University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2022
Assignment 10 – Solutions

Due 11:59pm Monday, March 14, 2022

Thanks in advance for following the submission guidelines on hw01. Please ask me by email if you have any trouble.

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1. **When is the QCD interaction attractive?**

   Write the amplitude for *tree-level* scattering of a quark and antiquark of different flavors (say u and $\bar{d}$) in the t-channel (in Feynman $\xi = 1$ gauge). Compare to the expression for $e\bar{\mu}$ scattering in QED.

   First fix the initial colors of the quarks to be different – say the incoming u is red and the incoming $\bar{d}$ is anti-green. Show that the potential is repulsive.

   Now fix the initial colors to be opposite – say the incoming u is red and the incoming $\bar{d}$ is anti-red – so that they may form a color singlet. Show that the potential is attractive.

   Alternatively or in addition, describe these results in a more gauge invariant way, by characterizing the potential in the color-singlet and color-octet channels.

   You can do this problem either by choosing a specific basis for the generators of $\mathsf{SU}(3)$ in the fundamental (a common one is called the Gell-Mann matrices), or using more abstract group theory methods.

   Schwartz p.512.

   The t-channel diagram is identical to the QED amplitude with replacement

   $$e^2 \to g^2 T^a_{3,ij} T^a_{\bar{3},\bar{k}\bar{l}}$$

   where T^a_3 and $T^a_{\bar{3}}$ are the generators of $\mathsf{SU}(3)$ in the fundamental and antifundamental representations, respectively. We saw on a previous homework that these are related by $T_{\bar{3}} = -T^\star_3$.

   A nice way to think about this is: The tensor product of 3 and $\bar{3}$ representations decomposes into irreducible representations as $3 \otimes \bar{3} = 1 \oplus 8$, where the former is the singlet and the latter is the adjoint.

2. **Where to find a Chern-Simons term.**

Consider a field theory in $D = 2 + 1$ of a massive Dirac fermion, coupled to a background $\mathsf{U}(1)$ gauge field A with action:

$$S[\psi, A] = \int d^3x \bar{\psi} \left(\mathbf{i} \not{D} - m \right) \psi$$

where $D_\mu = \partial_\mu - \mathbf{i} A_\mu$.

(a) Convince yourself that the mass term for the Dirac fermion in $D = 2 + 1$ breaks parity symmetry. By parity symmetry I mean a transformation $\psi(x) \to \Gamma \psi(Ox)$ where $\det O = -1$, and Γ is a matrix acting on the spin indices, chosen so that this operation preserves $\bar{\psi} \not{\partial} \psi$.

First: the definition of parity is an element of $\mathsf{O}(d, 1)$ that's not in $\mathsf{SO}(d, 1)$, *i.e.* one with $\det(g) = -1$. In three spatial dimensions this is accomplished by $(t, \vec{x}) \to (t, -\vec{x})$. But in two spatial dimensions, this transformation has only two minus signs and so has determinant one – it is just a π rotation. (Certainly $\bar{\psi}\psi$ is invariant under it. And in fact Peskin's argument for the transformation of the Dirac field goes through exactly – it picks up a γ^0.) Instead we must do something like $(t, x, y) \to (t, x, -y)$ (other transformations are related by composing with a rotation).

Now we must figure out what this does to the Dirac spinor. First recall that the clifford algebra in $D = 2 + 1$ can be represented by 2×2 matrices (*e.g.* the Paulis, times some factors of $\mathbf{i}$ to get the squares right) and there is no notion of chirality, since the product of the three Paulis is proportional to the identity. We want an operation on $\psi(t, x, -y)$ which gives back the (massless) Dirac equation:

$$0 = \left(\gamma^0 \partial_t + \gamma^1 \partial_x + \gamma^2 \partial_y \right) \psi(t, x, -y) = \left(\gamma^0 \partial_t + \gamma^1 \partial_x - \gamma^2 \partial_{\tilde{y}} \right) \psi(t, x, \tilde{y})$$

with $\tilde{y} \equiv -y$. Inserting $1 = -\gamma_2^2$ before ψ we have

$$0 = \left(\gamma^0 \partial_t + \gamma^1 \partial_x - \gamma^2 \partial_{\tilde{y}} \right) \left(-\gamma_2^2 \right) \psi(t, x, \tilde{y}) = \gamma_2 \left(\gamma^0 \partial_t + \gamma^1 \partial_x + \gamma^2 \partial_{\tilde{y}} \right) \gamma_2 \psi(t, x, \tilde{y})$$

which is proportional to $\not{\partial} \gamma^2 \psi(\tilde{x}) = 0$. We conclude that $P \psi(t, x, y) P = \gamma^2 \psi(t, x, -y)$ will work (there is a sign ambiguity in the definition of the transformation).

This gives $\bar{\psi}\psi \mapsto \left(\psi^\dagger \gamma^{2\dagger} \right) \gamma^0 \gamma^2 \psi = \bar{\psi} \left(\gamma^2 \right)^2 \psi = -\bar{\psi}\psi$, while $\bar{\psi} \not{D} \psi \to \bar{\psi} \not{D} \psi$. Here we used $\left(\gamma^\mu \right)^\dagger \gamma^0 = \gamma^0 \gamma^\mu$, and $A_\mu(t, x, y) \to (A_0(t, x, -y), A_x(t, x, -y), -A_y(t, x, -y))_\mu$.

(b) We would like to study the effective action for the gauge field that results from integrating out the fermion field

$$e^{-S_{eff}[A]} = \int [D\psi D\bar{\psi}] e^{-S[\psi, A]}.$$

Focus on the term quadratic in A:

$$S_{eff}[A] = \frac{1}{2} \int đ^D q A_\mu(q) \Pi^{\mu\nu}(q) A_\nu(q) + ...$$

We can compute $\Pi^{\mu\nu}$ by Feynman diagrams[1]. Convince yourself that Π comes from a single loop of ψ with two A insertions.

(c) Evaluate this diagram using dim reg near $D = 3$. Show that, in the low-energy limit $q \ll m$ (where we can't make on-shell fermions),

$$\Pi^{\mu\nu} = a \frac{m}{|m|} \epsilon^{\mu\nu\rho} q_\rho + ...$$

for some constant a. Find a. Convince yourself that in position space this is a Chern-Simons term with level $k = \frac{1}{2}\frac{m}{|m|}$.

[Hint: in $D = 2 + 1$, $\text{tr}\gamma^\mu\gamma^\nu\gamma^\rho = -2\epsilon^{\mu\nu\rho}$.]

The key ingredient is that in $D = 3$ we have $\text{tr}\gamma^\mu\gamma^\nu\gamma^\rho = -2\epsilon^{\mu\nu\rho}$, as you can check for the basis we chose above with the Pauli matrices. Note that this would have been zero in $D = 4$, as in Peskin's calculation on page 247-248. The answer in $D = 2 + 1$ is then the answer for general D plus this extra term, which also has a factor of m since it comes from expanding out the numerators of the electron propagators:

$$\Pi_2(q)^{\mu\nu} = ... - \frac{\mathbf{i}e^2}{(4\pi)^{D/2}} \int_0^1 dx \frac{\Gamma(2 - D/2)}{\Delta^{1/2}} \text{tr}\gamma^\mu\gamma^\nu\gamma^\rho m \left((p+q)_\rho - p_\rho\right) \quad (1)$$

$$= ... + \frac{\mathbf{i}e^2}{4\pi} \frac{m}{|m|} \epsilon^{\mu\nu\rho} q_\nu + ... \quad (2)$$

where the ... is all the terms that are there in other dimensions, plus also the terms from expanding in $m^2 \gg q^2$.

The effective action is then

$$S_{\text{eff}}[A] = \frac{1}{2} \int đ^3 q A_\mu(q) \Pi^{\mu\nu}(q) A_\nu(-q) \quad (3)$$

$$= \frac{e^2}{8\pi^2} \text{sign}(m) \int đ^3 A_\mu(q) A_\nu(-q) \epsilon^{\mu\nu\rho} q_\rho \quad (4)$$

$$= \frac{e^2}{8\pi^2} \text{sign}(m) \int A \wedge dA. \quad (5)$$

Clearly this shows that the mass term is odd under parity, since the Chern-Simons term it generates is proportional to $\text{sign}(m)$.

[1] The thing I've called $\Pi^{\mu\nu}$ here is actually twice the vacuum polarization. Sorry.

(d) Redo this calculation by doing the Gaussian path integral over ψ.

Roughly:

$$\int [D\psi D\bar{\psi}] e^{S[\psi,\bar{\psi},A]} = \det\left(\mathbf{i}\not{D} - m\right) = e^{\text{tr}\log\left(\mathbf{i}\not{D}-m\right)}.$$

Therefore

$$S_{\text{eff}}[A] = \text{Tr}\ \log\left(\mathbf{i}\not{\partial} - \not{A} - m\right) = \text{Tr}\ \log\left(\mathbf{i}\not{\partial} - m\right)\left(1 + \not{A}\left(\mathbf{i}\not{\partial} - m\right)^{-1}\right).$$

The trace Tr is over the space on which $\mathbf{i}\not{D} - m$ acts, which is the space of spinor-valued functions. So it includes the spinor trace tr as well as a sum $\int d^3x$ or $\int đ^d p$. Note that the term linear in A is the familiar tadpole diagram, which vanishes by charge conjugation symmetry or Furry's theorem. We need to expand this in A to second order to get Π, and, using

$$A(\hat{x}) = \int đp e^{-\mathbf{i}p\hat{x}}, \quad f(\mathbf{i}\partial) = \int đq |q\rangle\langle q| f(q)$$

the result is

$$S_{\text{eff}}[A] = ... + \frac{1}{2}\int d^3x \langle x|\, \text{tr}\not{A}\left(\mathbf{i}\not{\partial} - m\right)^{-1}\not{A}\left(\mathbf{i}\not{\partial} - m\right)^{-1}|x\rangle \tag{6}$$

$$= ... + \frac{1}{2}\int d^3x \int đ^3 p_{1,2} \int đ^3 q_{1,2} e^{-\mathbf{i}q_1 x}\langle x|p_1\rangle \underbrace{\langle p_1|\, e^{-\mathbf{i}q_2\hat{x}}\, |p_2\rangle}_{=\int d^3y e^{-\mathbf{i}q_2 y - \mathbf{i}p_1 y + \mathbf{i}p_2 y} = \not{\delta}^3(q_2 - p_1 + p_2)} \langle p_2|x\rangle$$

$$\text{tr}\left(\not{A}(q_1)\left(\not{p}_1 - m\right)^{-1}\not{A}(q_2)\left(\not{p}_2 - m\right)^{-1}\right) \tag{7}$$

$$= ... + \frac{1}{2}\int đ^d q A_\mu(q) A_\nu(-q) \int đ^d p\, \text{tr}\left(\gamma^\mu \frac{1}{\not{p} - m}\gamma^\nu \frac{1}{\not{p} - \not{q} - m}\right) \tag{8}$$

which is the same as the diagrammatic calculation above.

3. **A bit more about Chern-Simons theory.**

Consider again $\mathsf{U}(1)$ gauge theory in $D = 2+1$ dimensions with the Chern-Simons action

$$S[a] = \frac{k}{4\pi}\int_\Sigma a \wedge da.$$

(Here I've changed the name of the dynamical gauge field to a lowercase a to distinguish it from the electromagnetic field A which will appear anon.)

(a) Show that the Chern-Simons action is gauge invariant under $a \to a + d\lambda$, as long as there is no boundary of spacetime Σ. Compute the variation of the action in the presence of a boundary of Σ.

(b) [bonus] Actually, the situation is a bit more subtle than the previous part suggests. The actual gauge transformation is

$$a \to g^{-1} a g + \frac{1}{\mathbf{i}} g^{-1} dg$$

which reduces to the previous if we set $g = e^{\mathbf{i}\lambda}$. That expression, however, ignores the global structure of the gauge group (*e.g.* in the abelian case, the fact that g is a periodic function). Consider the case where spacetime is $\Sigma = S^1 \times S^2$, and consider a *large gauge transformation*:

$$g = e^{\mathbf{i} n \theta}$$

where θ is the coordinate on the circle. Show that the variation of the CS term is $-\mathbf{i}\frac{k}{4\pi} \int g^{-1} dg \wedge f$ (where $f = da$). Since the action appears in the path integral in the form $e^{\mathbf{i}S}$, convince yourself that the path integrand is gauge invariant if

(1) $\int_\Gamma f \in 2\pi\mathbb{Z}$ for all closed 2-surfaces Γ in spacetime, and

(2) $k \in 2\mathbb{Z}$ – the Chern-Simons level is quantized as an *even* integer.

The first condition is called flux quantization, and is closely related to Dirac's condition.

The quantization of the level k, i.e. the Chern-Simons coupling has a dramatic consequence: it means that this coupling constant cannot be renormalized by a little bit, only by an integer shift. This is an enormous constraint on the dynamics of the theory.

(c) [bonus] In the case where G is a non-abelian lie group, the argument for quantization of the level k is more straightforward. Show that the variation of the CS Lagrangian

$$\mathcal{L}_{CS} = \frac{k}{4\pi} \mathrm{tr} \left(a \wedge da + \frac{2}{3} a \wedge a \wedge a \right)$$

under $a \to g a g^{-1} - dg g^{-1}$ is

$$\mathcal{L}_{CS} \to \mathcal{L}_{CS} + \frac{k}{4\pi} \mathrm{d} \mathrm{tr} dg g^{-1} \wedge a + \frac{k}{12\pi} \mathrm{tr} \left(g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg \right).$$

The integral of the second term over any closed surface is an integer. Conclude that $e^{\mathbf{i}S_{CS}}$ is gauge invariant if $k \in \mathbb{Z}$.

The first term integrates to zero on a closed manifold. The second term is the winding number of the map $g : \Sigma \to \mathsf{G}$

(d) Now we return to the abelian case (for an extra challenge, redo this part in the non-Abelian case). If there is a boundary of spacetime, something must be done to fix up this problem. Consider the case where $\Sigma = \mathbb{R} \times \text{UHP}$ where $\mathbb{R}$ is the time direction, and UHP is the upper half-plane $y > 0$. One way to fix the problem is simply to declare that the would-be gauge transformations which do not vanish at $y = 0$ are not redundancies. This means that they represent physical degrees of freedom. Plug in $a = d\phi$ to the Chern-Simons action (where $\phi(x, y \to 0) \equiv \phi(x)$ is a scalar field, and d is the exterior derivative on the spatial manifold) to find the action for ϕ.

It was misleading of me to say 'plug in $a = d\phi$' for the following reason. The exterior derivative on this spacetime decomposes into $d = \partial_t dt + \tilde{d}$ where $\tilde{d}$ is just the spatial part, and similarly the gauge field is $a = a_0 dt + \tilde{a}$. Let us choose the gauge $a_0 = 0$. We must still impose the equations of motion for a_0 (in the path integral it is a Lagrange multiplier) which says $\tilde{d}\tilde{a} = 0$ (just the spatial part). This equation is solved by $\tilde{a} = \tilde{d}\phi$ (or rather $\tilde{a} = g^{-1}dg$ where g is a $\mathsf{U}(1)$-valued function). This is pure gauge except at the boundary. Plugging this into the CS term gives

$$S = \frac{k}{4\pi}\int_{\mathbb{R}\times D} \tilde{a} \wedge \left(dt\partial_t + \tilde{d}\right)\tilde{a} \tag{9}$$

$$= \frac{k}{4\pi}\int_{\mathbb{R}\times D} \tilde{d}\phi \wedge dt\partial_t \tilde{d}\phi \tag{10}$$

$$= \frac{k}{4\pi}\int_{\mathbb{R}\times D} \tilde{d}\left(\phi \wedge dt\partial_t \tilde{d}\phi\right) \tag{11}$$

$$\overset{\text{Stokes}}{=} \frac{k}{4\pi}\int_{\mathbb{R}\times \partial D} \phi dt \partial_t \tilde{d}\phi \tag{12}$$

$$= \frac{k}{4\pi}\int_{\mathbb{R}\times \partial D} dx dt \phi \partial_t \partial_x \phi \tag{13}$$

$$\overset{\text{IBP}}{=} -\frac{k}{4\pi}\int_{\mathbb{R}\times \partial D} dx dt \partial_x \phi \partial_t \phi. \tag{14}$$

We can also add local terms at the boundary to the action. Consider adding $\Delta S = g\int_{\partial\Sigma} a_x^2$ (for some coupling constant g). Find the equations of motion for ϕ.

This term evaluates to $\Delta S = \int_{\partial\Sigma} v \left(\partial_x \phi\right)^2$. Altogether we now have

$$S_{\text{edge}}[\phi] = \int_{y=0} dx dt \partial_x \phi \left(\frac{k}{4\pi}\partial_t \phi + g\partial_x \phi\right).$$

The EoM is then

$$\frac{\delta}{\delta\phi(x)} S_{\text{edge}}[\phi] = \partial_t \left(\frac{k}{4\pi}\partial_t \phi + g\partial_x \phi\right)$$

which is solved if $\frac{k}{4\pi}\partial_t\phi + g\partial_x\phi = 0$. This describes a dispersionless wave which moves only in the $\text{sign}k$ direction – a chiral bosonic edge mode.

I should mention that this physics is realized in integer quantum Hall states and incompressible fractional quantum Hall states. For more, I recommend the textbook by Xiao-Gang Wen.

Interpretation: the Chern-Simons theory on a space with boundary necessarily produces a chiral edge mode.

(e) Suppose we had a system in $2+1$ dimensions with a gap to all excitations, which breaks parity symmetry and time-reversal invariance, and involves a conserved current J^μ, with

$$0 = \partial^\mu J_\mu. \tag{15}$$

Solve this equation by writing $J^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\rho}\partial_\nu a_\rho$ in terms of a one-form $a = a_\mu dx^\mu$. Guess the leading terms in the action for a_μ in a derivative expansion. You may assume Lorentz invariance.

Well, the CS term has dimension 3 so is marginal. It has just the right symmetries. We can also add a Maxwell term, but that has dimension 4 so we can ignore it at low energies. This argument that the CS theory is the low-energy effective action for incompressible quantum Hall states is due to Wen and Zee.

(f) Now suppose the current J^μ is coupled to an external electromagnetic field A_μ by $S \ni \int J^\mu A_\mu$. Ignoring the Maxwell term for a, compute the Hall conductivity, σ^{xy}, which is defined by Ohm's law $J^x = \sigma^{xy}E^y$.

Using the action

$$S[a, A] = \int \left(\frac{k}{4\pi} a \wedge da + J^\mu A_\mu\right) = \int d^3x \frac{k}{4\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho + \epsilon^{\mu\nu\rho}\partial_\nu a_\rho A_\mu$$

we find the EoM

$$0 = \frac{\delta S}{\delta a} \propto \frac{k}{2\pi} f^{\mu\nu} + F^{\mu\nu}.$$

Using $J = \star da$ we can rewrite this as

$$F^{\mu\nu} = \frac{k}{2\pi}\epsilon^{\mu\nu\rho}J_\rho.$$

The components of this equation with $\mu, \nu = 0, i)$ say $E^i = \frac{k}{2\pi}\epsilon^{ij}J_j$ or

$$J_j = \frac{2\pi}{k}\epsilon_{ij}E^j$$

which says $\sigma^{xy} = \frac{4\pi}{k}$ (in natural units, which means $\sigma^{xy} = \frac{1}{k}\frac{e^2}{h}$).