Last time: Violation of scale inv by QM.

eg: $H = \frac{\vec{p}^2}{2} - g_0(\Lambda)\,\delta^2_\Lambda(\vec{q})$

has a boundstate w $E = -\epsilon_B$ iff

$$\boxed{1 = \int^\Lambda \frac{d^2p}{\hbar^2} \frac{g_0}{p^2/2 + \epsilon_B}}$$

If ϵ_B is fixed indep of Λ $\Longrightarrow$ $g_0(\Lambda) = \dfrac{2\pi\hbar^2}{\log\left(1 + \frac{\Lambda^2}{2\epsilon_B}\right)}$

Quantify violation of scale invariance:

Def: $\beta(g_0) \equiv \Lambda \frac{\partial}{\partial\Lambda} g_0(\Lambda)$

In our example:

$$\beta(g_0)_{\text{calculate}} = -\frac{g_0^2}{\pi\hbar^2}\Big(\underbrace{1}_{\text{perturbative}} - \underbrace{e^{-2\pi\hbar^2/g_0}}_{\text{non-perturbative}}\Big)$$

Flow eqns: $\quad \dot{g}_0 = \beta(g_0)$

$$\dot{A} \equiv \partial_s A \qquad s \equiv \log \Lambda/\Lambda_0$$

$$\Lambda = e^s \Lambda_0 \longrightarrow \frac{\partial}{\partial s} = \Lambda \frac{\partial}{\partial \Lambda}$$

$$\underbrace{\dot{g}_0 = -\frac{g_0^2}{\pi \hbar^2}\left(1 - e^{-2\pi\hbar^2/g_0}\right)}$$

nonlinear dynamical system.

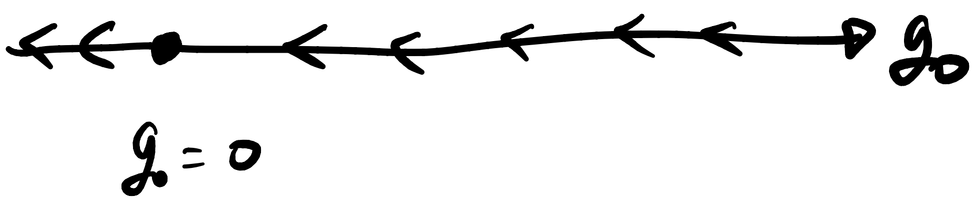
Def: A fixed point g_0^* of a flow is

$$0 = \dot{g}_0\Big|_{g_0^*} = \beta(g_0^*)$$

[Generalization: $\dot{g}_i = \beta_i(\vec{g}_0)$

$$0 = \beta_i(g_0^*) \qquad \forall i$$

In our example: $0 = \beta(g_0)\Big|_{g_0^*} \Longleftrightarrow g^* = 0$



for $\forall g_0$, $\dot{g}_0 = \beta(g_0) < 0$

ie for $g_0 > 0$, $g_0(\Lambda) \overset{\Lambda \gg \ldots}{\sim} \frac{1}{\log \Lambda^2} \to 0$

In the vicinity of an <u>Asymptotically Free</u> fixed point (ie $g_0(\Lambda) \to 0$ as $\Lambda \to \infty$)

the $e^{-1/g}$ stuff is small

$\Rightarrow$ $\beta(g_0)$ can be calculated perturbatively near an AF fixed point.

<u>Sign of β</u> : <u>Hep-th</u>: <u>arrows towards UV.</u> (large Λ)

<u>cond-mat/stat-mech</u>: arrows towards IR. $\beta^{cm} = -\beta^{Hep}$

$$S[q] = \int dt \left(\frac{\dot{q}^2}{2} + g_0 \delta^2(\vec{q}) \right)$$

has a scale symmetry $q(t) \mapsto s^\alpha q(st)$

1.2 A simple example of perturbative renormalization in QFT

$$\mathcal{L} = -\frac{1}{2}\phi \Box \phi - \frac{1}{2} m^2 \phi^2 - \frac{g}{4!}\phi^4$$

$0 = [S] \Longrightarrow [\phi] = \frac{D-2}{2}$ $\qquad [m] = 1$

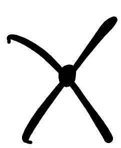
$[g] = \frac{4-D}{2}$

$= 0$ in $D=4$.

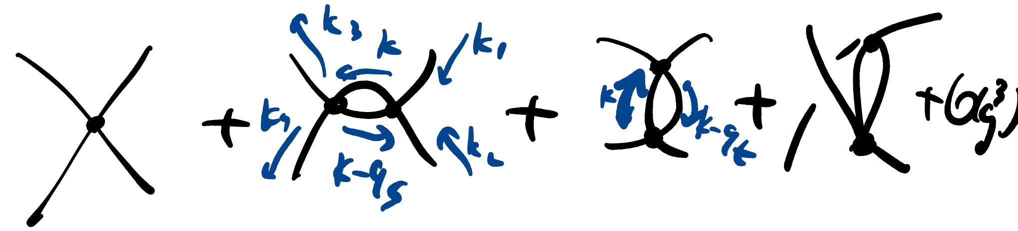
set $m = 0$.

Feynman rules:


$= \frac{i}{p^2 - m^2 + i\epsilon}$


$= -ig$

$$i\mathcal{M}_{2\leftarrow 2} =$$



$$q_s = k_1 + k_2 \qquad q_t = k_1 - k_3 \quad \cdots$$

$$= -ig + i\mathcal{M}_s + i\mathcal{M}_t + i\mathcal{M}_u + \mathcal{O}(g^3)$$

$$i\mathcal{M}_s = \frac{1}{2}(-ig)^2 \int d^4k \, \frac{i}{k^2 - m^2 + i\epsilon} \, \frac{i}{(q_s-k)^2 - m^2 + i\epsilon}$$

$$\sim \int^{\Lambda} \frac{d^4k}{k^4} \sim \log \Lambda .$$

<u>Parametrizing Ignorance</u>: many possible regulators. Find quantities that don't depend on choices.

<u>choose</u>: hard cutoff $\sum_{i=0}^{d} p_i^2 < \Lambda^2$.

Answer: $i\mathcal{M} = -ig + iCg^2 L + \mathcal{O}(g^3)$

$C = \frac{1}{16\pi^2}$. $\quad L \equiv \log\frac{\Lambda^2}{s} + \log\frac{\Lambda^2}{t} + \log\frac{\Lambda^2}{u}$

$\left(s = (k_1+k_2)^2 \quad t \equiv (k_1-k_3)^2 \quad u \equiv (k_1-k_4)^2 \right)$

Observables should be predicted in terms of other observables.

i.e. what is g?

Suppose we measure $i\mathcal{M}(s_0,t_0,u_0) \equiv -ig_P$.

$$\Rightarrow \quad -ig_P \overset{\text{measurement}}{=} i\mathcal{M}(s_0,t_0,u_0) \overset{\text{calc.}}{=} -ig + iCg^2 L_0 + \mathcal{O}(g^3)$$

$$\left(L_0 \equiv \log\frac{\Lambda^2}{s_0} + \log\frac{\Lambda^2}{t_0} + \log\frac{\Lambda^2}{u_0} \right) \qquad \star$$

solve $\star$ for g to eliminate g

$$-ig = -ig_P - iC g^2 L_0 + \mathcal{O}(g^3) \quad ②$$

$$\uparrow ③ \qquad \text{ie } g = g_P + \mathcal{O}(g^2)$$

$$g = g_P + \mathcal{O}(g^2) \qquad = g_P + \mathcal{O}(g_P^2)$$

$$\overset{(1.7)}{=} \underbrace{-ig_P - iC g_P^2 L_0 + \mathcal{O}(g_P^3)}_{\text{Known, meaningful}}$$

<u>eliminate g from $\mathcal{M}$:</u>

$$i\mathcal{M}(s,t,u) = -ig + iC g^2 L + \mathcal{O}(g^3)$$

$$\overset{(1.7)}{=} (-ig_P - iC g_P^2 L_0) + iC g_P^2 L + \mathcal{O}(g_P^3)$$

$$L - L_0$$

$$= \underbrace{\log\frac{\Lambda^2}{s} - \log\frac{\Lambda^2}{s_0}}_{} + (s\to t) + (s \to u)$$

$$= \log \frac{\cancel{\Lambda^2}}{s}\frac{s_0}{\cancel{\Lambda^2}} = \log\frac{s_0}{s}$$

$$i\mathcal{M}(s,t,u) = -ig_p + iCg_p^2 \log\frac{s_0}{s}\frac{t_0}{t}\frac{u_0}{u} + \mathcal{O}(g_p^3)$$

<u>Notice</u>: $$g = g(\Lambda) = g_p + C g_p^2 \log\frac{\Lambda^2}{s_0}\frac{\Lambda^2}{t_0}\frac{\Lambda^2}{u_0} + \mathcal{O}(g_p^3)$$

But also $g_p = g_p(s_0, t_0, u_0)$.

depends on on energy scale!

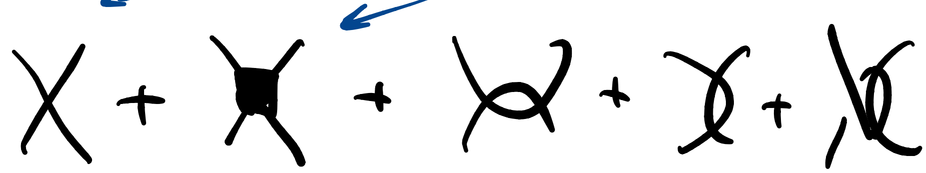
Renormalized perturbation theory (repackaging)

$$\mathcal{L} = -\frac{1}{2}\phi\Box\phi - \frac{g_p}{4!}\phi^4 - \frac{\delta_g}{4!}\phi^4$$

"counterterm".

<u>foresight</u>: $\delta_g \sim g_p^2$.

$$\rightarrow \mathcal{M}(s,t,u) = -g_p - \delta_g - C g_p^2 \left(\log\frac{stu}{\Lambda^6}\right) + \mathcal{O}(g_p^3)$$

=  + ...

"Renormalization condition": (def of g_p)

$$\mathcal{M}(s_0, t_0, u_0) \overset{!}{=} -g_p.$$

choose δ_g so that this is true.

$$\Rightarrow \quad \delta_g = -g_p^2\, C \left(\log \frac{s_0 t_0 u_0}{\Lambda^6}\right) + \mathcal{O}(g_p^3)$$

$$\Rightarrow \mathcal{M}(s,t,u) = -g_p - C g_p^2 \log \frac{stu}{s_0 t_0 u_0} + \mathcal{O}(g_p^3)$$

'bare coupling': $g_0 = g_p + \delta_g$ depends on Λ.

we can have one counterterm for each term in $\mathcal{L}$.

we need one renormalization condition for each counterterm.

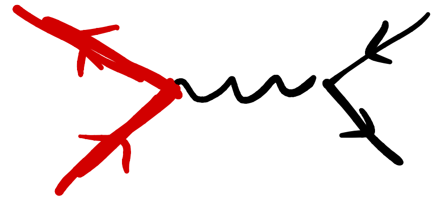
If we can remove all dependence on Λ at all orders w/ a finite # of counterterms then the theory is Renormalizable.

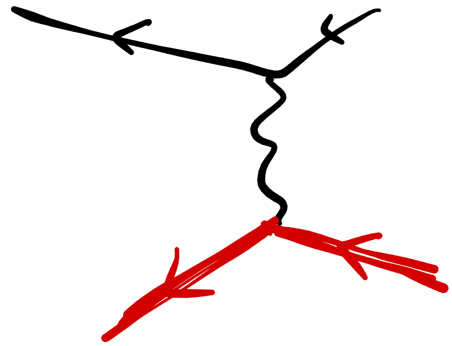
In this we learn nothing about short-distance physics.

Alternative: at each order in pert. thy more counterterms are required "effective field theory"

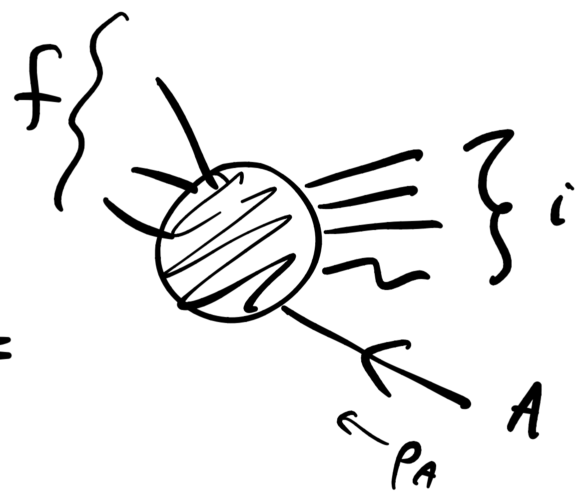
more precise answers teach us more high-energy physics.

1.3 Classical Interlude: Mott formula

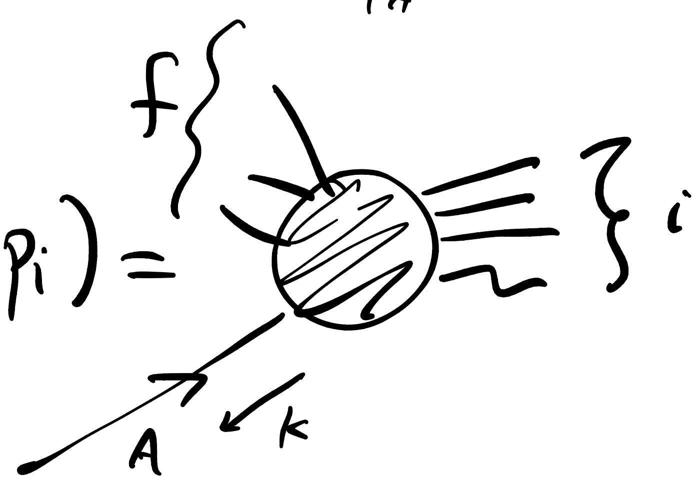
last quarter : $i\mathcal{M}_{\mu^+\mu^-\leftarrow e^+e^-} =$ 

$$i\mathcal{M}_{e^-\mu^-\leftarrow e^-\mu^-} =$$ 

Crossing symmetry :



$$i\mathcal{M}_{f\leftarrow iA}(p_f ; p_i ; p_A) =$$

$$= i\mathcal{M}_{\bar{A}f\leftarrow i}(p_f, k=-p_A ; p_i) =$$ 

$$\sum u_p \bar{u}_p = \not{p} + m$$

$$\sum v_k \bar{v}_k = \not{k} - m = -(\not{p} + m)$$

If A is a spinor,

$$\sum |\mathcal{M}|^2 = (-1)^{\#\text{ of fermions moved from in to out}} \sum |\mathcal{M}|^2$$

$$r p = -i r \frac{\partial}{\partial r}$$

$$r \frac{\partial}{\partial r} \left(r^{\Delta} \right) = \Delta \, r^{\Delta}$$

$$D \sim r \frac{\partial}{\partial r} + t \frac{\partial}{\partial t}$$