University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2022
Assignment 6 – Solutions

Due 11:59pm Monday, February 14, 2022

Thanks in advance for following the submission guidelines on hw01. Please ask me by email if you have any trouble.

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1. **Another consequence of the optical theorem.**

   A general statement of the optical theorem is:

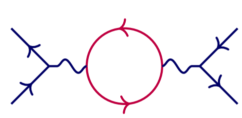
$$-\mathbf{i}\left(\mathcal{M}(a\to b)-\mathcal{M}(b\to a)\right)=\sum_f\int d\Phi_f\mathcal{M}^\star(b\to f)\mathcal{M}(a\to f)\ .$$

   Consider QED with electrons and muons.

   (a) Consider scattering of an electron (e^-) and a positron (e^+) into e^-e^+ (so $a=b$ in the notation above). We wish to consider the contribution to the imaginary part of the amplitude for this process which is proportional to $Q_e^2Q_\mu^2$ where Q_e and Q_μ are the electric charges of the electron and muon (which are in fact numerically equal but never mind that). Draw the relevant Feynman diagram, and compute the imaginary part of this amplitude $\mathrm{Im}\,\Pi_\mu(q^2)$ (just the $Q_e^2Q_\mu^2$ bit) as a function of $s\equiv(k_1+k_2)^2$ where $k_{1,2}$ are the momenta of the incoming e^+ and e^-.

   Check that the imaginary part is independent of the cutoff.

   There are a number of diagrams at this order, but the only one that contributes an imaginary part at finite s is the s-channel diagram with a muon loop, that is, where we insert into the photon propagator in the tree level s-channel diagram the contribution to the vacuum polarization from a muon loop (in red):



   The key ingredient we've calculated already:

$$\delta\Pi_2(q^2)=\Pi_2(q^2)-\Pi_2(0)=\frac{e^2}{2\pi^2}\int_0^1 dx x(1-x)\log\left(\frac{m^2-x(1-x)q^2}{m^2}\right).$$

(Note that without the fermion-loop minus sign, the sign would be opposite.) The imaginary part $\text{Im}\,\delta\Pi_2(q^2+\mathbf{i}\epsilon)$ comes from the locus where the argument of the log is negative (in which case the imaginary part is π), which happens when $m^2 - x(1-x)q^2 < 0$, which happens when $x \in [x_-, x_+] \equiv [\frac{1}{2} - \sqrt{1-m^2/q^2}, \frac{1}{2} + \sqrt{1-m^2/q^2}]$. So

$$\text{Im}\,\delta\Pi_2(q^2) = -\frac{e^2}{2\pi^2}\int_{x_-}^{x_+} dx x(1-x)\pi = -\frac{\alpha}{3}\sqrt{1-4m^2/q^2}\left(1+\frac{2m^2}{q^2}\right).$$

Note that there is also a t-channel diagram proportional to $Q_e^2 Q_\mu^2$, but it does not have an imaginary part.

(b) Use the optical theorem and the fact that the total cross section for $e^+e^- \to \mu^+\mu^-$ must be positive

$$\sigma(e^+e^- \to \mu^+\mu^-) \geq 0$$

to show that a Feynman diagram with a fermion loop must come with a minus sign. Check that with the correct sign, the optical theorem is verified.

Consider forward scattering of e^+e^-, and average over initial spins using

$$\frac{1}{4}\sum_{spins} \bar{u}(k)\gamma^\mu v(k_+)\bar{v}(k_+)\gamma_\mu u(k) = -k\cdot k_+ - 4m_e^2 \simeq -(k+k_+)^2 = -s.$$

(Notice that this is negative!) Recalling that $\Pi_2^{\mu\nu} = q^2\eta^{\mu\nu}\mathbf{i}\Pi_2(q^2)+$ longitudinal terms, gives

$$\text{Im}\,\mathcal{M} = -\frac{s^2}{s^2}\text{Im}\,\Pi_2(q^2) \tag{1}$$

$$= \frac{e^4}{12\pi}\sqrt{1-\frac{4m^2}{q^2}}\left(1+\frac{2m^2}{q^2}\right) = 2E_{cm}p_{cm}\sigma_{e^+e_-\to L^+L^-} \overset{E\gg m_e}{\simeq} 2s\sigma_{e^+e_-\to L^+L^-}. \tag{2}$$

If we left out the minus sign, we would get a negative cross section. In fact, this is how Feynman first figured out this particular Feynman rule.

2. **Bubble-chain approximation for bound states.**

In discussing the form of the spectral density for an operator which creates a massive particle, I mentioned that in addition to the single-particle delta function at $s = m^2$, and the continuum above $s = (2m)^2$, there could be delta functions associated with bound states at $m^2 < s < (2m)^2$. Here we'll get an idea how we might discover such a thing theoretically.

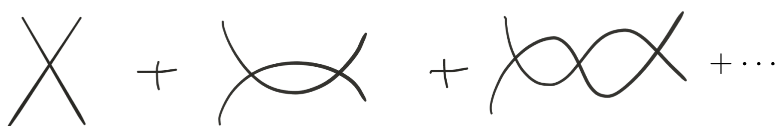
For this problem, we're going to work in $D = 2+1$ dimensions, so that we can avoid the problem of UV divergences. Consider the theory of a single real scalar with action

$$S[\phi] = \int d^3x \left(\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}m^2\phi^2 - \frac{g}{4!}\phi^4 \right)$$

where m, g are real. In this problem we will consider both signs of g, without worrying about questions of the stability of the vacuum.

I got this problem from Lawrence Hall.

(a) Consider the amplitude $\mathcal{M}(s)$ for elastic scattering $\phi\phi \to \phi\phi$, with $s = E_T^2$, the square of the total center of mass energy. Compute $\mathcal{M}(s)$ in the bubble-chain approximation, defined as the following infinite sum of Feynman diagrams:



Do not worry about justifying the validity of the approximation (it is not justified in this theory, though it is in a large-n version of the theory), and do not worry about convergence of the series. You can leave your answer as a Feynman parameter integral.

$$\begin{aligned}
\mathbf{i}\mathcal{M} &= -\mathbf{i}g + (-\mathbf{i}g)^2\mathcal{I}(s) + (-\mathbf{i}g)^3\mathcal{I}(s)^2 + \cdots &(3)\\
&= -\mathbf{i}g\left(1 + (-\mathbf{i}g)\mathcal{I}(s) + (-\mathbf{i}g)^2\mathcal{I}(s)^2 + \cdots\right) &(4)\\
&= \frac{-\mathbf{i}g}{1 - \mathbf{i}g\mathcal{I}(s)} &(5)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{I}(s) &\equiv \frac{1}{2}\int đ^D k \frac{\mathbf{i}}{k^2 - m^2 + \mathbf{i}\epsilon}\frac{\mathbf{i}}{(q-k)^2 - m^2 + \mathbf{i}\epsilon} &(6)\\
&= \frac{1}{2}\int_0^1 dx \int \frac{đ^D \ell}{(\ell^2 - \Delta)^2} &(7)\\
&= -\frac{1}{2}\int_0^1 dx \frac{\mathbf{i}}{(4\pi)^{D/2}}\frac{\Gamma\left(2 - \frac{D}{2}\right)}{\Gamma(2)}\Delta^{\frac{D}{2}-2} &(8)\\
&\overset{D=3}{=} \int_0^1 dx \frac{-\mathbf{i}}{16\pi\sqrt{\Delta}} &(9)
\end{aligned}$$

and $\Delta = m^2 - x(1-x)q^2 - \mathbf{i}\epsilon$. The factor of $\frac{1}{2}$ in (6) is a symmetry factor from exchanging the two internal lines of the bubble. So

$$\mathcal{M}(s) = \frac{-g}{1 - \mathbf{i}g\mathcal{I}} = \frac{-g}{1 + gX(s)} \tag{10}$$

with $X(s) \equiv \int_0^1 dx \frac{1}{16\pi\sqrt{\Delta}}$.

(b) Show, by explicit calculation, that the bubble chain approximation to the scattering amplitude obeys the optical theorem. [In elastic scattering in the center of mass frame in 3d, the element of solid angle $d\Omega$ is just an element of ordinary angle $d\theta$, and $d\sigma/d\theta = \frac{|\mathcal{M}|^2}{32\pi p E_T^2}$ where p is the magnitude of the spatial momentum of either particle.]

The cross section is

$$\frac{d\sigma}{d\theta} = \frac{|\mathcal{M}|^2}{32\pi p E_T^2}$$

independent of θ, so

$$\sigma = \pi \cdot \frac{|\mathcal{M}|^2}{32\pi p s}$$

where we only integrate θ from 0 to π because the two particles are identical. When $s < 4m^2$, the optical theorem is verified because the BHS is zero. The statement of the optical theorem for $s > 4m^2$ is

$$\text{Im}\,\mathcal{M} \stackrel{?}{=} 2p\sqrt{s}\sigma = \frac{|\mathcal{M}|^2}{16\sqrt{s}}.$$

It is useful to rewrite this (using $\text{Im}\,\mathcal{M}^{-1} = \frac{-\text{Im}\,\mathcal{M}}{|\mathcal{M}|^2}$) as

$$\text{Im}\,\mathcal{M}^{-1} = +\text{Im}\,X(s) \stackrel{?}{=} -\frac{1}{16\sqrt{s}}. \tag{11}$$

The imaginary part arises when the argument of the square root in X is negative. Actually, it's possible to get a closed-form expression for X (*e.g.* from Mathematica):

$$X = \frac{1}{16\pi}\int_0^1 dx \Delta^{-1/2} = \frac{1}{16\pi|m|}\int_0^1 dx\,\left(1 - 4x(1-x)w^2\right)^{-1/2} \tag{12}$$

$$= \frac{1}{16\pi m}\begin{cases} \frac{1}{2w}\log\left(\frac{1+w}{1-w}\right), & 0 < w < 1 \\ \frac{1}{2w}\left(\log\left(\frac{1+w}{1-w}\right) + \mathbf{i}\pi\right), & 1 < w \end{cases} \tag{13}$$

where we defined $w \equiv \frac{\sqrt{s}}{2m}$. So indeed (11) is verified.

(c) The interaction between the ϕ quanta could result in two of them forming a bound state of mass M_B. A signal of such a bound state is the appearance of a pole in $\mathcal{M}(s)$ at $s = M_B^2$ on the real axis, but below threshold ($0 < M_B^2 < 4m^2$). Find the values of g for which the bubble-chain approximation predicts bound states. [You are not asked to give an analytic expression for M_B.]

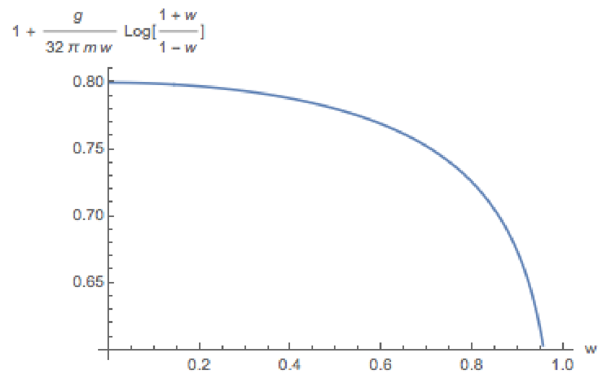
The condition for a boundstate is a pole in $\mathcal{M}(s)$ at $0 < s < 4m^2$ (which means $0 < w < 1$). Using (12) with $s < 4m^2$, this happens when the denominator in (10) is zero:

$$0 = 1 + gX(s) = 1 + \frac{g}{32\pi m w} \ln\left(\frac{1+w}{1-w}\right)$$

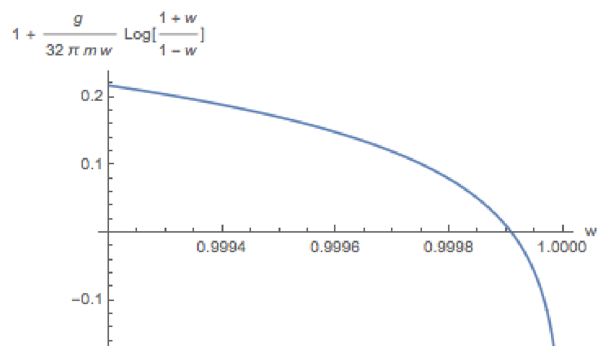
where $w \equiv \frac{\sqrt{s}}{2m} \in [0,1)$. We can rewrite this condition as

$$\frac{32\pi m w}{|g|} = \ln\left(\frac{1+w}{1-w}\right)$$

which is a bit too transcendental to solve analytically. Here is what the function $1 + \frac{g}{32\pi m w}\ln\left(\frac{1+w}{1-w}\right)$ looks like for $g/m = -.2 \times 16\pi$ (recall that g/m is dimensionless in $D = 3$):



For any $g < 0$, it is monotonically decreasing for $w \in (0,1)$. Looking closer near $w \sim 1$ we see that it crosses 0 at $w = w_\star(g)$ slightly less than 1:



The boundstate mass is $M_B = w_\star(g)2m$. It makes sense that we need $g < 0$ – an attractive interaction – to have a boundstate. When we make g too negative, the function starts out below the axis and stays there: for $g/m < -16\pi$, the interaction is too attractive and the boundstate mass^2 becomes negative, indicating that the boundstate would want to condense in the groundstate.