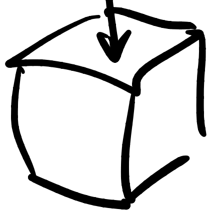
Homotopy groups, cont'd

$$\pi_q(X, x_0) \equiv \{\text{maps} : (I^q, \partial I^q) \to (X, x_0)\} / \simeq$$

homotopy equivalence

$I^q = \underbrace{I \times \cdots \times I}_{q \text{ times}}$



Basic facts: 0) $\pi_0(X) \equiv \text{maps}: pt \to X / \simeq$

$$\equiv [pt, X]$$

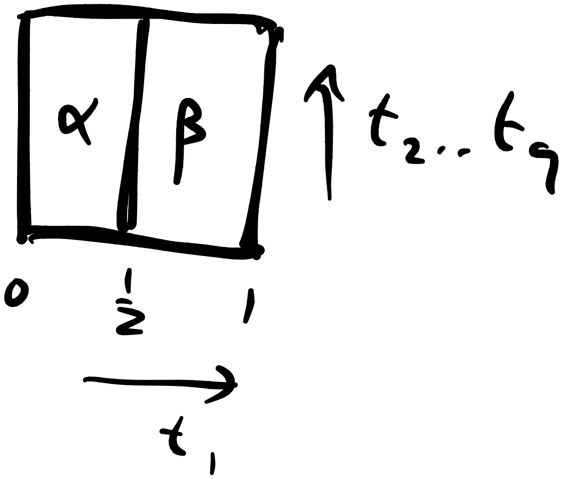
is not a group in general.

$= \{\text{path components of } X\}$

(if $X = G$, a lie group, $\pi_0(G) = G/G_0$ ($G_0 \equiv$ component w $\mathbb{1}$) is a group.)

1) $\pi_q(X)$, $q \geq 1$, is a group under $*$.

$$[\alpha][\beta] = [\alpha * \beta]$$

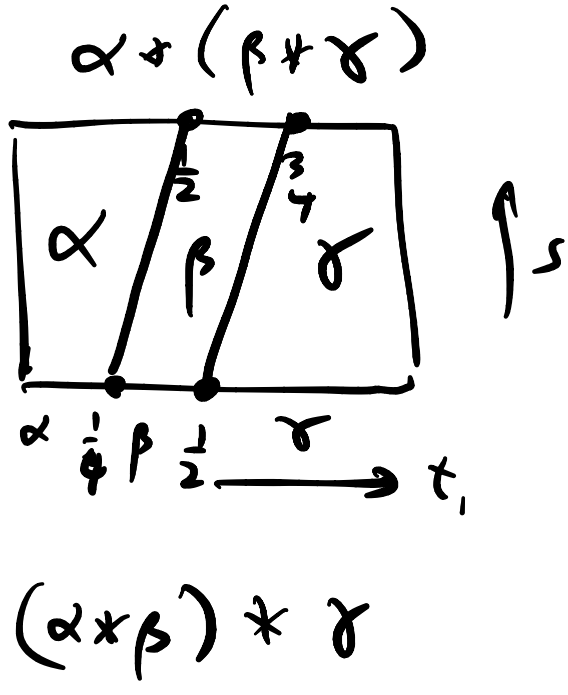


$$[f^{-1}(t, \dots)] = [f(1-t, \dots)]$$

$\mathbb{1}$ = constant map to the base pt, $\bar{p}$.

$$[\alpha * (\beta * \gamma)] = [(\alpha * \beta) * \gamma]$$

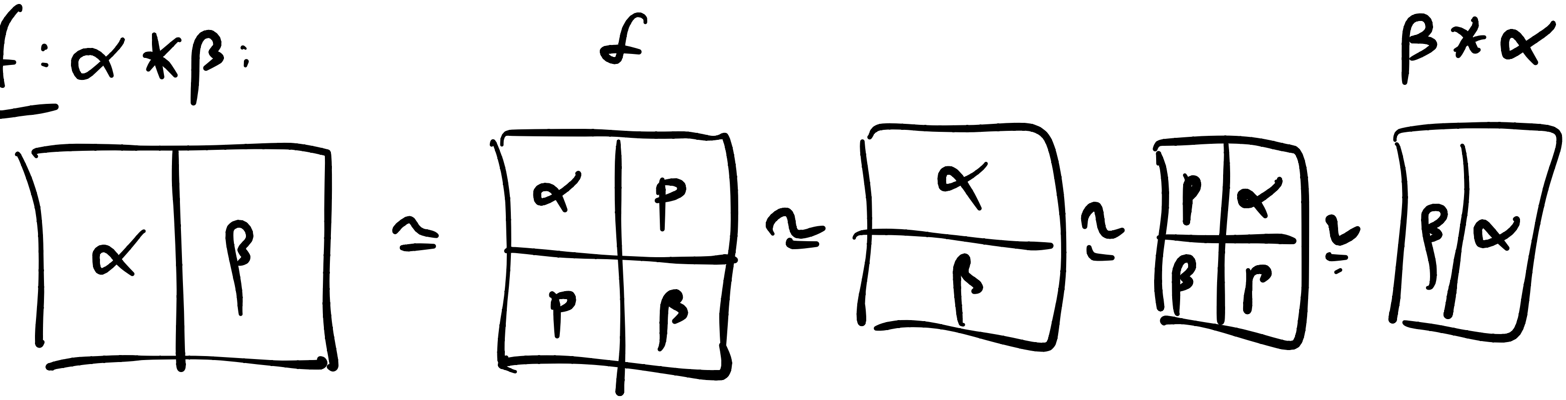
Pf:



2) $\pi_q(X)$ is abelian for $q > 1$.

$\pi_1(X) \cong$ fundamental group is special.

Pf: $\alpha * \beta$:



$$\delta(t_1, \ldots t_q) = \begin{cases} \alpha(2t_1, 2t_2 - 1, \ldots, t_q) & 0 \leq t_1 \leq \frac{1}{2},\ \frac{1}{2} \leq t_1 \leq 1 \\ \beta(2t_1 - 1, 2t_2 \ldots t_q) & \frac{1}{2} \leq t_1 \leq 1,\ 0 \leq t_2 \leq \frac{1}{2} \\ p & \text{otherwise} \end{cases}$$

3) If $X \simeq Y$ then $\pi_q(X) \cong \pi_q(Y)$

(warning: $\exists\ X, Y$ homeomorphic but have the same π_q.)

4) $\pi_q(X \times Y) = \pi_q(X) \times \pi_q(Y)$

a map $I^q \to X \times Y$

is (f_x, f_y) $\quad f_x : I^q \to X$, $f_y : I^q \to Y$

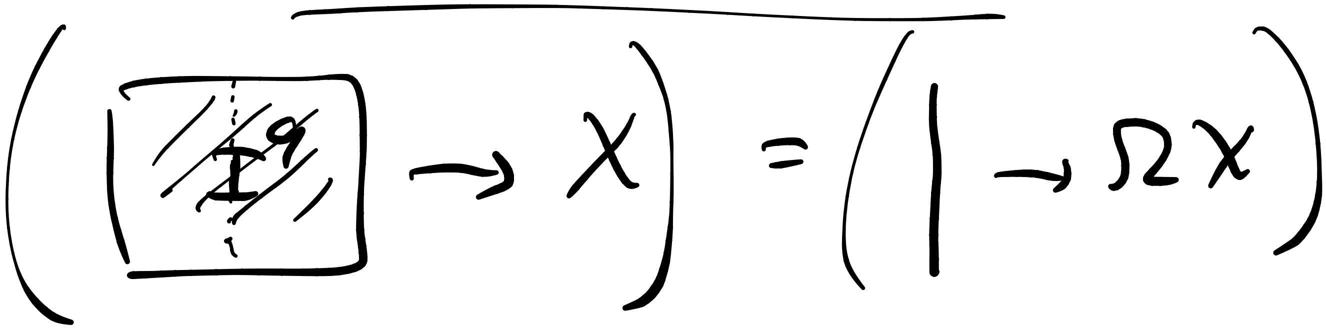
$(f_x, f_y) * (g_x, g_y) = (f_x * g_x, f_y * g_y)$.

5) Let $\Omega_p X \equiv \left\{ \begin{smallmatrix}\text{continuous}\\ \text{maps}\end{smallmatrix} : (I^1, \partial I^1) \to (X, p) \right\}$

$\equiv$ loop space of X
(pointed)

$\pi_1(X,p) \equiv \pi_0(\Omega_p X)$.

In general: $\underline{\underline{\pi_{q-1}(\Omega_p X) = \pi_q(X,p)}}$



$\underline{\text{corollary}}$: $\pi_1(\Omega X)$ is abelian.
$= \pi_2(X)$

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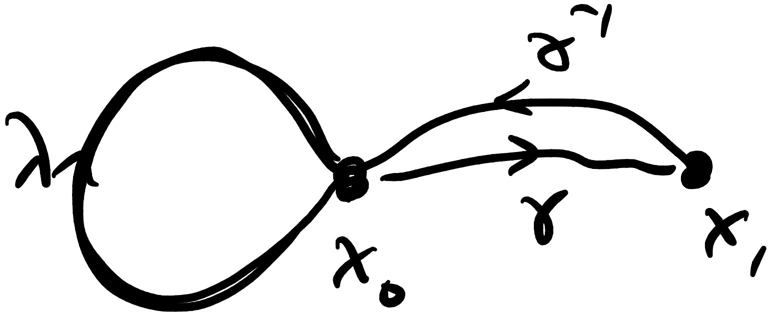
6) $\pi_q(X,p) \cong \pi_q(X,p')$ as groups...
if X is path connected.

A path from x_0 to x_1

gives a map



$\Omega_{x_0}X \to \Omega_{x_1}X$

$\lambda \mapsto \gamma * \lambda * \gamma^{-1}$

constant map to x_0

$$\Rightarrow \quad \gamma_* : \underbrace{\pi_{q-1}(\Omega_{x_0}X, \bar{x}_0)}_{= \pi_q(X, x_0)} \to \underbrace{\pi_{q-1}(\Omega_{x_1}X, \bar{x}_1)}_{= \pi_q(X, x_1)}$$

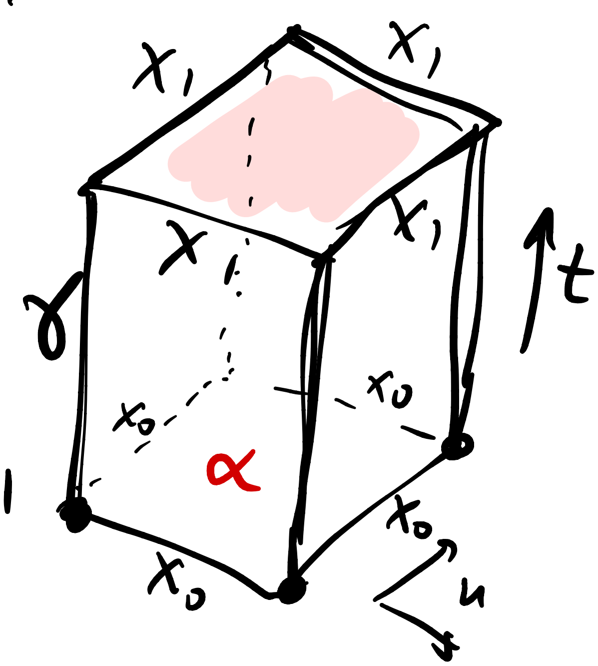
claim:

is an isomorphism.

Given $[\lambda] \in \pi_q(X, x_0)$

define $F : I^q \times I \longrightarrow X$
$\qquad\qquad (u, t)$

defines F on all but one face of I^{q+1}



such an F uniquely extends to I^{q+1}

$[F(u,1)] = \gamma_*[\lambda]$.

If we take $x_0 = x_1$ defines an action of $\pi_1(X, x_0)$ on $\pi_q(X, x_0)$.

$$\pi_q(X, x_0) / \pi_1(X, x_0) \cong [S^q, X]$$

inclusion "free homotopy"

Scariness of π_q:

table of $\pi_q(S^n)$

eg: $\pi_q(S^2) =$

q	2	3	4	5	6	7	8	9	...		
$\pi_q(S^2)$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	

$\cdots \; \mathbb{Z}_{12} \times \mathbb{Z}_2 \quad \mathbb{Z}_{84} \times \mathbb{Z}_2^2$

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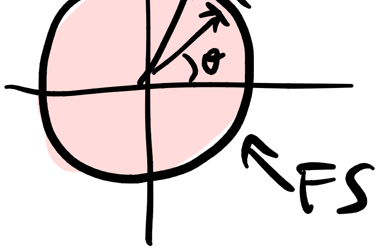
eg of ΩX in physics: "$U(1)^\infty$" symmetry

$n_p = c_p^\dagger c_p \qquad [n_p, H] \cong 0$

$\forall p \in FS$.

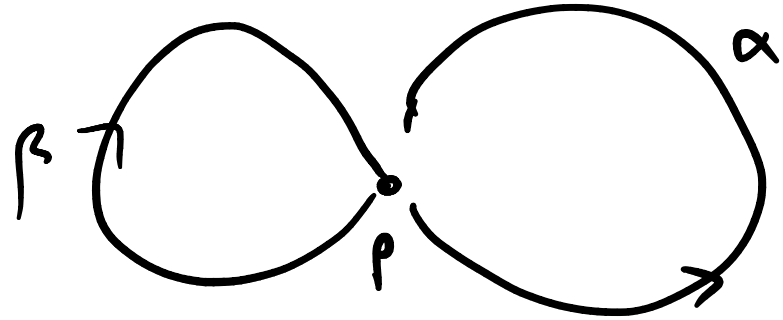
φ



Symmetry is $LU(1) =$ maps: $S^1 \to U(1)$

$\Omega U(1) = LU(1)/\sim$. [Senthil, Else]

Fundamental group: $\underline{\pi_1}(X) \ni [\alpha]$

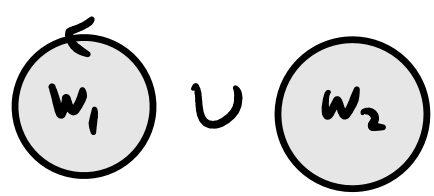


Rel'n between π_1 & H_1?

π_1 contains more info:

eg X w/ $\underbrace{H_1(X,\mathbb{Z})=0}_{\text{"acyclic"}}$ but $\pi_1(X) = I$.
$\neq 0$.

$X =$ 

glued to 

$\underline{\partial w_1 = a^5 b^{-3}}$ $\quad$ $\underline{\partial w_2 = b^3 (ab)^{-2}}$

$$0 \to \mathbb{Z}^2 \xrightarrow{\partial = M} \mathbb{Z}^2 \xrightarrow{0} \mathbb{Z} \to 0$$

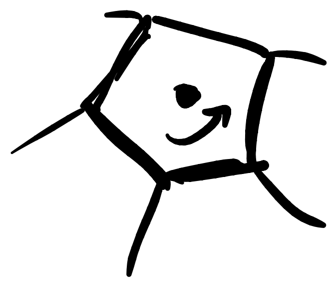
$$\partial_2 = M = \begin{pmatrix} 5 & -2 \\ -3 & 1 \end{pmatrix}$$

$$H_1 = \mathbb{Z}^2 / \operatorname{im} M \qquad \det M = -1$$
$$= 0. \qquad \Rightarrow \operatorname{im} M = \mathbb{Z}^2.$$

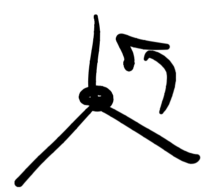
$$\pi_1(X) = \langle a, b \mid a^5 b^{-3} = 1,\ b^3 (ab)^{-2} = 1 \rangle$$
$$= \langle a, b \mid a^5 = b^3 = (ab)^2 \rangle$$
$$= A_5 = S_5 / \mathbb{Z}_2 \simeq I$$

icosahedral.

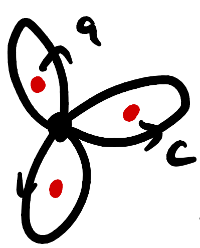
$a \longrightarrow$ $2\pi/5$ rotation about face



$b \longrightarrow$ $2\pi/3$ rot about vertex



why?

$\pi_1\Big($  $\Big) = \langle a, b, c \rangle$ free group on 3 el'ts.

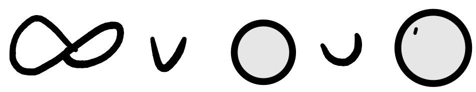
$= \pi_1(\mathbb{R}^2 \setminus \{3 \text{ pts}\})$

$\mathbb{F}_3$

BAD!

eg: $\mathbb{F}_q \not\supseteq \mathbb{F}_q \quad q > 1.$

- each disk introduce a rel'n.

$X =$  is related to

$M \equiv$ <u>homology sphere</u> $\quad H_\bullet(M,\mathbb{Z}) = H_\bullet(S^3,\mathbb{Z})$

but $\pi_1(M) = I'$

$$\begin{tikzcd} I' \arrow[d] & \subset & SU(2) \arrow[d] \\ I & \subset & SO(3) \end{tikzcd}$$

<u>Fact:</u> $H_1(X,\mathbb{Z}) \cong \pi_1(X) / [\pi_1(X), \pi_1(X)]$

$[G,G] \equiv \langle ghg^{-1}h^{-1},\ g,h \in G \rangle.$

"abelianization of π_1".

$$\pi_1 \xrightarrow[\text{order}]{\text{forgets}} H_1$$

<u>$aba^{-1}b^{-1} = 1.$</u>

<u>van Kampen Theorem</u> : π_1 analog of Mayer-Vietoris

$X = U \cup V$ open sets $W = U \cap V$.

Denote $\pi_1(Y) = \langle S_Y \mid r_Y \rangle$

set of generators

set of rel'ns

$Y = U, V, W$.

$i^{U/V} : W = U \cap V \to U/V$.

Given a map $f : \underline{X \to Y}$

$\Longrightarrow f_* : \pi_q(\underline{X,} x_0) \to \pi_q(\underline{Y,} f(x_0))$

$[\alpha] \longmapsto [f \circ \alpha]$

induced map on π_q.

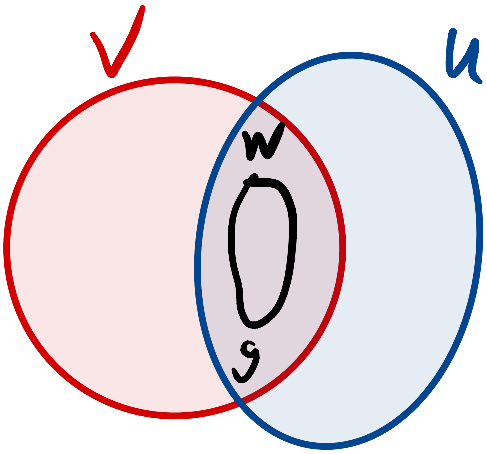
π_q is a <u>COVARIANT</u> functor :

top spaces $\to$ groups

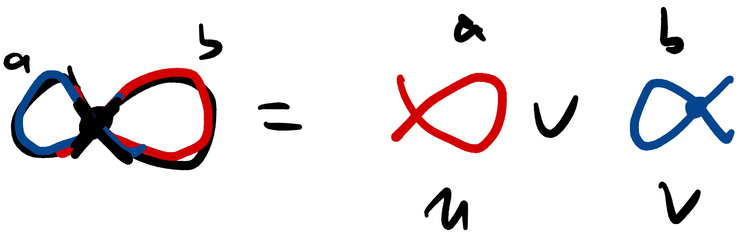
$i_*^{U/V} : \pi_1(W) \to \pi_1(U/V)$.

Take $p \in W \subset U, V$.

$$\pi_1(X) = \langle S_u \cup S_v \mid r_u \cup r_v \cup \{ i^u_*(g) = i^v_*(g) \}_{g \in S_w} \rangle$$



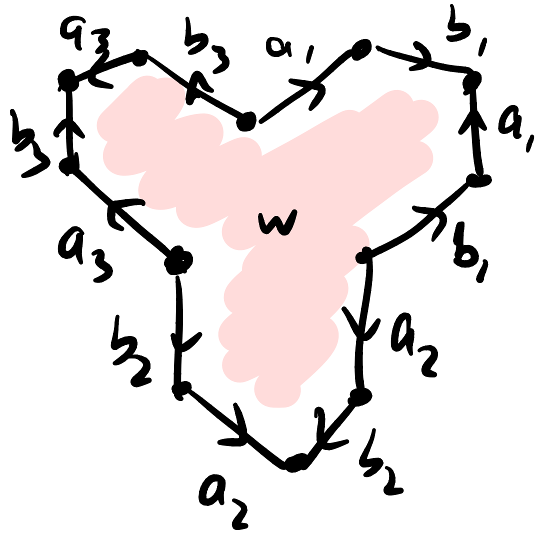
[Roberts' Knot knotes] §8.

Example 1: 

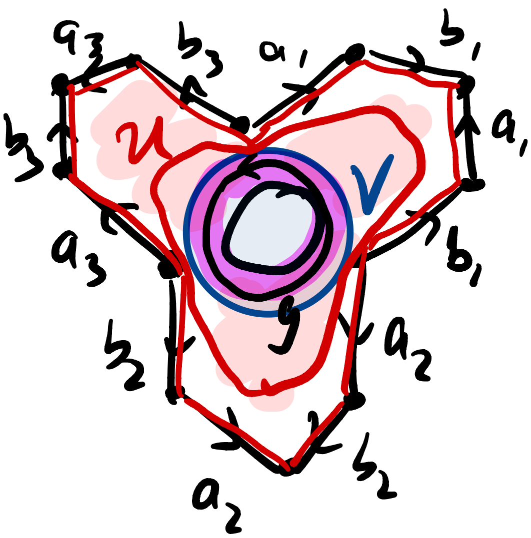
$W = X \simeq \bullet \Rightarrow \pi_1(W) = e.$

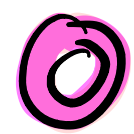
$$\pi_1(\infty) = \langle a, b \mid \text{no rel'ns} \rangle = \mathbb{F}_2 .$$

Example 2: $X =$  acc. M.

ex 3: $\Sigma_g =$ 

$U = \Sigma_g \setminus$ disk $\qquad V =$ bigger disk.



$W = U \cap V =$  $\simeq S^1$.

$\pi_1(W) = \langle g \rangle$.

$i_*^U(g) =$

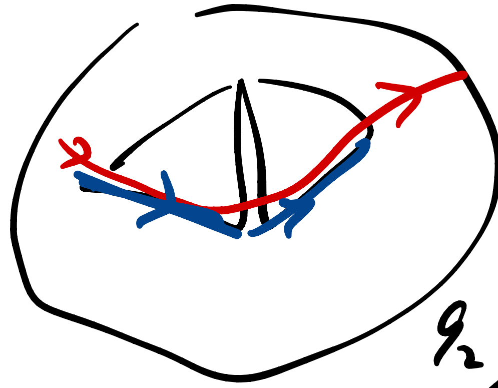
$\pi_1(V) \simeq 0$.

$\pi_1(U) = \langle a_1, b_1, a_2, b_2 \dots a_g b_g \rangle = F_{2g}$.

$\pi_1(\Sigma_g) = \left\langle \{a_i, b_i\}_{i=1}^{g} \;\middle|\; \prod_{i=1}^{g} [a_i, b_i] = 1 \right\rangle$

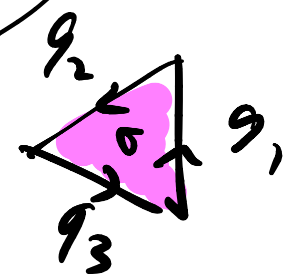
$[a,b] = aba^{-1}b^{-1}$.

Cellular approximation



$\Delta \equiv$ path-connected cell complex

$p \in \Delta_0$.



Let $G_\Delta \equiv \left\langle \{g_e\}_{e \in \Delta},\ \middle|\ \left\{ \prod_{e \in \partial\sigma} g_e = e \right\}_{\sigma \in \Delta_2} \right\rangle$

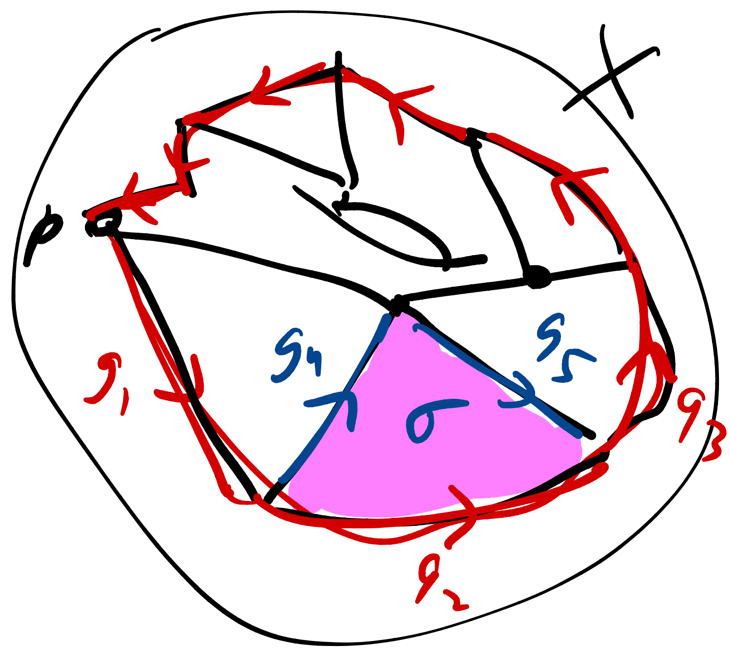
$\pi_1(\Delta) \cong \{$ words in G_Δ starting & end at p $\}$

$g_1 g_2 g_3 \dots$

idea:

$g_1 g_2 g_3 \dots$

$= g_1 g_4 g_5\text{-} g_3 \dots$



"Calculating thm" : (Nash & Sen)

$$\pi_1(\Delta) \cong \underline{\underline{G_\Delta}} \Big/ \underline{\underline{G_L}} = \left(\begin{array}{l} G_\Delta \text{ w/ } g_l = 1 \\ \quad \text{if } l \in L_1 \end{array} \right)$$

L = a contractible 1 dim'l subcomplex

$L_0 \supseteq \Delta_0$ contains all 0-cells.

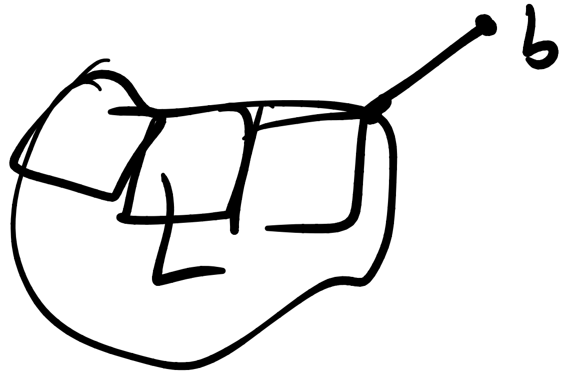
$\exists L$. L = maximal 1d contractible subcomplex.

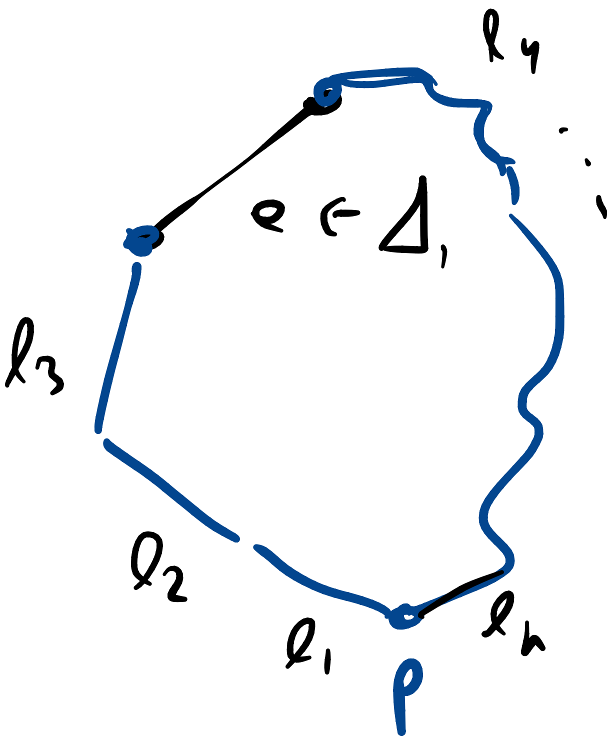
L contains all vertices :

if $\exists\, b \in \Delta_0 \setminus L_0$, $a \in L_0$

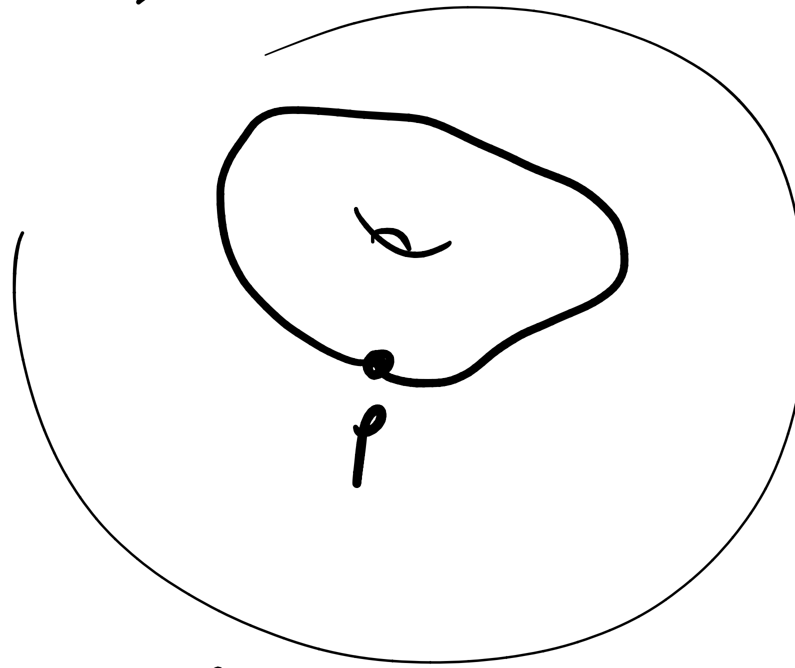
then $L \cup \langle ab \rangle \cup \{b\}$

is bigger & still contractible.





$\longleftrightarrow \quad [\alpha] \in \pi_1(\Delta)$



$$g_e = g_{l_1} g_{l_2} g_{l_3} g_e g_{l_4} \cdots g_{l_n}$$

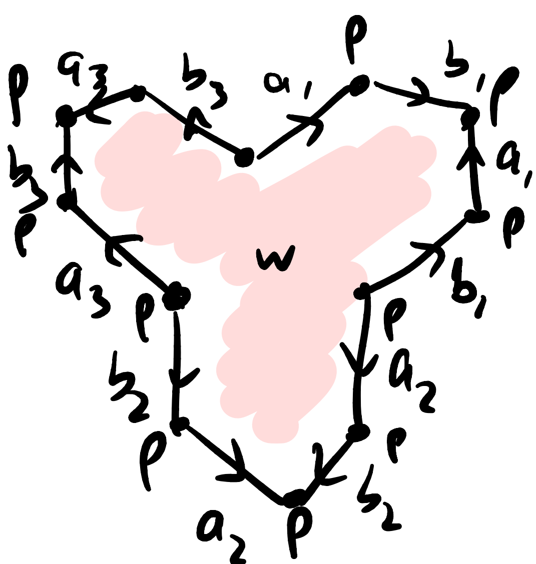
= loop starting & ending at p.

$$l_i \in L \implies g_e = 1.$$

eg:

$L_0 = p$

$L_1 = \emptyset$.



$\pi_1(\Sigma_g) = G_\Delta$.

$H_1(X, A) \longleftrightarrow$ g.s. of T.C.

$G = A$

$\underline{\underline{\pi_1(X) \to G}} / \sim \;\longleftrightarrow$ g.s. of Q.D.

G non-abelian

$\rho : \pi_1(X) \to G \longleftrightarrow$ flat G connections