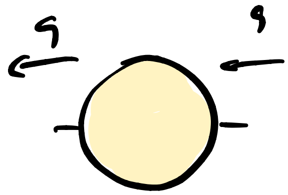
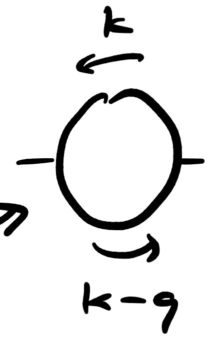
2.2 Cutting Rules & Optical Theorem (cont'd)

$$S[\phi] = \int d^D x \left[\frac{1}{2}(\partial\phi)^2 - m^2\phi^2 - \frac{g}{3!}\phi^3 \right]$$

$i\Sigma(q^2) =$ 

$=$ 

$+ \mathcal{O}(g^3)$

$$i\Sigma_2 = \frac{1}{2}(ig)^2 \int d^D k \, \frac{i}{k^2 - m^2 + i\epsilon} \, \frac{i}{(q-k)^2 - m^2 + i\epsilon}$$

$$\frac{1}{k^2 - m^2 + i\epsilon} = -i\pi\,\delta(k^2 - m^2) + \underbrace{P\frac{1}{k^2 - m^2}}_{\text{real}}$$

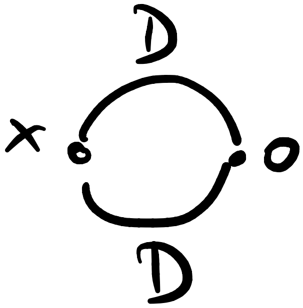
$$\equiv -i\Delta + P$$

$$\text{Im}\,\Sigma_2(q^2) = -\frac{1}{2} g^2 \int d\Phi \left(P_1 P_2 - \Delta_1 \Delta_2 \right)$$

$$d\Phi \equiv d^D k_1 \, d^D k_2 \, \delta^D(k_1 + k_2 - q) \qquad \bar{\delta}^D(..) = (2\pi)^D \delta^D(..)$$

$$i\Sigma_2(q^2) = \frac{1}{2}\int d^Dx\, e^{iqx}(-ig)^2\, iD(x)\, iD(x)$$

time-ordered position space propagator



$$= \frac{g^2}{2}\int d\Phi\, \frac{1}{k_1^2-m_1^2+i\epsilon}\; \frac{1}{k_2^2-m_2^2+i\epsilon}$$

$\to$ Time-ordered D.

$$0 = \frac{1}{2}\int d^Dx\, e^{iqx}(ig)^2\, i\underbrace{D_{adv}(x)}_{\substack{=0\\ \text{for } t\geq 0}}\, i\underbrace{D_{ret}(x)}_{\substack{=0\\ \text{for } t\leq 0}}$$

$$= \frac{1}{2}g^2\int d\Phi\, \underbrace{\frac{1}{k_1^2-m_1^2-\sigma_1 i\epsilon}}_{\to \text{adv}}\; \underbrace{\frac{1}{k_2^2-m_2^2+\sigma_2 i\epsilon}}_{\to \text{ret.}}$$

$$\sigma_{1,2} \equiv \text{sign}(k^0_{1,2})$$

$$\Rightarrow \text{Im}(i0) = \frac{1}{2}g^2\int d\Phi\, (P_1P_2 + \sigma_1\sigma_2\Delta_1\Delta_2)$$

$$\Rightarrow \mathrm{Im}\,\Sigma_2(q^2) = \frac{1}{2} g^2 \int d\Phi \left(\underbrace{(1+\sigma_1\sigma_2)}_{\text{only nonzero when } \mathrm{sign}\,k_1^0 = \mathrm{sign}\,k_2^0} \underbrace{\Delta_1 \Delta_2}_{\text{only nonzero when } k_i^2 = m_i^2} \right)$$

$\ni \delta(q^0 = k_1^0 + k_2^0)$

$*$

WLOG: $q^0 > 0$ $\Longrightarrow$ $k_1^0 > 0,\ k_2^0 > 0$.

$*$

$*$ $\Longrightarrow$ Real 2-particle state in intermediate state!

$$\mathrm{Im}\,\Sigma_{12} = \frac{1}{2} g^2 \int d\Phi\, 2\, \theta(k_1^0)\theta(k_2^0)\, \Delta_1 \Delta_2$$

$$= \frac{1}{2} g^2 \int d\Phi\, 2\, \theta(k_1^0)\pi\delta(k_1^2 - m^2)\ \theta(k_2^0)\pi\delta(k_2^2 - m^2)$$

$$\int d^D k_1\, \theta(k_1^0)\pi\delta(k_1^2 - m^2) = \frac{1}{2}\int \frac{d^d k_1}{2\omega_1}$$

$$= \frac{1}{2} g^2 \underbrace{\frac{1}{2} \int \frac{d^d k_1}{2\omega_1} \int \frac{d^d k_2}{2\omega_2}\, \delta^D(k_1 + k_2 - q)}_{= d\Pi_{LI}}$$

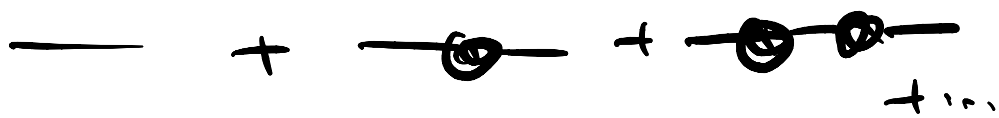
$$\mathrm{Im}\,\Sigma(q) = \frac{1}{2}\sum_{\substack{\text{actual states } n \\ \text{into which } \phi \\ \text{w/ momentum } q \\ \text{can decay}}}' \|A_{\phi\to n}\|^2 = m\Gamma$$

$\uparrow$ if $q = (m, \vec{0})$

in the rest frame:

$$\Gamma = \int \frac{d\Pi_{LI}}{2m} \|A\|^2$$

(above $A = ig$)

$G(q) =$ 

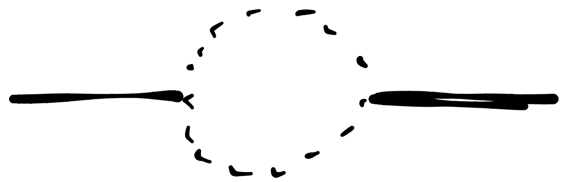
$$= \frac{i}{q^2 - m^2 - \underbrace{\Sigma(q^2)}}$$

Kinematical annoyance: if $m>0$ $\phi \to \phi\phi$ is forbidden by $q = k_1 + k_2$.

eg to which our calc. literally applies:

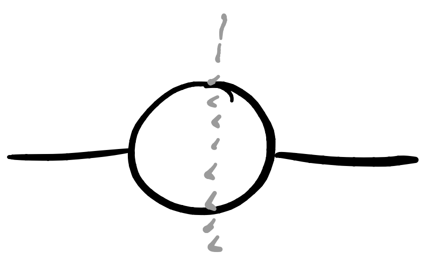
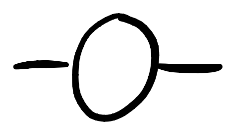
$$\mathcal{L} = \frac{1}{2}\left((\partial\Phi)^2 - M^2\Phi^2 + (\partial\phi)^2 - m^2\phi^2 - g\,\phi^2\Phi\right)$$

w/ $m < M$

Σ_Φ 

now $\Phi \to \phi\phi$ is allowed.

Cutkowsky Cutting Rules:

$\mathrm{Im}\Big($  $\Big) =$  $\Big|_{\text{cut propagators}}$

If >1 possible cut separating I & F,

$$\mathrm{Im}(\;) = \sum_{\text{cuts}} (\ldots)$$

$$\frac{i}{p^2-m^2} \rightsquigarrow \theta(p^0)\,2\pi\delta(p^2-m^2) = \theta(p^0)\,\frac{2\pi\delta(p^0-\epsilon_p)}{2\epsilon_p}.$$

(true also w/o Lorentz)

Why: unitarity.

$$S_{fi} = \langle f | e^{-iHT} | i \rangle \Big|_{T\to+\infty} \equiv \underline{(\mathbb{1}+iT)_{fi}}$$

$$H = H^\dagger \Rightarrow \mathbb{1} = S^\dagger S$$

$$= (\mathbb{1} - iT^\dagger)(\mathbb{1} + iT)$$

$$= \mathbb{1} + i\underbrace{(T - T^\dagger)}_{= -2\,\mathrm{Im}\,T} + T^\dagger T$$

$$\Rightarrow \boxed{2\,\mathrm{Im}\,T = T^\dagger T}$$

optical theorem.

$$2\,\mathrm{Im}\,T_{fi} = \sum_n T^\dagger_{fn} T_{ni}$$

$$\mathbb{1} = \sum_n |n\rangle\langle n|$$

BIG HORRIBLE sum.

eg: ϕ^3 at $O(g^2)$: $\displaystyle\sum_n \sim \int \frac{d^dk_1}{2\omega_1}\frac{d^dk_2}{2\omega_2} f(k_T)$.

In a basis of scattering states:

$$\langle f|T|i\rangle \equiv T_{fi} = \delta^D(p_f - p_i)\,\mathcal{M}_{fi}$$

$$\langle f|T^\dagger|i\rangle \equiv T^\dagger_{fi} = \delta^D(p_f - p_i)\,\mathcal{M}^*_{if}$$

$$\mathbb{1} = \sum_N \left(\prod_{f=1}^{N} \int \frac{d^d q_f}{2E_f}\right) |\{q_f\}^N\rangle\langle\{q_f\}^N|$$

N-particle state.

$$\langle F|T^\dagger \mathbb{1}\, T|I\rangle =$$

$$\sum_N \underbrace{\langle F|\, T^\dagger \prod_f \int \frac{d^d q_f}{2E_f} |\{q_f\}\rangle}\underbrace{\langle\{q_f\}|T|I\rangle}$$

$$= \sum_N \prod_{f=1}^{N} \int \frac{d^d q_f}{2E_f}\ \delta^D\Big(p_F - \sum_f q_f\Big)\mathcal{M}^*_{\{q_f\}F}$$

$$\delta^D\Big(p_I - \sum_f q_f\Big)\mathcal{M}_{\{q_f\}I}$$

$$\prod_{f=1}^{N} \int \frac{d^d q_f}{2E_f}\ \delta^D\Big(p_F - \sum_f q_f\Big) = \int d\Pi_N$$

N-particle phase sp.

$$\Rightarrow i\left(\mathcal{M}^*_{IF} - \mathcal{M}_{FI}\right) = \sum_N \int d\Pi_N \, \mathcal{M}^*_{\{q_f\}F} \mathcal{M}_{\{q_f\}I}$$

Now: consider forward scattering: $I = F$.

$$S = \mathbb{1} + iT = \mathbb{1} + i\delta(p_T)\mathcal{M}$$

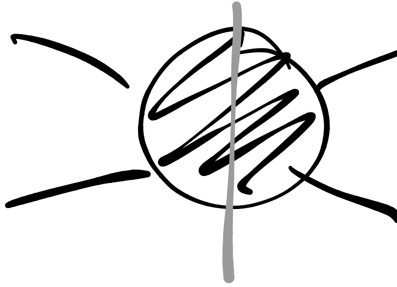
$$\Rightarrow 2\,\mathrm{Im}\,\mathcal{M}_{II} = \sum_N \int d\Pi_N \left|\mathcal{M}_{\{q_f\}_N I}\right|^2$$

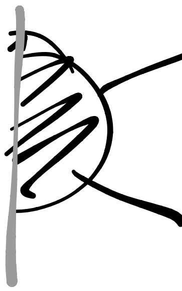
$$\propto \sigma(I \to \text{anything})$$

eg: $I =$ 2-particle state

$$\sigma_{\text{com}}(k_1 k_2 \to \text{any}) = \underbrace{\frac{1}{2\omega_1}\frac{1}{2\omega_2}\frac{1}{|v_1 - v_2|}} \sum_N \int d\Pi_N \left|\mathcal{M}_{k_1 k_2 \to N}\right|^2$$

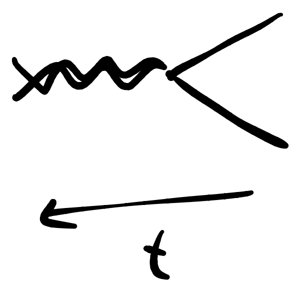
$$\overset{\text{com}}{=} \frac{1}{4|k_1||E_1|} \, 2\,\mathrm{Im}\,\mathcal{M}_{k_1 k_1 \to k_1 k_2}$$

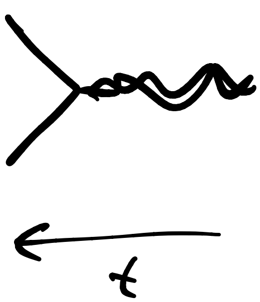
Im 

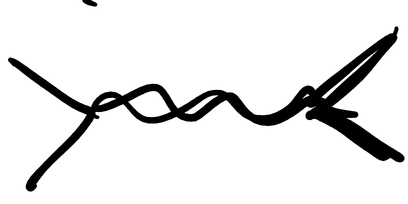
$= \sum_{\substack{\text{on-shell}\\ \text{intermediate}\\ \text{states}}} \Big\|$  $\Big\|^2 \propto \sigma(2 \to \text{any})$

= anything that 2 particles can turn into

Simpler Application: Resonances

some particles don't live long enough to separate production: 

from decay: 

Instead: Im()

$$\delta^D(p_T)\, i\mathcal{M}_{FI} = \langle F | iT | I \rangle$$

special case: $|F\rangle$, $|I\rangle$ are 1-particle states.

LSZ:

$$\mathcal{M} = +(\sqrt{Z})^2 \left(-i \underset{\text{amplitude}}{\overset{\text{amputated}}{\rightarrow}} \right)$$

$$= -Z \cdot \Sigma_1 .$$

Let $\Sigma(p) = A(p^2) + iB(p^2)$

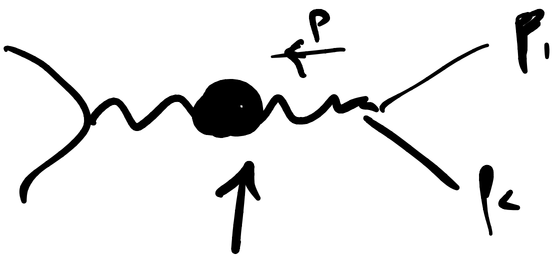
near the 1-particle pole

$$\tilde{G}^{(2)}(p) = \frac{i}{p^2 - m_0^2 - \Sigma(p)} \simeq \frac{i}{(p^2-m^2)\underbrace{(1-\partial_{p^2}A|_{m^2})}_{Z^{-1}} - iB}$$

$$= \frac{iZ}{(p^2-m^2) - iBZ} \qquad \Gamma_w \equiv -\frac{Z B(m^2)}{m}$$

$$\overset{p^2=m^2}{=} \frac{iZ}{(p^2-m^2) + im\Gamma_w} .$$

Suppose:

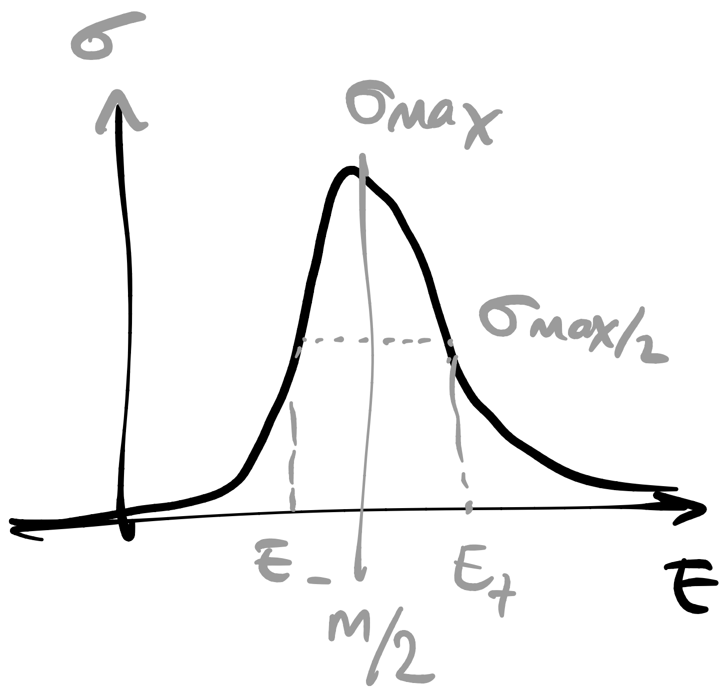


$\tilde{G}^{(2)}(p)$

$p_1 + p_2 = p$

$$\Rightarrow \sigma_{2\leftarrow 2}(p^2) \propto \left| \tilde{G}^{(2)}(p) \right|^2 = \left| \frac{iZ}{p^2 - m^2 - im\Gamma_W} \right|^2$$

$$= \frac{Z^2}{(p^2-m^2)^2 + m^2\Gamma_W^2}$$



Lorentzian

Breit-Wigner distributn

$$\text{CM} \begin{cases} p_1^\mu = (E, \vec{p}_1)^\mu \\ p_2^\mu = (E, -\vec{p}_1)^\mu \end{cases}$$

$2E = m$

$$E_\pm = \sqrt{\frac{m(m \pm \Gamma_W)}{4}}$$

$\Gamma_W \ll m$

$$\simeq \frac{m}{2} \pm \frac{\Gamma_W}{4}.$$

$\Gamma_W \doteq$ width.

optical thm says:

$$\Gamma_W = -\frac{BZ}{m} \overset{LSZ}{=} -\frac{1}{m}\left(-\operatorname{Im}\mathcal{M}_{1\to 1}\right)$$

optical thm

$$= \frac{1}{2m}\sum_N \int_f d\Pi_N \left|\mathcal{M}_{\{f\}_N \leftarrow 1}\right|^2$$

(Z is real) $B = \operatorname{Im}\Sigma$

$$= \Gamma$$

decay rate in the cm frame.

- (if $\Delta E \ll \Gamma$.)

- if not $\Gamma \ll m$, broad resonance

must keep $B(p^2) = B(m^2) + (p^2 - m^2)B'(m^2) + \ldots$

Unitarity & High energy physics:

① $\operatorname{Im}\mathcal{M}$ had better not depend on the cutoff.

② $0 \le$ probabilities ≤ 1.

claim: $\sigma_{total}(s) \le C \ln^2 s$ Froissant Bound.

In particular σ can't grow polynomially in s. like s^α $\alpha > 0$ $\times$.

suppose $\underline{[G] = k}$. $A_{tree} \sim G$

$$\sigma_{tree} \sim |A_{tree}|^2 \propto G^2$$

$$[\sigma] = -2$$

if $E \gg$ everybody $\Rightarrow$ $\sigma(E \gg \ldots) \sim G^2 E^{-2-2k}$

$$= G^2 s^{-1-k}$$

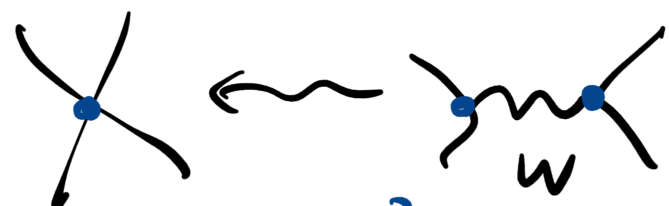
if $k \le -1$ $\sigma \sim s^{\text{large positive}}$ violates Froissant.

Non-Renormalizable interaction ($[G] \le -1$) Requires new high-energy dof to "restore unitarity".

eg1: 4 fermi theory: $S[\psi] = \int d^4x \left[\bar\psi \partial\!\!\!/ \psi - \frac{G_F}{\sqrt{2}} \bar\psi\psi\bar\psi\psi \right]$

$[G_F] = -2$

$G_F \propto \frac{g^2}{M_W^2}$



$$\frac{g^2}{M_W^2} \xleftarrow{p^2 \ll M_W^2} \frac{g^2}{p^2 - M_W^2}$$

eg2: gravity $[G_N] = -2$

$G_N = \frac{1}{M_{pl}^2}$?

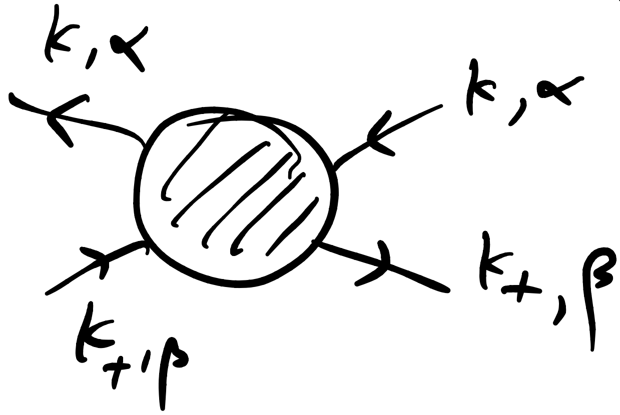
$\Rightarrow$ QFT of gravitons w EH action becomes strongly coupled at $E \sim M_{pl}$.

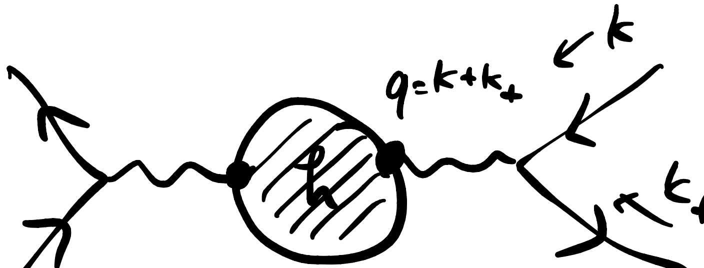
2.3 How to study hadrons using perturbative QCD

(application of both §2.1 & §2.2)

$$\sigma^{\text{anything} \leftarrow e^+e^-}(s=(k+k_+)^2 \gg m_e^2) \overset{\text{optical thm}}{=}$$

$$\frac{1}{2s}\,\mathrm{Im}\,\mathcal{M}(e^+e^- \leftarrow e^+e^-)_{\underline{\underline{\text{forward}}}}$$



$i\mathcal{M}_h =$ 

...

$$\sigma^{\text{hadrons} \leftarrow e^+e^-} = \frac{1}{2s}\,\mathrm{Im}\,\mathcal{M}_h$$

$F(x)$

<u>Find</u> : $x\,\partial_x F(x) = \text{cst}$

$\uparrow$

$L^4/T \to \infty$

C, u depend on $\frac{1}{\log T}$

$u \equiv U/T$.

$$\sum_{\text{pols}} \epsilon^{\mu\nu}(k)\, \epsilon^{*\rho\sigma}(k)$$

$$= \left(\mathbb{1} \text{ on the space of 1-particles states w momentum } k \right)^{\mu\nu\rho\sigma}$$

$$= \left(\text{numerator of propagator} \right)^{\mu\nu\rho\sigma}$$

+ terms that vanish in σ

$$\langle h^{ij} \quad h^{kl} \rangle = \frac{i}{p^2} \, (>0)$$

$$\int \prod dx \quad x_i \, x_j \quad e^{-x_i A_{ij} y_j}$$

$$\propto (A^{-1})_{ij}$$

$i = \mu\nu = \nu\mu$
$j = \rho\sigma = \sigma\rho$

$$(A^{-1})_{ij} \, (A)_{jk} = \delta_{ik}$$

$$\mathcal{L} = +\frac{1}{2} \partial_\mu A^i \, \partial^\mu A^i$$