1.5.2 Q: Where is the UV sensitivity?

Goals:
- use dim'l analysis to answer this Q
- relevant / irrelevant operators
  renormalizable / non-renormalizable

Def: A diagram A has a "superficial degree of divergence"

$$D_A \equiv [A] = \text{\# of powers of mass.}$$

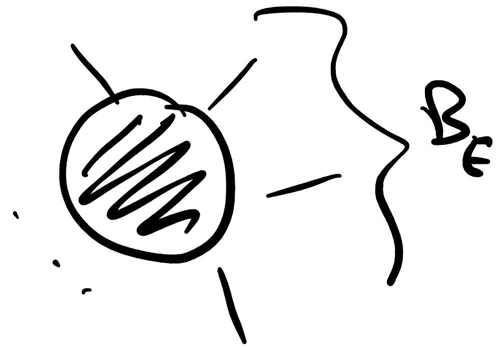
$$A \overset{\Lambda \gg \text{everything}}{\simeq}{}^{*} \; \Lambda^{D_A}$$

(If $D_A = 0$, expect $\log \Lambda$.)

* if no cancellations.

eg: ϕ^4 thy in $D = 4$.

$\mathcal{A}$ is a connected diagram w/ B_E external scalar lines

$\mathcal{A} =$ 

claim: $D_{\mathcal{A}} = 4 - B_E$.

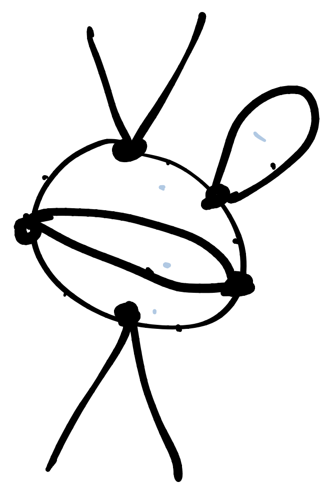
why doesn't it depend explicitly on

$B_I \equiv$ # of internal lines

$V \equiv$ # of ϕ^4 vertices

$L \equiv$ # of loops

?



$B_I = 8$

$B_E = 4$

$V = 5$

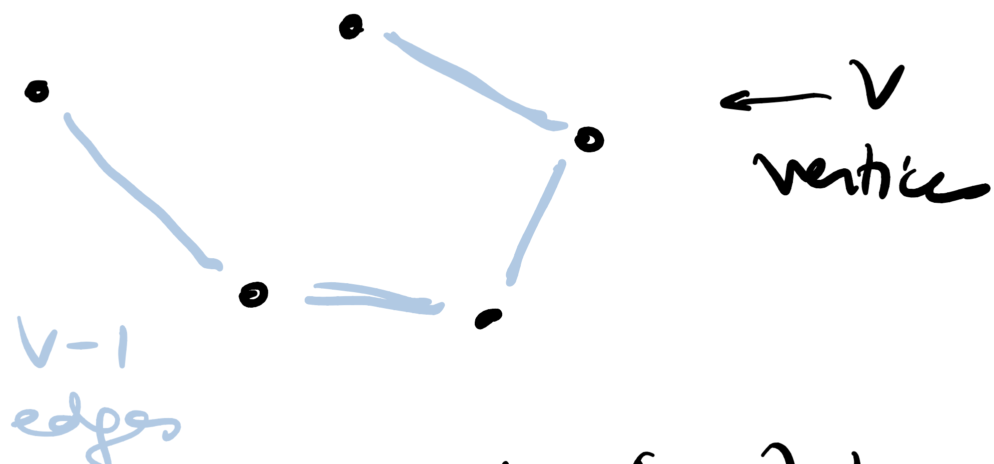
$L = 4$

Graph theory fact # 1 : $L = B_I - (V-1)$ connected!

$$A \propto \prod_{\alpha=1}^{B_I} \int d^D q_\alpha \prod_{I=1}^{V} \delta^D\left(\sum q\right) \cdots$$

$$\propto \delta^D(P_{total}) \prod^{L} \int d^D q$$

Pf:



each edge beyond the $(V-1)$st adds one loop.

In the example: $4 = 8 - (5-1)$ ✓

Graph Thy Fact #2 :

# of ends of lines attached to vertices

$$= 4V = B_E + 2B_I$$

ϕ^4 thy

- each ext line comes out of a vertex
- each internal line connects two vertices
- In ϕ^4 thy each vertex has 4 lines coming out.

In the example: $4\cdot 5 = 4 + 2\cdot 8$ ✓

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eliminate $B_I = 2V - B_E/2$

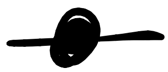
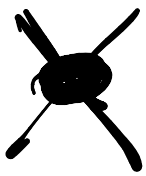
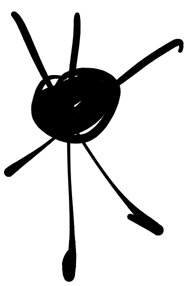
$$A \overset{\Lambda \gg \text{everybody}}{\sim} \prod_{\alpha=1}^{L} \int^{\Lambda} d^D k_\alpha \prod_{a=1}^{B_I} \frac{1}{k_a^2}$$

$$D_A = [A] = DL - 2B_I \quad \text{(general for relativistic scalars)}$$

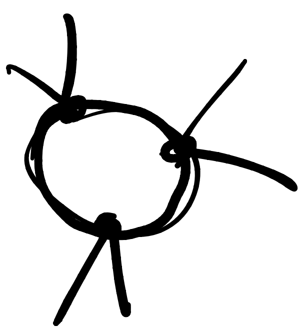
$$\overset{①}{=} B_I(D-2) - D(V-1)$$

$$D_A \overset{(2)}{=} D + \frac{2-D}{2} B_E + \underline{V(D-4)}$$

$$\xrightarrow{D\to 4} \quad 4 - B_E \,.$$

	B_E	D_A	
$\delta m^2 \sim$  $\sim \Lambda^2$	2	2	
$\delta g \sim$  $\sim \log \Lambda$	4	0	
$\delta g_6 \sim$  $\sim \Lambda^{-2} \xrightarrow{\Lambda\to\infty} 0$	6	-2	finite.

$\underline{\text{eg}}$:



$$\sim \int^{\Lambda} \frac{d^4 p}{p^6} \sim \Lambda^{-2}$$

$\Longrightarrow$ counterterms required at ~~one~~ leading loop order are enough for all orders

Not a proof: $\mathcal{I} = \int^{\Lambda} \frac{d^4p}{(p^2+m^2)^5} \int^{\Lambda} d^4k$

$$D_{\mathcal{I}} = 4+4-10 = -2$$

$$= \left(\text{finite bit} + \frac{1}{\Lambda^6}\right)\Lambda^4$$

$$\xrightarrow{\Lambda \gg \ldots} \left(\frac{1}{m^6}\right)\left(\Lambda^4\right)$$

(can't happen)

not finite.

eg #2:

$$S[\phi,\psi] = -\int d^Dx \Big[\tfrac{1}{2}\phi(\Box + m_\phi^2)\phi + \bar{\psi}(-\not{\partial} + m_\psi)\psi + y\phi\bar{\psi}\psi + \frac{g}{4!}\phi^4 \Big]$$

$$A = \prod^{L} \int^{\Lambda} d^Dk \prod^{B_I} \frac{1}{k^2} \prod^{F_I} \frac{1}{k}$$

each scalar propagator

each spinor propagator

$$D_A = DL - 2B_I - F_I$$

\# of ends of fermion lines $= 2V_y$

$\overset{2_F}{=} F_E + 2F_I$

\# of ends of boson lines $= V_y + 4V_g$

$\overset{2_B}{=} B_E + 2B_I$

\# of loops : $L = \underbrace{B_I + F_I}_{\text{all internal lines}} - (\underbrace{V_y + V_g}_{\text{all vertices}} - 1)$

$$D_A = D + (D-4)\left(V_g + \tfrac{1}{2}V_y\right) + B_E\left(\frac{2-D}{2}\right) + F_E\left(\frac{1-D}{2}\right)$$

$$(D-4) \xrightarrow{D \to 4} 0$$

$$\Delta\mathcal{L} = G\,(\bar\psi\psi)(\bar\psi\psi)$$
$$\text{or}\quad G_V\,(\bar\psi\gamma^\mu\psi)(\bar\psi\gamma_\mu\psi)$$
$$\text{or}\quad G_A\,(\bar\psi\gamma^5\gamma^\mu\psi)(\bar\psi\gamma^5\gamma_\mu\psi)$$

$$\longrightarrow\quad D_A \overset{D=4}{=} 4 - B_E - \frac{3}{2}F_E + 2V_G$$

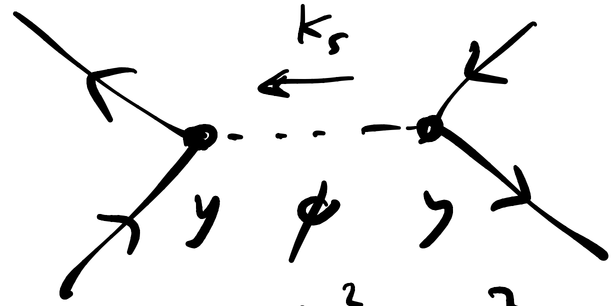
$$[G] = -2.$$

$$\left(\text{vs}\quad [g]_{D=4} = [y]_{D=4} = 0\right)$$

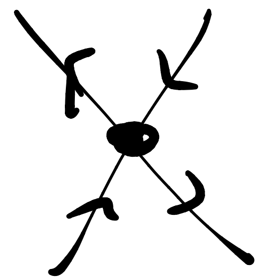
In this 4-fermion (Fermi) theory:
for any B_E, F_E $\;\exists\, V_G$ s.t. $D_A > 0$.

$\Rightarrow$ we need counterterms $\delta_n\,(\bar\psi\psi)^n$ $\;\forall n$!!

Consider:



$$\sim y^2 \frac{i}{k_s^2 - m_\phi^2} \overset{k_s^2 \ll m_\phi^2}{\sim} -\frac{i y^2}{m_\phi^2}$$

$\sim$ 

$$G = \frac{y^2}{m_\phi^2}.$$

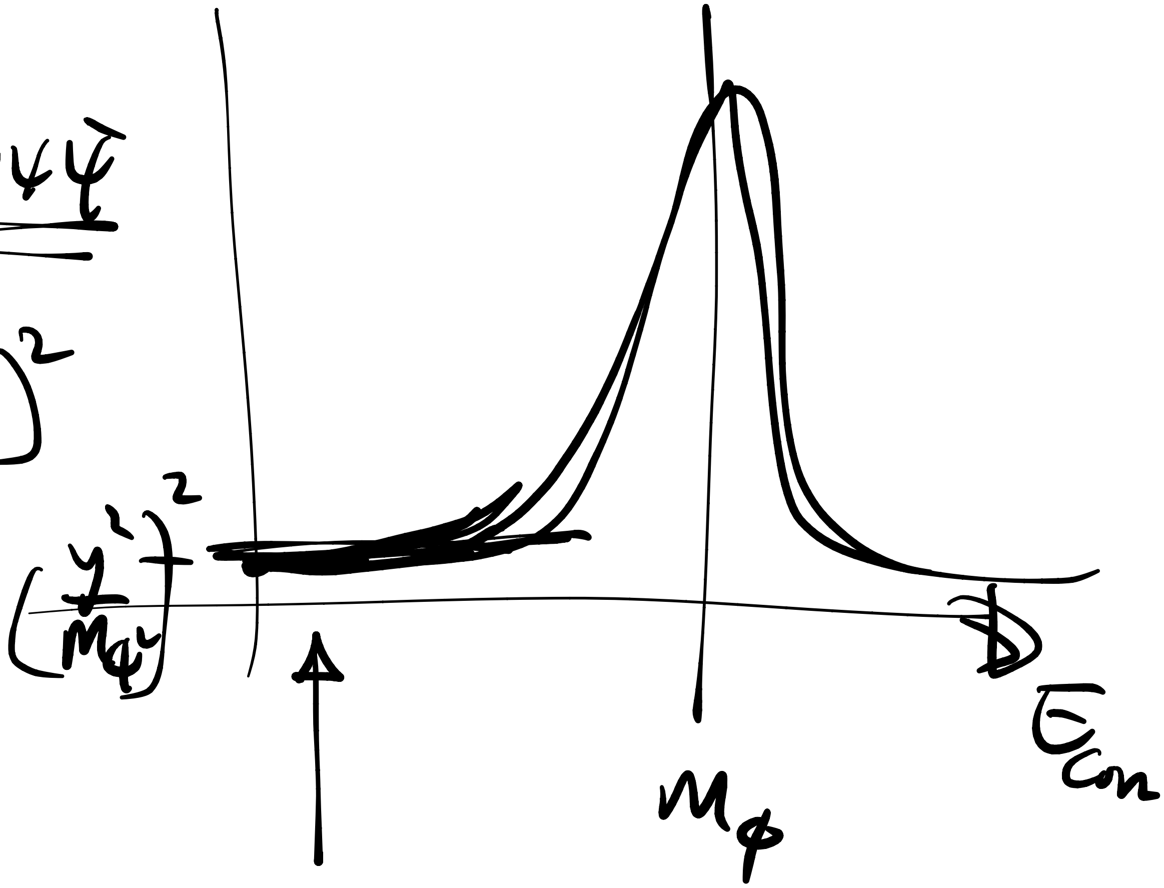
G is "non-renormalizable"
(still useful.)

1.5.3 Naive scale invariance in QFT

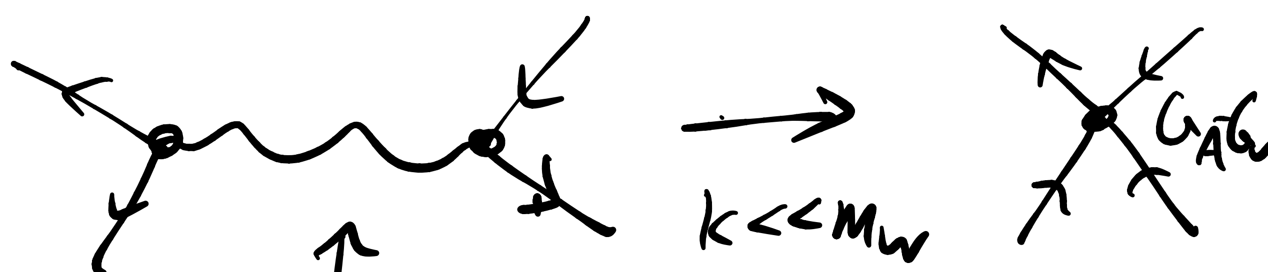
eg 4: $S[\phi] = \int d^D x \left(\frac{(\partial\phi)^2}{2} - g\phi^p \right)$

$$\sigma_{\psi\bar{\psi}\to\psi\bar{\psi}}$$

$$\sim \frac{1}{(E^2-m_\phi^2)^2}$$



Fermi theory: $G=0,\ G_F = G_A = -G_V.$



$$g^2 \frac{-\eta_{\mu\nu}}{k^2-M_W^2}$$

$$G_F = \frac{g^2}{M_W^2}$$

eg ϕ: $S[\phi] = \int d^D x \left(\frac{(\partial\phi)^2}{2} - g\phi^p \right)$

$[S] = [\hbar] = 0 \qquad [x] = -1 = [dx] \qquad [\partial] = +1$

$0 = [S_{kin}] = -D + 2 + 2[\phi]$

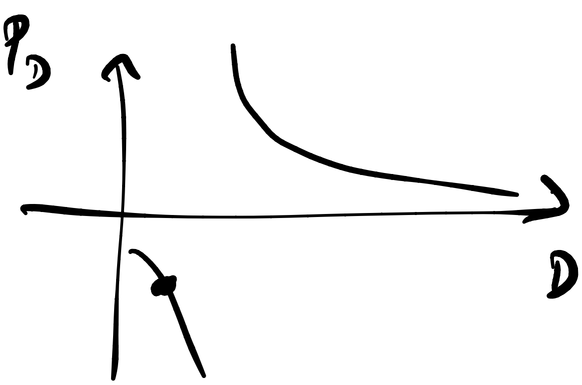
$$[\phi] = \frac{D-2}{2}$$

$0 = [S_{int}] = -D + [g] + p[\phi]$

$$\Rightarrow [g] = D + p\,\frac{2-D}{2}$$

g is dim'less when

$$p = p_D = \frac{2D}{D-2}$$



D	1	2	3	4	5	6	$\cdots$	∞
$[\phi]$	$-1/2$	0	$1/2$	1	$3/2$	2	$\cdots$	∞
p_D	-2	*	6	4	$10/3$	3	$\cdots$	2

[$1/r^2$ an

$$\Delta S = \int d^D x \ \frac{m^2}{2} \phi^2 \Longrightarrow [m] = 1 \ \forall D.$$

* In $D=2$ ϕ is dimless

$$\Longrightarrow S[\phi] = \int d^D x \left[g_{IJ}(\phi)\, \partial^\mu \phi^I \partial_\mu \phi^J + V(\phi^I) \right]$$

is classically scale invariant $\forall\, g_{IJ}(\phi),\ V(\phi)$

"non-linear σ-model".

For $D > 2$ an interaction $g\phi^p$ has

$$[g] = D\, \frac{p_D - p}{p_D} = \begin{cases} < 0 & \text{for } p > p_D \quad \text{non-renormalizable/irrelevant} \\ = 0 & p = p_D \quad \text{renormalizable/marginal} \\ > 0 & \text{for } p < p_D \quad \text{super-renormalizable/relevant} \end{cases}$$

$$f = \sum_{n=0}^{\infty} g^n c_n$$

c_n ind of g.

$$[f] = n[g] + [c_n]$$

$$[c_n] = [f] - n[g]$$

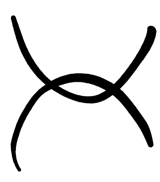
if $[g] < 0$ $\quad c_n \overset{\Lambda \gg \ldots}{\sim} \tilde{c}_n (\Lambda)^{-n[g]}$

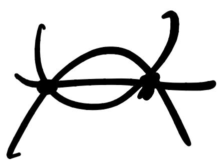
$\tilde{c}_n$: dimless

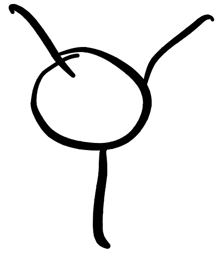
diverges as $\Lambda \to \infty$.

if $[g] = 0$ $\quad c_n = \tilde{c}_n \log^{\nu(n)}\left(\frac{\Lambda_{UV}}{\Lambda_{IR}}\right)$

$\Lambda_{IR} \sim m.$

ϕ^4 in $D=4$: $\delta g_4 \sim$  $\sim \int \frac{d^4k}{(k^2)^2}$

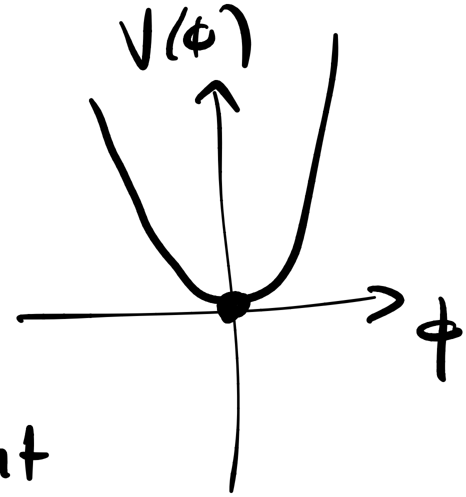
ϕ^6 in $D=3$: $\delta g_6 =$  $\sim \int \frac{(d^3k)^2}{(k^2)^3}$

ϕ^3 in $D=6$: $\delta g_3 =$  $\sim \int \frac{d^6 k}{(k^2)^3}$

in $D=2$: $\langle \phi(x)\phi(0)\rangle \sim \int^{\Lambda_{uv}} \frac{d^2 k \, e^{-ikx}}{k^2}$

$$\sim \log \frac{|x|}{\Lambda_{uv}}.$$

cultural baggage.



eg: $\delta\mathcal{L} = \frac{c}{M^{24}}\phi^{28}$ irrelevant

vs $\delta\mathcal{L} = m^2\phi^2$ relevant (matters at $E < m$).

$$D_k(x,y) = \langle \phi(k,x)\, \phi(-k,y) \rangle_0$$

$$= \int d^D z\, e^{i k_\mu z^\mu} \underbrace{\langle \phi(z,x) \phi(0,y) \rangle_0}$$

$$= \int d^{D+2} p\, \frac{e^{i\left(p^M (x-y)_M + p_\perp^i z_i\right)}}{p_\perp^2 + p^M p_M}$$

$$= \left(\frac{1}{\partial_M \partial^M} \right)$$