University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215B QFT Winter 2022
Assignment 5 – Solutions

Due 11:59pm Monday, February 7, 2022

Thanks in advance for following the submission guidelines on hw01. Please ask me by email if you have any trouble.

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1. If you didn't do problem 5 on homework 4 go back and do it now.

   I put my solution with homework 4.

2. Verify for yourself that the one-loop vacuum polarization amplitude in QED satisfies the Ward identity, *i.e.* is proportional to $q^\mu q^\nu - \eta^{\mu\nu} q^2$. It's up to you how much of this to hand in.

   The calculation is done using dim reg on pages 251-252 of Peskin and using PV in Zee (2d ed) pages 202-204.

3. **Soft gravitons?** [optional] Photons are massless, and this means that the cross sections we measure actually include soft ones that we don't detect. If we don't include them we get IR-divergent nonsense.

   Gravitons are also massless. Do we need to worry about them in the same way? Here we'll sketch some hints for how to think about this question.

   (a) Consider the action
   $$S_0[h_{\mu\nu}] = \int d^4x \frac{1}{2} h_{\mu\nu} \Box h^{\mu\nu}.$$
   This is a kinetic term for (too many polarizations of a) two-index symmetric-tensor field $h_{\mu\nu} = h_{\nu\mu}$ (which we'll think of as a small fluctuation of the metric about flat space: $g_{\mu\nu} \simeq \eta_{\mu\nu} + h_{\mu\nu}$, and this is where the coupling below comes from). Like with the photon, we'll rely on the couplings to matter to keep unphysical polarizations from being made. Write the propagator for h. We still raise and lower indices with $\eta_{\mu\nu}$. [1]

[1] A warning: I've done two misdeeds in the statement of this problem. First, the Einstein-Hilbert term is $\int d^4x \frac{1}{8\pi G_N}\sqrt{g}R = \int d^4x \frac{1}{8\pi G_N}(\partial h)^2 + ...$ – it has a factor of G_N in front of it. R has units of $\frac{1}{\text{length}^2}$, and g is dimensionless, so G_N has units of length^2 – it is $8\pi G_N = \frac{1}{M_{\text{Pl}}^2}$, where M_{Pl} is the Planck mass. I've absorbed a factor of $\sqrt{G_N}$ into h so that the coefficient of the kinetic term is unity. Second, the $(\partial h)^2$ here involves various index contractions, which I haven't written. Some gauge fixing (de Donder gauge) is required to arrive at the simple expression I wrote above, and one more thing –

The propagator is imply the inverse of the kinetic term. After the gauge fixing (implicit in the expression I gave) it is indeed invertible, just like in Maxwell theory. The graviton propagator you'll find on Wikipedia is the propagator for $h_{\mu\nu}$, rather than $\bar{h}_{\mu\nu}$.

(b) Couple the graviton to the electron field via

$$S_G = \int d^4x \; G h^{\mu\nu} T_{\mu\nu}$$
$$T_{\mu\nu} \equiv \bar{\psi} \left(\gamma_\mu \partial_\nu + \gamma_\nu \partial_\mu \right) \psi. \tag{1}$$

What are the engineering dimensions of the coupling constant G? What is the new Feynman rule?

$G = \sqrt{G_N}$ has dimensions of one over mass. This factor pops out of $L \sim \frac{1}{G_N}\sqrt{g}R + T_{\mu\nu}h^{\mu\nu}$ upon rescaling h to give it canonical kinetic terms.

(c) Draw a (tree level) Feynman diagram which describes the creation of gravitational radiation from an electron as a result of its acceleration from the absorption of a photon ($e\gamma \to eh$). Evaluate it if you dare. Estimate or calculate the cross section (hint: use dimensional analysis).

The two diagrams that contribute at tree level are similar to those appearing in Compton scattering. The external graviton in the final state comes with a polarization tensor $\epsilon^\star_{\mu\nu}$. Since we don't measure the polarization of the soft graviton, we want the polarization-summed cross-section. As for photons, the polarization sum

$$\sum_r \epsilon_r^{\mu\nu}(q)\epsilon_r^{\rho\sigma}(q) = \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} + \text{terms with } q^\mu \text{ or } q^\nu$$

takes the same form as the numerator of the propagator, up to ambiguous terms that vanish because of the Ward identity for the gauge invariance described below.

The amplitude has a single factor of G, so the probability goes like G^2 which has dimensions of length-squared, which is already the right dimensions for a cross section. Apparently, for energies large compared to the electron mass, this cross section is constant in energy.

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the $h_{\mu\nu}$ I've written is actually the 'trace-reversed' graviton field

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}$$

where $h \equiv \eta^{\mu\nu} h_{\mu\nu}$ is the trace. (I didn't write the bar.) For the details of this, which are not needed for this problem, see chapter 10 of my GR notes.

(d) Now the main event: study the one-loop diagram by which the graviton corrects the QED vertex. Is it IR divergent? If not, why not?

There are extra powers of the momentum in the numerator from the derivative coupling. This paper shows that this is not enough to prevent an IR divergence. So indeed, if we wish to include the (very small!) radiative corrections from gravitons, we must study inclusive amplitudes that allow for soft gravitons.

(e) If you get stuck on the previous part, replace the graviton field by a massless scalar $\pi(x)$. Compare the case of derivative coupling $\lambda \partial_\mu \pi \bar{\psi} \gamma^\mu \psi$ with the more direct Yukawa coupling $y \pi \bar{\psi} \psi$. [Warning: though this example has some similarities with the graviton case, the conclusion is different.]

In this case, the extra powers of the momentum in the numerator from the derivative coupling do prevent an IR divergence.

(f) Quite a bit about the form of the coupling of gravity to matter is determined by the demand of coordinate invariance. This plays a role like gauge invariance in QED. Acting on the small fluctuation, the transformation is

$$h_{\mu\nu}(x) \to h_{\mu\nu}(x) + \partial_\mu \lambda_\nu(x) + \partial_\nu \lambda_\mu(x).$$

What condition does the invariance under this (infinitesimal) transformation impose on the object $T_{\mu\nu}$ appearing in (1).

The variation of the action is

$$\delta S = \int d^4x \left(\partial_\mu \lambda_\nu(x) + \partial_\nu \lambda_\mu(x) \right) T^{\mu\nu} \overset{\text{IBP}}{=} -2 \int d^4x \lambda_\mu \partial_\nu T^{\mu\nu}$$

which vanishes if $\partial_\nu T^{\mu\nu} = 0$, *i.e.* if $T^{\mu\nu}$ is a conserved stress tensor.

4. **Equivalent photon approximation.** [optional] Consider a process in which very high-energy electrons scatter off a target. At leading order in α, the electron line is connected to the rest of the diagram by a single photon propagator. If the initial and final energies of the electron are E and E', the photon will carry momentum q with $q^2 = -2EE'(1-\cos\theta)$ (ignoring the electron mass $m \ll E$). In the limit of forward scattering ($\theta \to 0$), we have $q^2 \to 0$, so the photon approaches its mass shell. In this problem, we ask: To what extent can we treat it as a real photon?

(a) The matrix element for the scattering process can be written as

$$\mathcal{M} = -\mathbf{i} e \bar{u}(p') \gamma^\mu u(p) \frac{-\mathbf{i} \eta_{\mu\nu}}{q^2} \hat{\mathcal{M}}^\nu(q)$$

where $\hat{\mathcal{M}}^\nu$ represents the coupling of the virtual photon to the target. Let $q = (q^0, \vec{q})$ and define $\tilde{q} = (q^0, -\vec{q})$. The contribution to the amplitude from the electron line can be parametrized as

$$\bar{u}(p')\gamma^\mu u(p) = Aq^\mu + B\tilde{q}^\mu + C\epsilon_1^\mu + D\epsilon_2^\mu$$

where ϵ_α are unit vectors transverse to $\vec{q}$. Show that B is at most of order θ^2 (dot it with q), so we can ignore it at leading order in an expansion about forward scattering. Why do we not care about the coefficient A?

Dotting with q, the terms with ϵ vanish since the polarizations are transverse and the Ward identity gives $0 = Aq^2 + Bq_\mu\bar{q}^\mu$, but $q^2 = -2EE'(1-\cos\theta) \sim \theta^2$ when $\theta \ll 1$. Since $q_\mu\bar{q}^\mu$ is order 1, B must be order θ^2. The term with A drops out when we contract this with $\hat{\mathcal{M}}_\mu$, again by the Ward identity.

(b) Working in the frame with $p = (E, 0, 0, E)$, compute

$$\bar{u}(p')\gamma \cdot \epsilon_\alpha u(p)$$

explicitly using massless electrons, where $\bar{u}$ and u are spinors of definite helicity, and $\epsilon_{\alpha=\parallel,\perp}$ are unit vectors parallel and perpendicular to the plane of scattering. Keep only terms through order θ. Note that for $\epsilon_\parallel$, the (small) $\hat{3}$ component matters.

Choose definite helicity states, say

$$u^+(p) = \sqrt{2E}\begin{pmatrix}0 & 0 & 1 & 0\end{pmatrix}^t, u^-(p) = \sqrt{2E}\begin{pmatrix}0 & 1 & 0 & 0\end{pmatrix}^t$$

for $\vec{p} \propto \hat{z}$, $i.e. p^\mu = (E, 0, 0, E)^\mu$. The two definite-helicity spinors for momentum $p' = (E', 0, E'sin\theta, E'\cos\theta)$ are related by a spinor rotation by angle θ, so

$$u^+(p') = \sqrt{2E'}\begin{pmatrix}0 & 0 & \cos\theta/2 & \sin\theta/2\end{pmatrix}^t \simeq \sqrt{2E'}\begin{pmatrix}0 & 0 & 1 & \theta/2\end{pmatrix}^t,$$

$$u^-(p') = \sqrt{2E'}\begin{pmatrix}-\sin\theta/2 & \cos\theta/2 & 0 & 0\end{pmatrix}^t \simeq \sqrt{2E'}\begin{pmatrix}-\theta/2 & 1 & 0 & 0\end{pmatrix}^t.$$

We must also find expressions for the polarization vectors:

$$\epsilon_\perp = (0,0,1,0), \epsilon_\parallel \propto (0, p'\cos\theta - p, 0, p'\sin\theta) \propto (0, 1, 0, \frac{E'}{E'-E}\theta).$$

Then

$$\bar{u}'_\pm\gamma \cdot \epsilon_\parallel u_\pm = \pm\mathbf{i}\sqrt{EE'}\theta, \bar{u}'_\pm\gamma \cdot \epsilon_\parallel u_\pm = -\sqrt{EE'}\frac{E+E'}{E-E'}\theta,$$

and $\bar{u}'_\pm\gamma \cdot \epsilon_\alpha u_\mp = 0$. The key conclusion is that all the nonzero entries in $\bar{u}_\pm(p')\gamma \cdot \epsilon_\alpha u_\pm(p)$ are order θ.

(c) Now write the expression for the electron scattering cross section, in terms of $|\hat{\mathcal{M}}^\mu|^2$ and the integral over phase space of the target. This expression must be integrated over the final electron momentum $\vec{p}'$. The integral over $p^{3'}$ is an integral over the energy loss of the electron. Show that the integral over $p'_\perp$ diverges logarithmically as $p'_\perp$ or $\theta \to 0$.

We find $|\mathcal{M}|^2 \propto \theta^2$. Then

$$\sigma \propto \int_0 d\theta \sin\theta \frac{|\mathcal{M}|^2}{q^4} \sim \int_0 d\theta \frac{\theta^3}{\theta^4}$$

is log divergent.

(d) The divergence as $\theta \to 0$ is regulated by the electron mass (which we've ignored above). Show that reintroducing the electron mass in the expression

$$q^2 = -2(EE' - pp'\cos\theta) + 2m^2$$

cuts off the divergence and gives a factor of $\log(s/m^2)$ in its place.

Just replace the denominator q^4 with this regulated expression.

(e) Assembling all the factors, and assuming that the target cross sections are independent of photon polarization, show that the largest part of the electron-target cross section is given by considering the electron to be the source of a beam of real photons with energy distribution given by

$$N_\gamma(x)dx = \frac{dx}{x}\frac{\alpha}{2\pi}(1 + (1-x)^2)\log\frac{s}{m^2}$$

where $x \equiv E_\gamma/E$. This is the Weiszäcker-Williams equivalent photon approximation. It is a precursor to the theory of jets and partons in QCD.

5. **Spectral representation at finite temperature.**

In lecture we have derived a spectral representation for the two-point function of a scalar operator in the vacuum state

$$-\mathbf{i}\mathcal{D}(x) = \langle 0|\, \mathcal{T}\mathcal{O}(x)\mathcal{O}^\dagger(0)\, |0\rangle$$

Derive a spectral representation for the two-point function of a scalar operator in thermal equilibrium at a nonzero temperature T:

$$-\mathbf{i}\mathcal{D}_\beta(x) \equiv \mathrm{tr}\frac{e^{-\beta\mathbf{H}}}{Z_\beta}\mathcal{T}\mathcal{O}(x)\mathcal{O}^\dagger(0) = \frac{1}{Z_\beta}\sum_n e^{-\beta E_n}\langle n|\, \mathcal{T}\mathcal{O}(x)\mathcal{O}^\dagger(0)\, |n\rangle\,.$$

Here $Z_\beta \equiv \mathrm{tr} e^{-\beta\mathbf{H}}$ is the thermal partition function. Check that the zero temperature ($\beta \to \infty$) limit reproduces our previous result. Assume that $\mathcal{O} = \mathcal{O}^\dagger$ if you wish.

The idea is again to insert a resolution of the identity in between the operators. All the steps as for the vacuum correlators go through, the only difference being that instead of arriving at a sum of squares of matrix elements of the operator between the vacuum and an arbitrary state, we get matrix elements between pairs of states:

$$\mathbf{i}D(x) = Z_\beta^{-1} \sum_n e^{-\beta E_n} \sum_m \| \mathcal{O}_{nm} \|^2 \left(e^{\mathbf{i}x(p_n - p_m)} \theta(t) + e^{\mathbf{i}x(p_m - p_n)} \theta(-t) \right) \ .$$

From here, the momentum space representation follows as before. When $\beta \to \infty$, only the groundstate contributes (assume it is nondegenerate) and $Z_\beta \to e^{-\beta E_0}$.