University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2022
Assignment 7

Due 11:59pm Monday, February 21, 2022

Thanks in advance for following the submission guidelines on hw01. Please ask me by email if you have any trouble.

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1. **Yukawa couplings in QED.** [optional] Consider adding to QED an additional scalar field of (physical) mass m, coupled to the electron by

$$L_Y = \lambda \phi \bar{\psi} \psi.$$

Verify that the divergent contribution to the electron wavefunction renormalization factor Z_2 from a virtual ϕ equals the divergent contribution to the QED vertex Z_1 from the one loop correction to the vertex with a virtual ϕ. For an added challenge, verify that the finite parts agree as well.

2. **Another consequence of unitarity of the S matrix.**

   (a) Show that unitarity of S, $S^\dagger S = \mathbb{1} = SS^\dagger$, implies that the transition matrix is *normal*:
$$\mathcal{T}\mathcal{T}^\dagger = \mathcal{T}^\dagger \mathcal{T} \ . \tag{1}$$

   (b) What does this mean for the amplitudes $\mathcal{M}_{\alpha\beta}$ (defined as usual by $\mathcal{T}_{\alpha\beta} = \oint(p_\alpha - p_\beta)\mathcal{M}_{\alpha\beta}$)?

   (c) The probability of a transition from α to β is

$$P_{\alpha\to\beta} = |S_{\beta\alpha}|^2 = VT\oint(p_\alpha - p_\beta)|\mathcal{M}_{\alpha\beta}|^2$$

which is IR divergent. More useful is the transition rate per unit time per unit volume:

$$\Gamma_{\alpha\to\beta} \equiv \frac{P_{\alpha\to\beta}}{VT}.$$

Show that the the total decay rate of the state α is

$$\Gamma_\alpha \equiv \int d\beta \Gamma_{\alpha\to\beta} = 2\mathrm{Im}\mathcal{M}_{\alpha\alpha}.$$

(d) Consider an ensemble of states p_α evolving according to the evolution rule

$$\partial_t p_\alpha = -p_\alpha \Gamma_\alpha + \int d\beta p_\beta \Gamma_{\beta\to\alpha}. \tag{2}$$

$S[p] \equiv -\int d\alpha p_\alpha \ln p_\alpha$ is the Shannon entropy of the distribution. Show that

$$\frac{dS}{dt} \geq 0$$

as a consequence of (1). This is a version of the Boltzmann H-theorem.

(e) [Bonus] Notice that we are doing something weird in the previous part by using classical probabilities. This is a special case; more generally, we should describe such an ensemble by a density matrix $\rho_{\alpha\beta}$. Generalize the result of the previous part appropriately.

3. **An application of effective field theory in quantum mechanics.**

Consider a model of two canonical quantum variables ($[\mathbf{x}, \mathbf{p}_x] = \mathbf{i} = [\mathbf{y}, \mathbf{p}_y], 0 = [\mathbf{x}, \mathbf{p}_y] = [\mathbf{x}, \mathbf{y}]$, etc) with Hamiltonian

$$\mathbf{H} = \mathbf{p}_x^2 + \mathbf{p}_y^2 + \lambda \mathbf{x}^2 \mathbf{y}^2.$$

(This is similar to the degenerate limit of the model studied in lecture with two QM variables where both natural frequencies are taken to zero.)

(a) Based on a semiclassical analysis, would you think that the spectrum is discrete or continuous?

(b) Study large, fixed x near $y = 0$. We will treat x as the slow (= low-energy) variable, while y gets a large restoring force from the background x value. Solve the y dynamics, and find the groundstate energy as a function of x:

$$V_{\text{eff}}(x) = E_{\text{g.s. of y}}(x).$$

(c) [Bonus] Presumably you did the previous part using your knowledge of the spectrum of the harmonic oscillator. Redo the previous part using path integral methods.

(d) The result for $V_{\text{eff}}(x)$ is not analytic in x at $x = 0$. Why?

(e) Is the spectrum of the resulting 1d model with

$$\mathbf{H}_{\text{eff}} = \mathbf{p}_x^2 + V_{\text{eff}}(\mathbf{x})$$

discrete? Is this description valid in the regime that matters for the semiclassical analysis?

[Bonus: determine the spectrum of $\mathbf{H}_{\text{eff}}$.]