Recall: Fiber bundle $E$

$$\pi : E \to B$$

① $\pi^{-1}(\mathcal{U}) \overset{\Phi}{\cong} \underline{\underline{\mathcal{U} \times F}}$

$\mathcal{U} \subset B$

② 
$$\begin{tikzcd}
\pi^{-1}(\mathcal{U}_\alpha) \arrow[r, "\phi_\alpha"] \arrow[dr, "\pi"'] & \mathcal{U}_\alpha \times F \arrow[d, "\text{forget } F"] \\
& \mathcal{U}_\alpha
\end{tikzcd}$$

A Vector bundle is a fiber bundle w/

③ $F$ is a vector space, $V$ of dim $r$ over $\mathbb{F}$

④ transition f'ns act linearly

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \times V \to \mathcal{U}_\alpha \cap \mathcal{U}_\beta \times V$$

$$(x, v) \mapsto (x, g_{\alpha\beta}(x) v)$$

$$g_{\alpha\beta} : \mathcal{U}_{\alpha\beta} \longrightarrow \underline{\underline{GL(r, \mathbb{F})}}$$

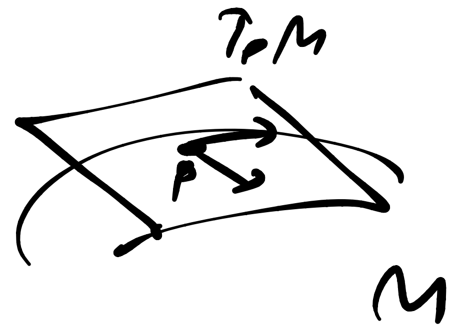
i.e. $r \times r$ Matrix nonsingular.

$r \equiv$ rank of v.b.

structure group of $E = \langle g_{\alpha\beta} \rangle \subset GL(n, \mathbb{F})$

eg: $TM$ = tangent bundle to a smooth $M$

$V = \pi^{-1}(p \in M) = T_pM$.




structure group of $TM$

$\equiv$ holonomy of $M$. $\subset O(n)$

if $M$ oriented $\subset SO(n)$.

complex mfld has holonomy $\subset U(n)$

Calabi-Yau mfld " " $\subset SU(n)$.

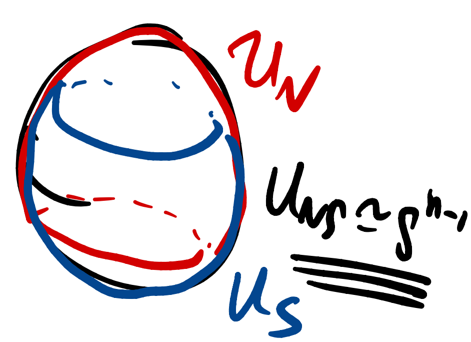
A v.b. is specified by $g_{\alpha\beta}$ on $U_{\alpha\beta}$

eg: A v.b. on $S^n$

is specified by a map $g_{NS} : U_{NS} \to G$

$\simeq g_{NS} : S^{n-1} \to G$.

$\longleftrightarrow \pi_{n-1}(G) / \pi_0(G)$



- If we choose $V$ to be a cx v.s.

$$g_{\alpha\beta}(x) \in GL(r, \mathbb{C})$$

$E$ is a $\underline{\underline{\text{cx}}}$ v.b.

- $E_1, E_2 \longrightarrow E_1 \oplus E_2$,

$E_1 \subset E_2 \longrightarrow E_2/E_1$

[ Def: Do these v.s. on the fibers. ]

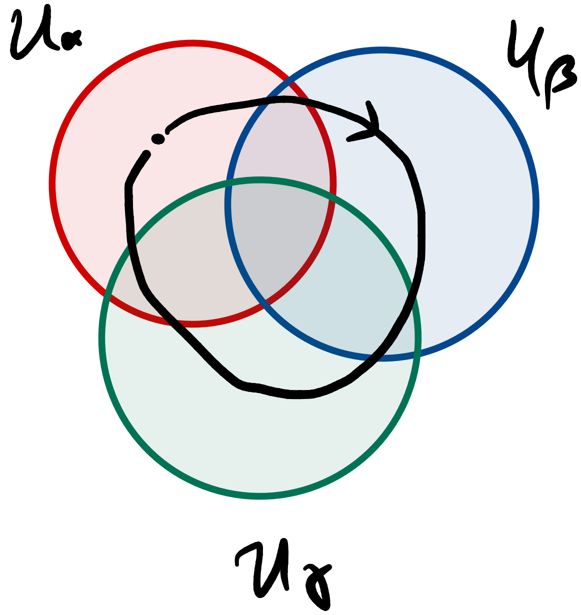
- Given a vb w/ some $\{g_{\alpha\beta}\}$ ← "principal G-bundle"

and a Rep/ of $G$ (the structure group) $D$

$$\begin{cases} V \longrightarrow \text{carrier space of } D & \dim(D) \\ g_{\alpha\beta}(x) \longrightarrow D(g_{\alpha\beta}(x))_{ij} & \end{cases}$$

$i, j = 1 .. \dim D$

"associated bundle".




$\exists\ U_{\alpha\beta\gamma}$

$\Rightarrow C$ is contractible.

$$g_{\gamma\alpha}\, g_{\beta\gamma}\, g_{\alpha\beta} = e$$

$$= (\delta g)_{\alpha\beta\gamma}$$

cocycle condition on $\{g_{\alpha\beta}\}$.

$$g_{\alpha\beta} \longrightarrow f_\alpha\, g_{\alpha\beta}\, f_\beta^{-1}$$

gives the same v.b.

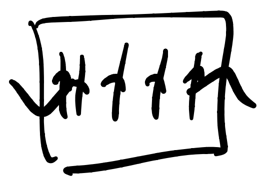
transition f'ns $\longleftrightarrow$ 1-st Čech cohomology w/ coeffs in $\underline{\underline{G}}$.

($G$ non-abelian ??)

Given $\{ g_{\alpha\beta} : U_{\alpha\beta} \to GL(r, \mathbb{F}) \}$

satisfying $(\delta g)_{\alpha\beta\gamma} = 1$ on $U_{\alpha\beta\gamma}$

$$E = \coprod_{\alpha} U_\alpha \times \mathbb{F}^r \Big/ (x,v) \sim (x, g_{\alpha\beta}(x)\cdot v) \text{ on } U_{\alpha\beta}.$$



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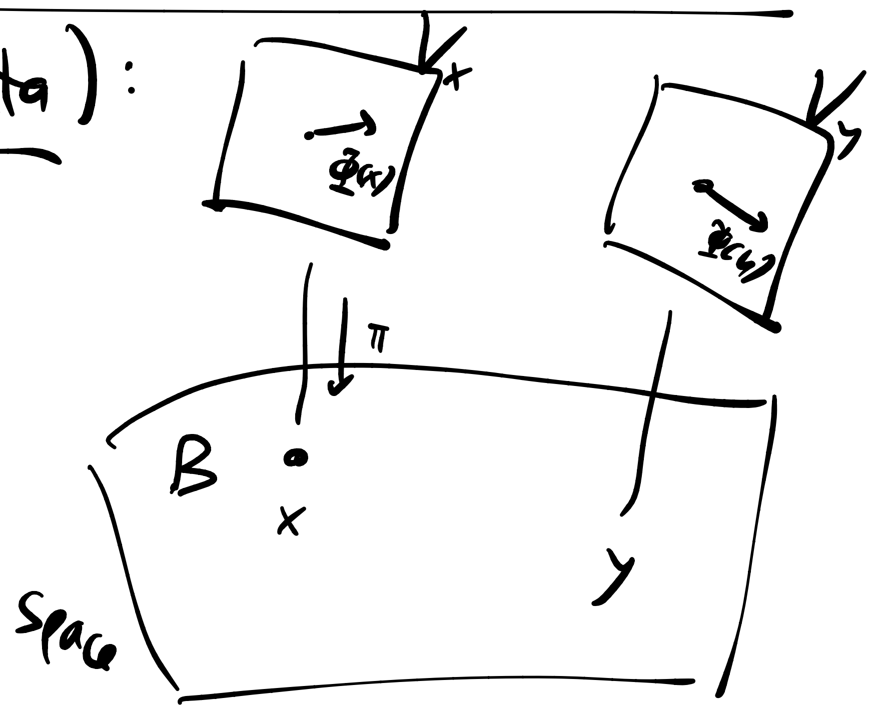
Connection (extra data):

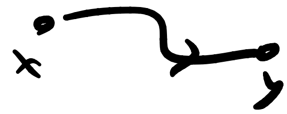
$\Phi_\alpha(x) \mapsto \Lambda_{\alpha\beta}(x)\,\Phi_\beta(x)$

$\in V_x$.




need to compare $\Phi(x)$ w $\Phi(y)$

to do this requires a comparator or connection:

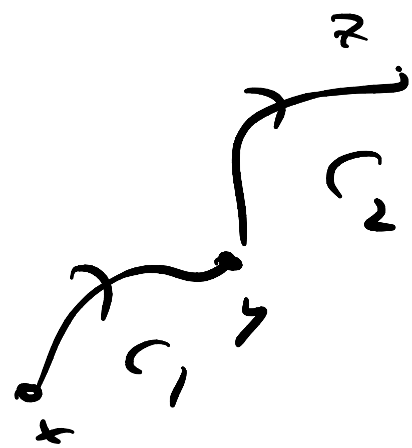
$\underline{W_{xy} \in G}$ 

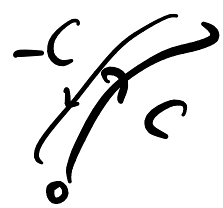

transforms like: $W_{xy} \mapsto \Lambda(x)\, W_{xy}\, \Lambda(y)^{-1}$ (*)

$\Rightarrow \Phi^\dagger(x)\, W_{xy}\, \Phi(y)$ is invt. under

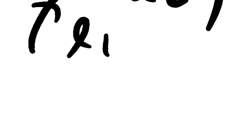
$W_{xy}$ associates an el't of $G$ to each __path__ between $x$ & $y$.

s.t.
- $W(\underset{\text{from } x \text{ to } x}{\phi}) = \mathbb{1}$
- $W(C_2 \circ C_1) = W(C_2)\, W(C_1)$
- $W(-C) = W^{-1}(C)$




on a cell complex a connection is $W_e$ for each 1-cell.

$W($  $) = W_{e_2} W_{e_1}$

A fiber over each 0-cell

$$(W_{\langle ij\rangle})_{\alpha\beta} \xrightarrow{\sim} \Lambda_{\alpha\gamma}^{(i)} (W_{\langle ij\rangle})_{\gamma\delta} (\Lambda^{-1})_{\alpha\gamma}^{(j)}.$$

A system of dofs $= \{W_e\}/\sim$

$\equiv$ lattice gauge theory

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Continuum : $G$ continuous : $x \to x+\Delta x$

$B$ "

$$\lim_{\Delta x \to 0} \frac{W(x, x+\Delta x)\Phi(x+\Delta x) - \Phi(x)}{\Delta x} \equiv D\Phi(x)$$

$$= \left(\partial + \frac{ie}{\hbar c} A\right)\Phi(x)$$

gauge field

in terms of which

$$W_C = P\, e^{\frac{ie}{\hbar c}\int_C A}$$

A is a Lie-algebra-valued 1-form.
(on patches)

$$F_{\mu\nu} \equiv [D_\mu, D_\nu]$$

describes local path-dependence of $W$.


$$F_{\mu\nu}(x) \longrightarrow \Lambda(x) \left(F_{\mu\nu}(x)\right) \Lambda^{-1}(x)$$

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G abelian & continuous $\quad (G = U(1))$.

$$W_C = e^{i\int_C A}$$


$$W_{C'} = W_C \, e^{ie\oint_{C'-C} A}$$

$$\overset{\text{stokes}}{=} W_C \, e^{ie\oint_{\partial S} A}$$

$$= W_C \, e^{ie\int_S F}$$

$F = dA$. $\Rightarrow dF = 0$ where $A$ makes sense.

$$e^{ie\oint_{\underset{=\partial V}{S'-S}} F} \overset{\text{stokes}}{=} \underbrace{e^{ie\int_V dF}}$$

$$\overset{!}{=} 1 \iff e\int_V dF \in 2\pi\mathbb{Z}$$

$$dF = *j_m$$

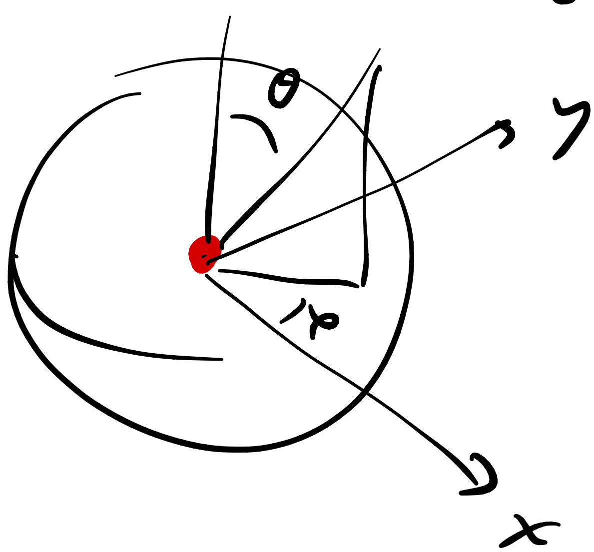
$\Rightarrow \int_V dF$ = magnetic charge in $V \equiv g$ 4\pi $= eg$ 4\pi $\quad$ Dirac qtzn.

example: Dirac (Wu-Yang) monopole

$= $ Hopf bundle $S^3 \to S^2$.

Goal: Find "A" on $S^2(1)$ s.t. $\vec{\nabla} \cdot \vec{B} = 4\pi g\, \delta^3(x)$

$$= (*dF)_t = (*d^2 A)_t$$




$$\begin{cases} U_N = S^2 \setminus S \\ U_S = S^2 \setminus N \end{cases}$$

on $U^N$: $A^N = g(1-\cos\theta)\,d\varphi$

on $U^S$: $A^S = g(-1-\cos\theta)\,d\varphi = A^N - 2g\,d\varphi$

$$\boxed{\begin{aligned} dA^N = dA^S = F &= g \sin\theta \, d\theta \wedge d\varphi \\ &= \frac{g}{4\pi} (\text{vol of } S^2) \end{aligned}}$$

$= g_{NS}^{-1}(A^N + id)\,g_{NS}$ gauge transf.

with $g_{NS}(\theta,\varphi) = e^{2gi\varphi}$.

on $U_{NS}$

$g_{NS}$ is two things: – transition f'n on a
cx vector bundle
of rank 1
"≡ line bundle"
w/ structure group $U(1)$.

– gauge transf relating
two gauges for vector potential.

$\varphi \simeq \varphi + 2\pi$.  $g_{NS}(\varphi + 2\pi) \overset{!}{=} g_N(\varphi)$

$\Longleftrightarrow$ $4\pi g \in 2\pi \mathbb{Z}$

$\Rightarrow 2g \in \mathbb{Z}$.

QM: wavefn of a charged particle
$\psi(x) \longrightarrow g_{NS}\,\psi(x)$

is a phase $\in U(1)$.

→ <u>flux quantization</u> A v.b. w/ compact structure group has fluxes w/ quantized <u>periods</u>

$$\underline{\oint_S \frac{F}{2\pi} \in \mathbb{Z}.}$$

compact 2d submfd of $B$

$$\oint_{S^2} \frac{F}{2\pi} = \frac{1}{2\pi}\left[\int_{H_N} dA^N + \int_{H_S} dA^S\right] = \frac{1}{2\pi}\oint_e \underline{\underline{(A^N - A^S)}}$$


$$= \oint_e \frac{-i}{2\pi} \underbrace{g_{NS}^{-1}\, dg_{NS}}_{d\log g_{NS}}$$

$g_{NS} = e^{2gi\varphi}$

$$= \int_0^{2\pi} \frac{d\varphi}{2\pi}\, 2g \in \mathbb{Z}.$$

Dirac

a $\underline{G = U(1)}$.

$\frac{F}{2\pi} \equiv c_1$ first Chern class of $E$

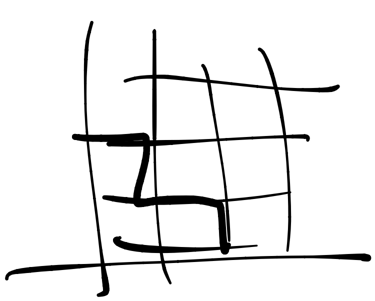
$[c_1]$ indep of connection. (property of v.b.)

ie $\left(\oint_S F/2\pi\right)$

More general line bundle on a more general space

$$\{\mathcal{U}_{\alpha\beta}\} \longrightarrow -\frac{i}{2\pi} d \log g_{\alpha\beta}$$

$$\in C^1(\mathcal{U}, \Omega^1)$$

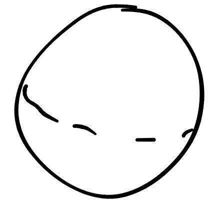


is $d$-closed and $\delta$-closed

$\Longrightarrow \exists$ a global 2-form

$\underline{\underline{c_1}}$.

a line bundle on $S^2$



$\longleftrightarrow \; g_{NS} : \underset{\simeq U_{NS}}{\text{equator}} \longrightarrow S^1 = G$

$[g_{NS}] \in \pi_1(S^1) = \mathbb{Z}$. $\longleftarrow c_1(E) = [g_{NS}]$.

$\Longrightarrow$ a line bundle on $S^2 \simeq \mathbb{CP}^2$

is specified by $c_1$.

**Hopf = Dirac** : $g_{NS} \in U(1) \simeq S^1$

make a new bundle w same $g_{NS}$

w $F = S^1$.

section of Hopf: $s : B \to E$ s.t. $\pi \circ s = \mathbb{1}$ on $B$.

$\pi : z \longrightarrow \underbrace{z^\dagger \vec{\sigma} z}_{\in S^2}$

$= (z_0, z_1) \in \mathbb{C}^2$

s.t. $|z_0|^2 + |z_1|^2 = 1$

Given $\hat{r} \in S^2$ ($\subset \mathbb{R}^3$) let $\rho = \frac{1}{2}(1 + \hat{r}\cdot\vec{\sigma})$

$\rho \geq 0$, $\mathrm{tr}\,\rho = 1$, $\rho^2 = \rho$.

$$\Rightarrow \rho = |z\rangle\langle z|$$

w/ $|z\rangle = z_0|0\rangle + z_1|1\rangle$

$(\rho_{\alpha\beta} = z_\alpha z_\beta^\dagger)$

$\hat{r}\cdot\vec{\sigma}\,|z\rangle = |z\rangle$.

$$\begin{cases} z = s^N(\theta,\varphi) = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2\, e^{i\varphi} \end{pmatrix} & \text{on } U_N \\[2ex] z = s^S(\theta,\varphi) = \begin{pmatrix} \cos\theta/2\, e^{-i\varphi} \\ \sin\theta/2 \end{pmatrix} & \text{on } U_S \end{cases}$$

$$s^S = e^{i\varphi} s^N = g_{NS}^{(\theta,\varphi)} s^N$$

transition fn for $2g = 1$ minimal monopole.

local sections & connection:

$$\begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = e^{i\psi/2} \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\varphi/2} \\ \sin\frac{\theta}{2}\, e^{i\varphi/2} \end{pmatrix}$$

$\theta, \psi, \varphi$ coords on $S^3$

$\psi \in [0, 4\pi)$

$$\Rightarrow s^{N/S} : S^2 \to S^3$$
$$(\theta, \varphi) \mapsto (\theta, \varphi, \psi = \pm\varphi)$$

$$\pi : S^3 \to S^2$$
$$(\theta, \varphi, \psi) \mapsto (\theta, \varphi).$$

on $S^3$: $\Omega^1(S^3) \ni \bar{z}_\alpha dz_\alpha = \frac{i}{2}(d\psi - \cos\theta\, d\varphi)$

$$(s^{N/S})^* : \Omega^\bullet(S^3) \to \Omega^\bullet(U^{N/S})$$

$$(s^{N/S})^*(z^\dagger dz) = A^{N/S}.$$

local section

local connection

Hopf invariant: given a smooth map

$$f : S^3 \to S^2$$

$$f^*(\alpha) = d\omega \Leftarrow f^* : \underbrace{\Omega^2(S^2)}_{H^2(S^2) = \langle\alpha\rangle = \mathbb{Z}} \to \Omega^2(S^3)$$

$H^2(S^3) = 0$

$$H[f] = \int_{S^3} \omega \wedge d\omega$$

is ind. of $\omega \to \omega + d\lambda = \omega'$

$$\int \omega \wedge d\omega - \int \omega' \wedge d\omega'$$

$$= \int_{S^3} (\omega - \omega') \wedge d\omega \overset{\text{Stokes}}{=} 0.$$

$$f : S^{2n-1} \to S^n \qquad f^*(\alpha) = d\omega$$

$$\omega \in \Omega^{n-1}(S^{2n-1})$$

$$H[f] = \int_{S^{2n-1}} \omega \wedge d\omega.$$

for odd $n$ $\quad H[f] = 0 \quad \omega \wedge d\omega = d\Big(\overbrace{\frac{\omega \wedge \omega}{2}}^{\neq 0}\Big)$

- $f \simeq g \Rightarrow H[f] = H[g]$.

[Bott-Tu
p. 227]

- $H(f) =$ linking # of
  $f^{-1}(p)$ and $f^{-1}(q)$
  $p, q \in S^2$.

$$\boxed{H(\pi) = 1.}$$

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For bundles of general rank on $M$.
general chern classes are

$$\frac{\operatorname{tr} F}{2\pi}, \quad \frac{\operatorname{tr} F \wedge F}{16\pi^2} \quad \ldots$$

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$S^4 = \mathbb{R}^4 \cup \{\infty\}$.

$G$ = simple compact Lie group.

put a connection, demand $F \xrightarrow{r \to \infty} 0$

$$F \to 0 \quad \Rightarrow \quad A \xrightarrow{r \to \infty} g^{-1} dg$$

("instanton")

$g : S^3 \to G$.

$\to g$: $G = SU(2)$

$g^0(x) = \mathbb{1}$

$$g^1(x) = \frac{x_4 \mathbb{1} + i \vec{x} \cdot \vec{\sigma}}{r} \in SU(2)$$

(generalizes

$$g_{NS}(x) = \frac{x + iy}{r} \in U(1) \,)$$

$g^\nu(x) = (g^1(x))^\nu$

$\nu \in \mathbb{Z}$.

to be continued...