\mathcal{L}_{global} = \frac{1}{2}\rho^2(\partial_\mu\theta)^2 + \frac{1}{2}(\partial\rho)^2 - V(\rho)$$

spontaneously breaks the global U(1) sym.

$$\Phi \to e^{i\alpha}\Phi$$



θ is the goldstone mode.

$$\overset{E \ll V''(v)}{\sim} \frac{1}{2}v^2(\partial\theta)^2 .$$

SUPERFLUID.

Gauge the $U(1)$ symmetry:

$$\mathcal{L}_h \equiv -\frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}|D_\mu \Phi|^2 - V(|\Phi|)$$

$$D_\mu \Phi \equiv (\partial_\mu - iqA_\mu)\Phi$$

'minimal coupling'.

(Abelian Higgs Model)

has a gauge redundancy:

$$\begin{cases} \Phi(x) \to e^{iq\lambda(x)}\Phi(x) \\ A_\mu \to A_\mu + \partial_\mu \lambda \end{cases}$$

$$\left(D_\mu\Phi \to e^{iq\lambda(x)} D_\mu \Phi \right)$$

① $$\Gamma_0 = \frac{\delta S}{\delta A_0} = \underline{\underline{\vec{\nabla}\cdot\vec{E}}} - \underline{\underline{4\pi\rho}} \equiv \hat{\Pi}(x)$$

$$\hat{\Pi}\,|\text{phys}\rangle \overset{!}{=} 0 .$$

$$\int DA_0 \; e^{\int A_0 \Pi} = \delta[\Pi]$$

② Π is the generator of gauge transf.

$$\delta\mathcal{O}(y) = i\int d^Dx\, \lambda(x)\left[\Pi(x), \mathcal{O}(y)\right]$$

③

$$\Rightarrow \langle\text{phys}1|\,(\mathcal{O}+\delta\mathcal{O})\,|\text{phys}2\rangle$$

$$= \langle\text{phys}1|\,\mathcal{O}\,|\text{phys}2\rangle$$

gauging a symmetry is a quotient

S is action of a group $G : s_1 \to g(s_1)$

$$\rightsquigarrow \quad S/G = \left\{ \begin{array}{c} \text{equivalence classes} \\ \text{mod } G \end{array} \right\}$$

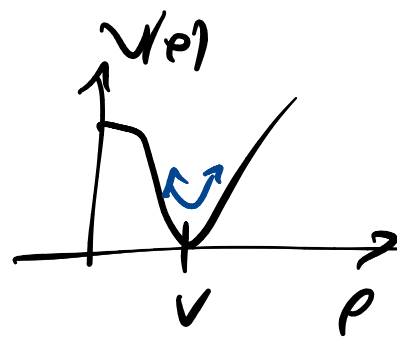
$$[s_1] = [g(s_1)]$$

In classical physics: $S =$ config space.

In polar coords $\Phi = \rho e^{i\theta}$

$$\mathcal{L}_h = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} q^2 \rho^2 (A_\mu - \partial_\mu \theta)^2$$

$$+ \underbrace{\frac{1}{2} (\partial \rho)^2 - V(\rho)}_{\rho \,\equiv\, \text{Higgs mode}}$$



$\kappa \gg 1$ $\quad \partial_\rho^2 V|_{\rho=v} = m_\rho^2 = 8\kappa v^2 \gg m_A^2$ $\quad$ forget ρ

$$\rightsquigarrow \mathcal{L}_h = -\frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} + \frac{m_A^2}{2}(A_\mu - \partial_\mu \theta)^2$$

$$m_A^2 = \underline{\underline{q}}^2 v^2 .$$

can choose unitary gauge to set $\theta = 0$

$\equiv \mathcal{L}_B$

Anderson-Higgs mechanism.

In order for γ to get a mass, we needed:

$$\delta\Pi_{\mu\nu}(k) = \text{~~~}\cdots\cdots\text{~~~} \quad \xleftarrow{k} \quad 1/k^2$$

$$\mathcal{L} \ni A_\mu \partial^\mu \theta$$

gives γ a mass consistent w/ gauge invariance.

would-be goldstone mode θ is eaten, provides the third polarization.

advantages: ① can describe transitions by varying v^2.

② A renormalizable description.

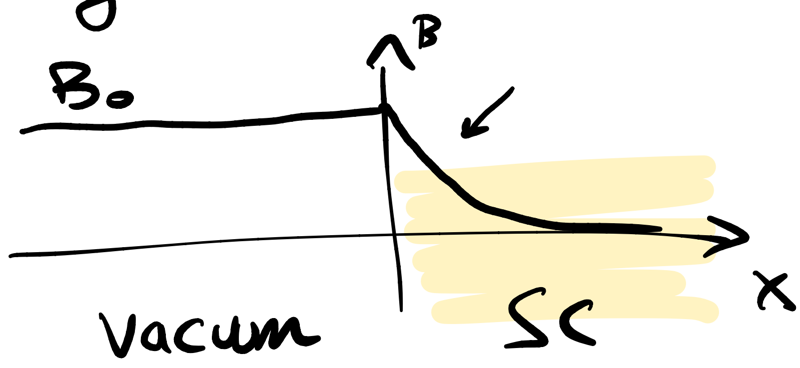
③ Same works for non-abelian case. (Electroweak theory.)

④ This is a description of a superconductor.

condensate Φ is charged under E&M.

$\xrightarrow{\langle\Phi\rangle\neq 0}$ photon gets a mass

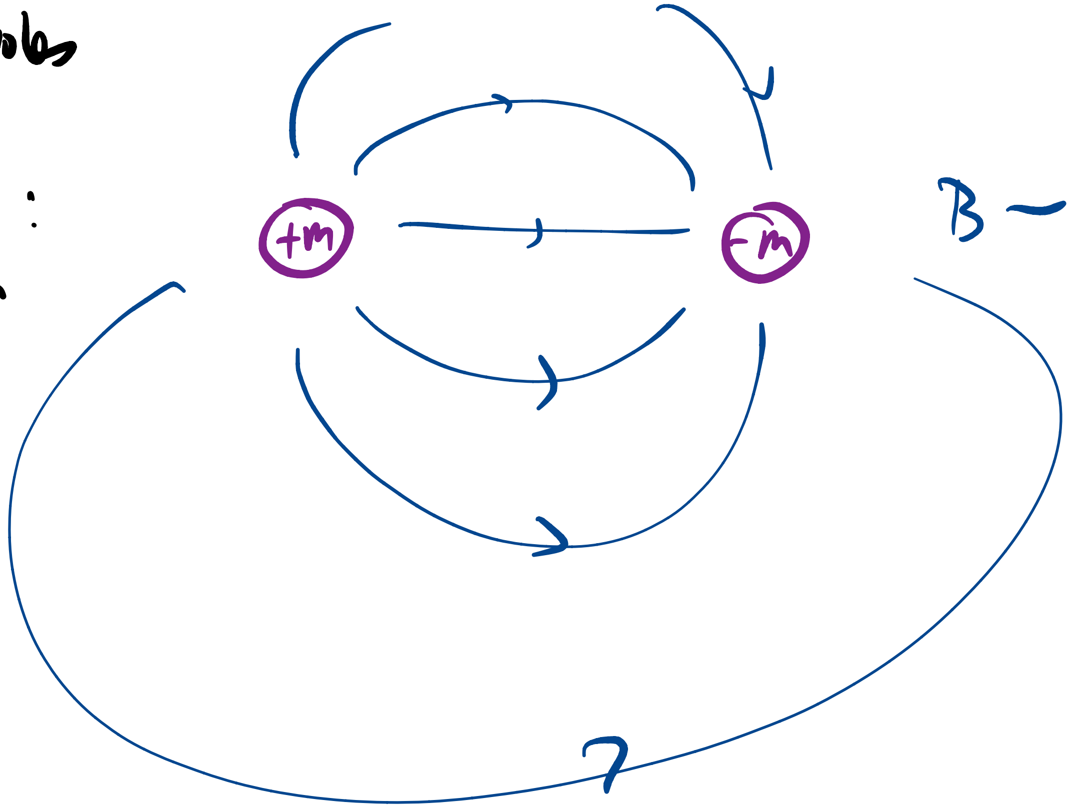
Meissner effect



$$0 = \partial_\mu F^{\mu\nu} - \underline{\underline{m^2 A^\nu}}$$

$$\Rightarrow B(x) = B_0 e^{-mx}$$

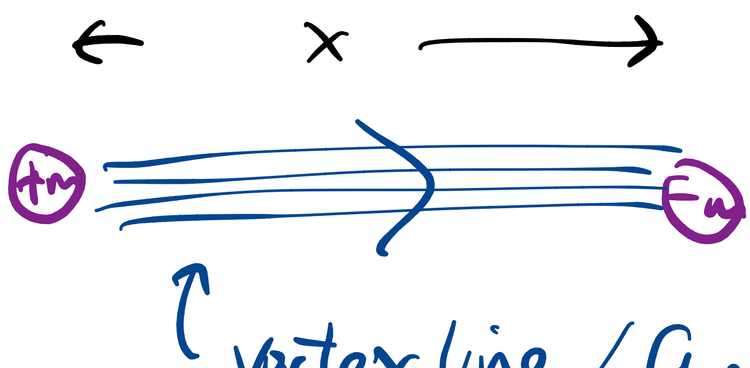
magn. monopoles in vacuum:



$B \sim \frac{1}{r^3}$

magn. monopoles in a superconductor:

$B \sim 0$



vortex line / flux tube.

$\Rightarrow$ monopoles are CONFINED inside a s.c.

$\Rightarrow$ energy $E \sim E_0 + \sigma |x|$

E_0: rest energy

σ: flux tube tension.

<u>EM duality</u>: $\begin{cases} \vec{E} \to \vec{B} \\ \vec{B} \to -\vec{E} \\ e \leftrightarrow 1/e \\ j_m^\mu \leftrightarrow j_e^\mu \end{cases}$

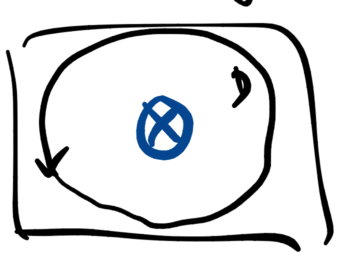
Higgs phenomenon (superconductor) $\longleftrightarrow$ Confinement.

= condensation of electric charge $\langle \Phi_e \rangle \neq 0$ $\longleftrightarrow$ consensation of magnetic charge $\langle \Phi_m \rangle \neq 0$.

In real superconductors $q=2$

Φ = cooper pair field $\sim \epsilon_{\alpha\beta} \psi_\alpha(x) \psi_\beta(x)$

collimated magnetic flux line = vortex line.



<u>claim</u>: $\int_D B$ = # of units of flux

$\overset{\text{stokes}}{=} \oint_{\partial D} A \overset{\min E}{=} \# \oint_{\partial D} \partial\theta =$ winding # of Φ.

4.2 Festival of gauge invariance

$$\mathcal{L}_{global} \to \sum_{\alpha=1}^{N} \partial_\mu \Phi_\alpha^* \partial^\mu \Phi_\alpha - V(\Phi_\alpha^* \Phi_\alpha)$$

$$\left(\text{or} \quad \mathcal{L} = \bar{\Psi}_\alpha \not{\partial} \Psi_\alpha - \ldots \right)$$

is invariant under the $U(N)$ transf

$$\Phi_\alpha \longmapsto \Lambda_{\alpha\beta} \Phi_\beta \quad \text{if} \quad \underline{\Lambda^\dagger \Lambda = \mathbb{1}}.$$

Recall:
$$\Lambda = \Lambda(\lambda) = e^{i \sum_{A=1}^{N^2-1} \lambda^A T^A} \underbrace{e^{i\lambda^0}}_{\substack{\text{generates} \\ U(1) \subset U(N)}}$$

$SU(N) \ni \Lambda(\lambda^0 = 0)$

has $\det \Lambda(\lambda^0=0) = e^{tr(T^A)\lambda^A}$

$= 1$

$\lambda^0 = 0$ ✕

$SU(N)$ Lie algebra

$A = 1 .. N^2-1$

$\underline{T^A}$: $\quad tr T^A = 0$, $\quad [T^A, T^B] = i f_{ABC} T^C$.

$\underline{N=2}$: $SU(2)$ $\begin{cases} T^A = \frac{1}{2}\sigma^A & A = 1,2,3. \\ f_{ABC} = \epsilon_{ABC} \end{cases}$

$\underline{\forall \lambda \ll 1}$: $\Phi_\alpha \mapsto \Phi_\alpha + i\lambda^A T^A_{\alpha\beta}\Phi_\beta$

$(\alpha, \beta = 1 \ldots N.)$

<u>fundamental</u> (N-dim'l) rep. of $SU(N)$

other representations : a collection of $d \times d$ matrices T^A

s.t. $\mathrm{tr}\, T^A = 0$, $[T^A, T^B] = i f_{ABC} T^C$

$\rightsquigarrow$ d-dim'l rep of $SU(N)$

<u>eg:</u> $(T^B_{adj})_{AC} = -i f_{ABC}$ generate the <u>adjoint</u> (N^2-1) rep.

Gauge the SU(N) sym.

In the abelian case:

$$\begin{cases} \Phi \to e^{iq\lambda(x)}\Phi(x) \\ A_\mu \to A_\mu + \partial\lambda \end{cases}$$

$$\partial_\mu \Phi \rightsquigarrow D_\mu \Phi = (\partial - iqA)_\mu \Phi \to e^{iq\lambda(x)} D_\mu \Phi .$$

$$\partial_\mu \Phi_\alpha \rightsquigarrow \partial_\mu \Phi_\alpha - i A_\mu^A T^A_{\alpha\beta} \Phi_\beta$$

$$\Phi \to \Phi + i\lambda^A(x) T^A \Phi$$

$$A_\mu^A \to A_\mu^A + \partial_\mu \lambda^A - \underbrace{f_{ABC}\lambda^B A_\mu^C(x)}_{}$$

new ingredient: gauge field is a matrix.

$\wedge$

$N\times N$

$$A \equiv \sum_B A^B_\mu T^B$$

$$\mathcal{L}_{Max} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$\mathcal{L}_{YM} = -\frac{1}{4g^2} \sum_A \Big(\underbrace{\partial_\mu A_\nu^A - \partial_\nu A_\mu^A + f_{ABC} A_\mu^B A_\nu^C}_{\equiv F_{\mu\nu}^A = -F_{\nu\mu}^A} \Big)^2$$

$$= -\frac{1}{4g^2} \operatorname{tr} F_{\mu\nu} F^{\mu\nu}$$

$$F \equiv \sum_A F^A T^A$$

$$\operatorname{tr} F \cdot F = \sum_{A,B} F^A F^B \underbrace{\operatorname{tr} T^A T^B}_{\delta^{AB}}$$

CLAIM:

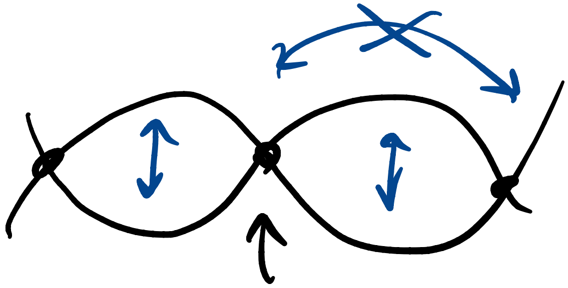
$$\begin{cases} F \mapsto \Lambda F \Lambda^{-1} \\ \Phi \mapsto \Lambda \Phi \end{cases}$$

Inf'l transf:

$$F^A_{\mu\nu} \mapsto F^A_{\mu\nu} + f_{ABC}\lambda^B F^C_{\mu\nu}$$

$$= F^A_{\mu\nu} + i\lambda^B \left(T^B_{adj}\right)_{AC} F^C_{\mu\nu}.$$

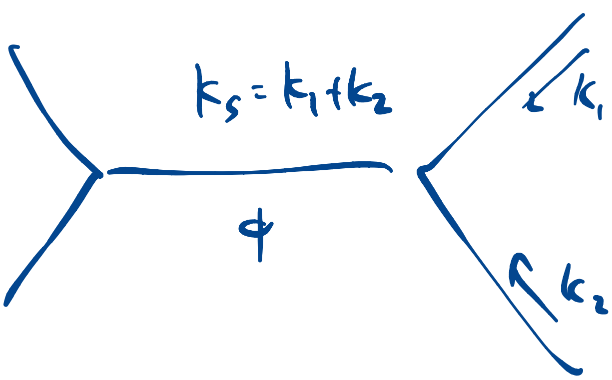
transforms in the adjoint rep of $SU(N)$



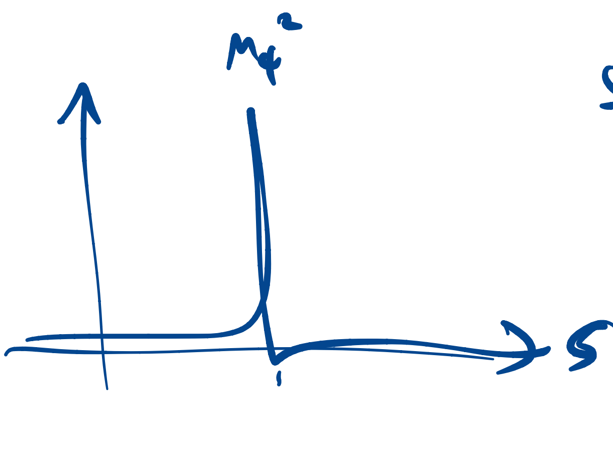
$$2^{-2} g^3$$

$$\langle 0| \phi\phi \, T e^{i\int V} \phi\phi |0\rangle$$

$$\phi\phi \quad \phi^4 \quad \phi^4 \quad \phi^4 \quad \phi\phi \qquad \frac{1}{3!}\left(\frac{-ig}{4!}\right)^3$$

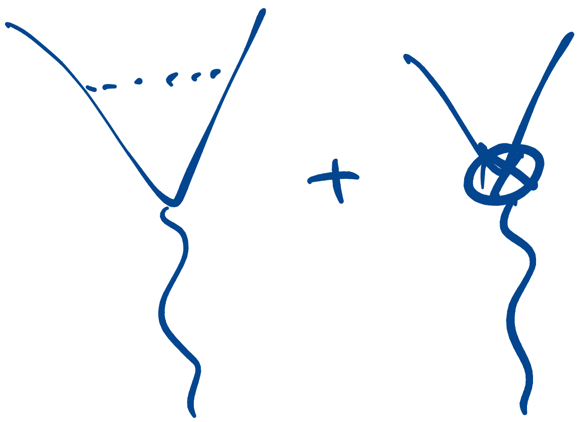


$$\sim \frac{1}{s - m_\phi^2 + i\epsilon}$$



$$s = (k_1 + k_2)^2$$

$$\times\, \underbrace{\not{p}}_{\delta Z_2} + m$$



$$= \gamma^\mu \left(\cdots \quad + \; Z_1 \right)$$

$$\bar{\psi}(\underline{\not{\partial}} + \underline{A\!\!\!/}\,)\psi + m\bar{\psi}\psi - \frac{1}{4e^2}F^2$$

$$+ \; \phi\bar{\psi}\psi$$