Last time: Čech cohomology

$\mathcal{U} = \{ \mathcal{U}_\alpha \}_\alpha$ is a good cover of M.

[↑ each $\mathcal{U}_{\alpha_0 \dots \alpha_p} \equiv \mathcal{U}_{\alpha_0} \cap \mathcal{U}_{\alpha_1} \cdots \cap \mathcal{U}_{\alpha_p}$ are $\simeq$ ball]

to each $\mathcal{U}_{\alpha\beta}$ associate

a $\mathcal{H} = \operatorname{span} \{ | \sigma_{\alpha\beta} \rangle ,\ \sigma_{\alpha\beta} \in A \}$

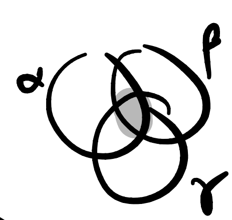
↑ abelian group

eg $A = \mathbb{Z}_2$

[A 1-cochain $\sigma \in C^1(\mathcal{U}, A)$ specifies a basis state]

Take $$H = - \sum_{\mathcal{U}_\alpha} A_\alpha - \sum_{\mathcal{U}_{\alpha\beta\gamma}} B_{\alpha\beta\gamma}$$

$$B_{\alpha\beta\gamma} | \{\sigma\} \rangle = (-1)^{(\delta\sigma)_{\alpha\beta\gamma}} | \{\sigma\} \rangle$$

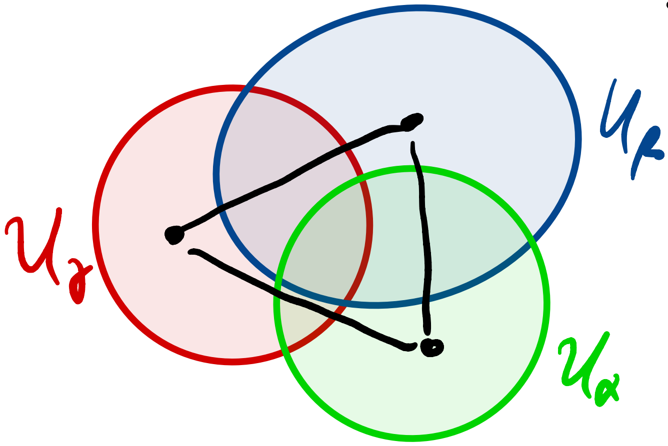


$$\delta\sigma : C^1 \to C^2$$
$$\sigma \mapsto (\delta\sigma)_{\alpha\beta\gamma} = \sigma_{\alpha\beta} + \sigma_{\beta\gamma} + \sigma_{\gamma\alpha} .$$

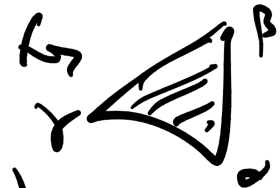
$$B_{\alpha\beta\gamma}|\{\sigma\}\rangle = |\{\sigma\}\rangle$$

says σ is a 1-cocycle

$$\delta\sigma = 0 \mod 2.$$



$B_{\alpha\beta\gamma} \sim$ plaq. op



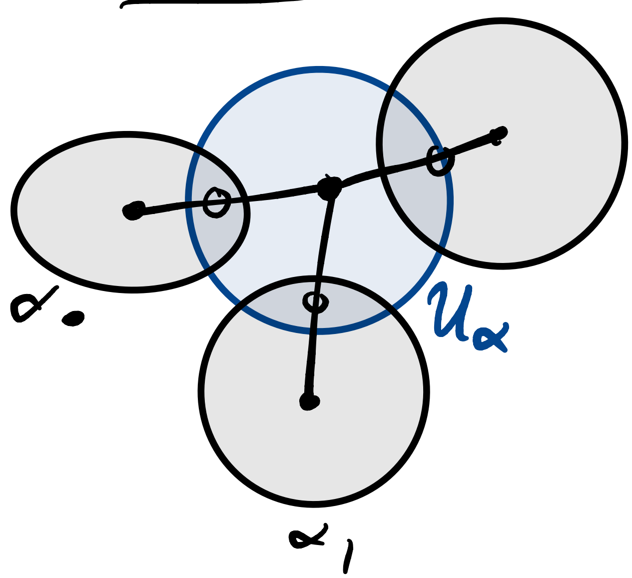
$$\underline{\underline{A_\alpha}}|\{\sigma\}\rangle = |\{\underline{\underline{\sigma_{\alpha_0\alpha_1}}} + \overbrace{\underline{\underline{\delta_{\alpha\alpha_0} - \delta_{\alpha\alpha_1}}}}^{\delta(\lambda_\alpha)}\}_{\alpha_0\alpha_1}\rangle$$

$$\lambda_\alpha \equiv \begin{cases} 1 & \text{on } U_\alpha \\ 0 & \text{else} \end{cases}$$

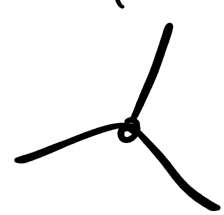
$$\boxed{\delta^2 = 0}$$

$\Rightarrow [A, B] = 0.$

same as T.C.



star operator



$\Rightarrow$ Čech cohomology $\simeq$ cellular homology.

Nice thing abt Čech complex:

we can take weird coefficients

Proof of equivalence of $H^\bullet_{dR}(M) \cong \check{H}^\bullet(\mathcal{U}, \mathbb{R})$

$\uparrow$ good cover of M.

$$M \leftarrow \coprod_\alpha U_\alpha \overset{i_0}{\underset{i_1}{\leftleftarrows}} \coprod_{\alpha_0\alpha_1} U_{\alpha_0\alpha_1} \Lleftarrow \coprod_{\alpha_0\alpha_1\alpha_2} U_{\alpha_0\alpha_1\alpha_2} \cdots$$

$\Rightarrow$ generalized Mayer Vietoris seq on $\Omega^\bullet_{dR}(M)$

$$0 \to \Omega^\bullet(M) \xrightarrow{res} \bigoplus_\alpha \Omega^\bullet(U_\alpha) \xrightarrow{\delta} \bigoplus_{\alpha\beta} \Omega^\bullet(U_{\alpha\beta}) \xrightarrow{\delta} \bigoplus_{\alpha\beta\gamma} \Omega^\bullet(U_{\alpha\beta\gamma}) \to \cdots$$

$i_\alpha : U_{\alpha\beta\gamma} \to U_{\beta\gamma}$

$\leadsto \quad \delta_\alpha : \Omega^\bullet(U_{\beta\gamma}) \to \bigoplus_\alpha \Omega^\bullet(U_{\alpha\beta\gamma})$

$$\delta \equiv \sum_i (-1)^i \delta_{\alpha_i}$$

$$(\delta\omega)_{\alpha_0 \ldots \alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \omega_{\alpha_0 \ldots \hat{\alpha}_i \ldots \alpha_{p+1}}$$

a form on $U_{\alpha_0 \ldots \alpha_{p+1}}$.

(Prev. M-V seq. was the special case $M = U \cup V$.)

Facts: ① $\delta^2 = 0$

② seq is **exact**.

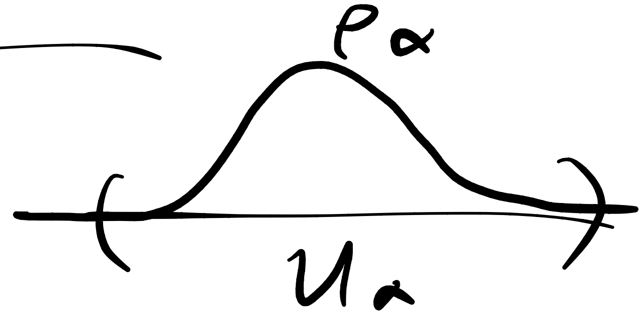
why: make a homotopy operator

$$K : \bigoplus_{\alpha_0 \ldots \alpha_p} \Omega^\bullet(U_{\alpha_0 \ldots \alpha_p}) \to \bigoplus \Omega^\bullet(U_{\alpha_0 \ldots \alpha_{p-1}})$$

$$(K\omega)_{\alpha_0 \ldots \alpha_{p-1}} = \sum_\alpha \rho_\alpha \, \omega_{\alpha \alpha_0 \ldots \alpha_{p-1}}$$

where $\sum_\alpha \rho_\alpha = 1$ partition of unity.

claim $\underline{K\delta + \delta K = 1}$.



(**Note:** Can't do this on $C^\bullet$!)

$$K^{p,q} \equiv \underbrace{C^p(\mathcal{U}, \Omega^q)}_{\substack{\text{Cech cochains} \\ \text{w/ values in } \Omega^q}} \equiv \bigoplus_{\alpha_0 \dots \alpha_p} \Omega^q(\mathcal{U}_{\alpha_0 \dots \alpha_p})$$

<u>Cech-de Rham double complex</u>

deRham complex

$$\begin{tikzcd}
 & \vdots & q \text{ form index} & & & \\
 & \Omega^2(M) \arrow[u] \arrow[r, "r"] & K^{02} \arrow[u, "d"] \arrow[r] & K^{12} \arrow[u, "d"] \arrow[r] & K^{22} \arrow[u] & \\
0 \arrow[r] & \Omega^2(M) \arrow[from=3-1, phantom] & & & & \\
\end{tikzcd}$$

$$\begin{tikzcd}
 & \vdots & & & & \\
0 \arrow[r] & \Omega^2(M) \arrow[u] \arrow[r, "r"] & K^{02} \arrow[u, "d"] \arrow[r] & K^{12} \arrow[u, "d"] \arrow[r] & K^{22} \arrow[u] & \\
0 \arrow[r] & \Omega^1(M) \arrow[u, "d"] \arrow[r, "r"] & K^{01} \arrow[u, "d"] \arrow[r, "\delta"] & K^{11} \arrow[u, "d"] \arrow[r, "\delta"] & K^{21} \arrow[u] \arrow[r, "\delta"] & {} \\
0 \arrow[r] & \Omega^0(M) \arrow[u, "d"] \arrow[r, "r"] & K^{00} \arrow[u, "d"] \arrow[r, "\delta"] & K^{10} \arrow[u, "d"] \arrow[r, "\delta"] & K^{20} \arrow[u] \arrow[r, "\delta"] & {} \\
 & 0 \arrow[u] & & & &
\end{tikzcd}$$

p

Cech index

Rows are M.V exact seqs.

New coboundary operator $D = \delta + \underline{\underline{(-1)^p}} d$

$D^2 = 0$ <u>B/C</u>:

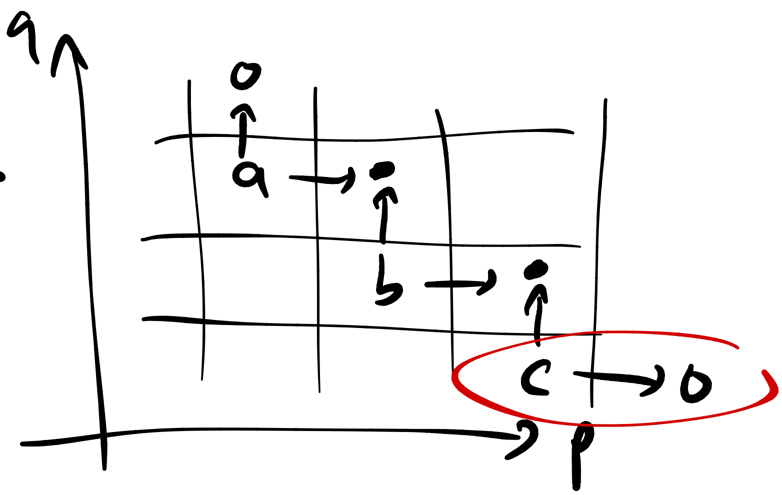
$$\begin{tikzcd}
\bullet \arrow[r, "\delta"] & \bullet \\
\bullet \arrow[u, "d"] \arrow[r, "\delta"'] \arrow[ur, phantom, "\pi"] & \bullet \arrow[u, "-d"']
\end{tikzcd}
\quad p$$

$d\delta - \delta d = 0.$

Lemma: If rows are exact then

$$H^\bullet_{D=\delta+(-1)^p d} \cong H^\bullet_{\text{left column}} \cong H^\bullet_{dR}$$

Pf:

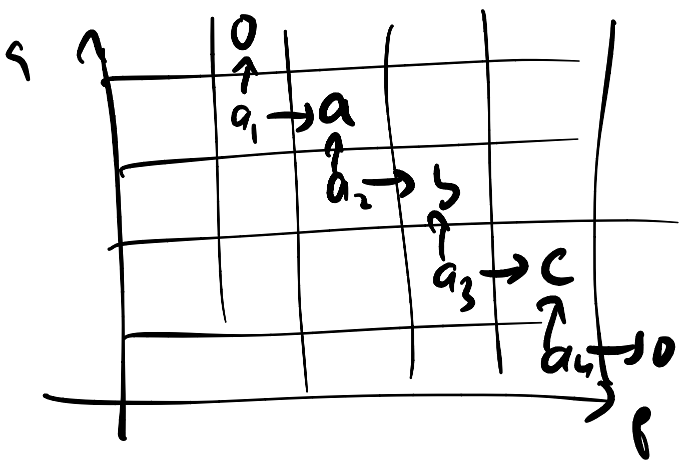


A D-cocycle:

$$\phi = a + b + c$$

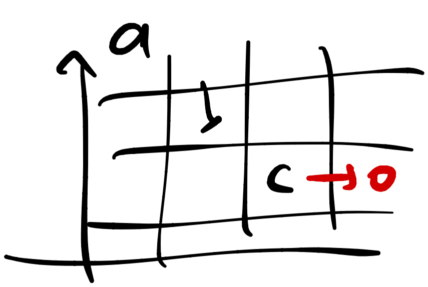
w/
$$\begin{cases} da = 0 \\ \delta a = (-1)^{p+1} db \\ \delta b = (-1)^p dc \\ \delta c = 0 \end{cases}$$

A D coboundary is $\phi = a + b + c$



$$a = \delta a_1 \pm d a_2$$

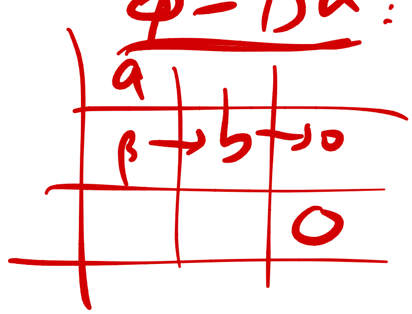
$$b = \delta a_2 \pm d a_3$$



Given ϕ, a D-cocycle

$\delta c = 0 \Rightarrow c = \delta \alpha$

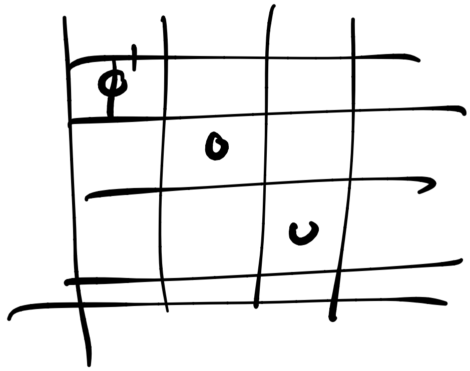
$\phi - D\alpha$:



$\delta b = 0$

$\Rightarrow b = \delta \beta$

$\exists$ a representative $[\underline{\phi}] = [\underbrace{\phi - D\alpha - D\beta \ldots}_{= \phi'}]$



$d\phi' = 0$

$\delta\phi' = 0$

$\Rightarrow \underline{\phi'}$ is a global form on M. $\square$

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deRham complex

Calculus of diff forms on M

q form index

$$
\begin{tikzcd}
{} & \vdots & {} & {} & {} & {} \\
0 \arrow[r] & \Omega^2(M) \arrow[u] \arrow[r, "r"] & K^{02} \arrow[u, "d"] \arrow[r] & K^{12} \arrow[u, "d"] \arrow[r] & K^{22} \arrow[u] & {} \\
0 \arrow[r] & \Omega^1(M) \arrow[u, "d"] \arrow[r, "r"] & K^{01} \arrow[u, "d"] \arrow[r, "\delta"] & K^{11} \arrow[u, "d"] \arrow[r, "\delta"] & K^{21} \arrow[u] \arrow[r, "\delta"] & {} \\
0 \arrow[r] & \Omega^0(M) \arrow[u, "d"] \arrow[r, "r"] & K^{00} \arrow[u, "d"] \arrow[r, "\delta"] & K^{10} \arrow[u, "d"] \arrow[r, "\delta"] & K^{20} \arrow[u] \arrow[r, "\delta"] & {} \\
{} & 0 \arrow[u] \arrow[r] & C^0(\mathcal{U},\mathbb{R}) \arrow[u, "i"] \arrow[r, "\delta"] & C^1(\mathcal{U},\mathbb{R}) \arrow[u] \arrow[r, "\delta"] & C^2(\mathcal{U},\mathbb{R}) \arrow[u] \arrow[r, "\delta"] & {}
\end{tikzcd}
$$

p Cech index

Cech complex

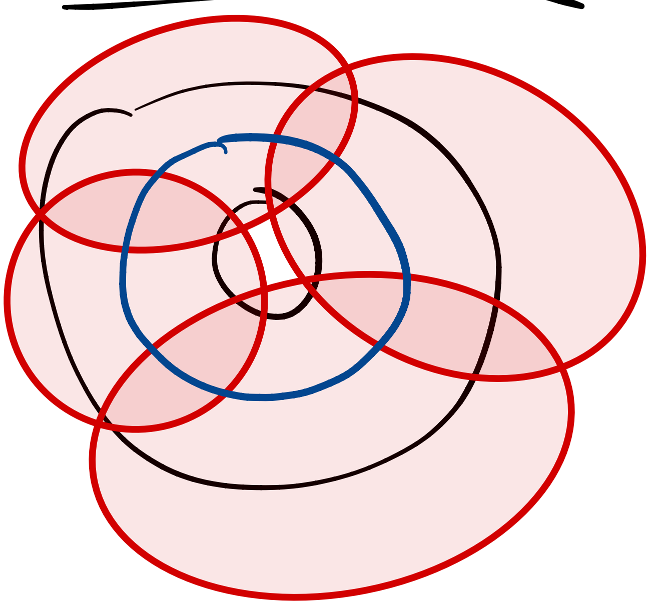
p-rels combinatorics of good cover.

$$H^\bullet_{dR}(M) \simeq H^\bullet_D \simeq \check{H}^\bullet(\mathcal{U}, \mathbb{R})$$

$\Longrightarrow$ Cech cohomology is ind of good cover!

2.7 Local reconstructability (of quantum states)

Key feature of T.O.: reduced density matrices on patches $\equiv$ marginals don't determine the global state



$\{$ density matrices $\}$ = convex set.

can't use this as the coefficients

$\rho_1 + \rho_2$ is not a density matrix

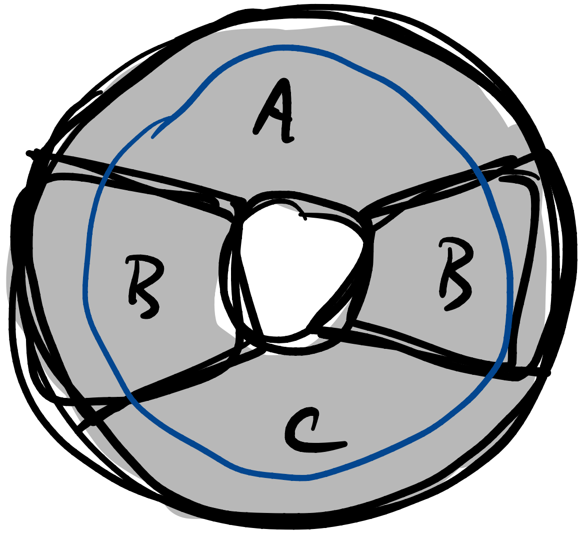
$(\operatorname{tr} \rho = 1)$

Obstruction to global reconstructability:

$$I(A:C|B) \equiv S_{AB} + S_{BC} - S_B - S_{ABC}$$

Q: from $\rho_{AB}, \rho_{BC}, \rho_B$

can we reconstruct ρ_{ABC}?



If $I(A:C|B) = 0$

$\rho_{ABC} \equiv$ "Markov chain"

can be uniquely reconstructed acting locally on B. [Petz]

If $S_A = |\partial A|\Lambda - \gamma\, b_0(\partial A)$

for some γ ind. of A.

$\Longrightarrow I(A:C|B) = 2\gamma \geq 0$ (strong subadditivity)

TEE (top. entanglement entropy)

① Ind. of $_{\wedge}$ deformations of the regions
continuous

② Ind of defs. of the state keeping
the correlation length << size of regions.

③ It can be related to the anyon data.

$$\gamma = \log \sum_a d_a^2$$

$d_a \equiv$ quantum dim of anyon a.

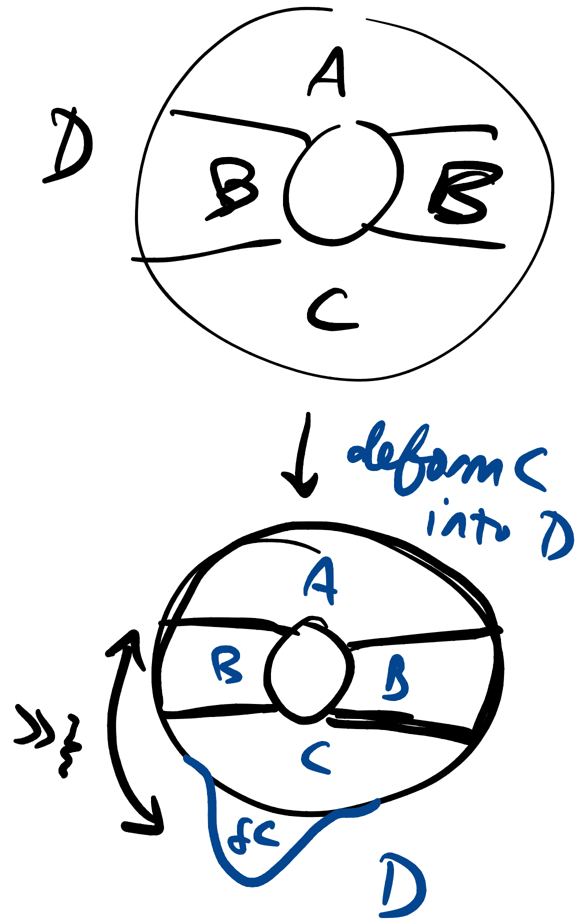
$$\left[\dim \mathcal{H}_{k \text{ anyons of type } a} \overset{k\to\infty}{\simeq} d_a^{\,k} \right]$$

$$-2\gamma = -I(A:C|B) = \underbrace{(S_{ABC} - S_{BC})}_{\Delta S \text{ from closing } BC \text{ on top}} - \underbrace{(S_{AB} - S_B)}_{\Delta S \text{ from closing } B \text{ on top}}$$

Pf of 1 : let $D = \overline{ABC}$

$\Rightarrow ABCD$ is a pure state.

$\Rightarrow$ $S_A = S_{BCD}$.



Under δC :

$S_{AB} - S_B$ can't change.

$S_{ABC} = S_{BC} \overset{\text{pure}}{=} S_D - S_{AD}$

expect $\Delta_{\delta C}(S_D - S_{AD}) = 0$

Suppose state is g.s. of a local H.

Pf of 2 : ΔH localized inside some region doesn't $S_{A,AB} \dots$

If ΔH localized on overlap of A & B :

move A & B so that $\Delta H \in A$.

$\boxtimes$

Fusion rules: $a \times b = \sum_c N_{ab}^c \, c$

$\Rightarrow a \times \underbrace{a \cdots \times a}_{k \text{ times}}$

$d_a d_b = \sum_c N_{ab}^c d_c$

$= \sum_c (N_a)^c{}_b \, d_c$

d_a are positive eigenvectors of $(N_a)^c{}_b$.

3. (Quantum Double Model and) Homotopy (Groups)

Def. A homotopy is a family of maps

$$f_t : X \to Y \qquad t \in I = [0,1]$$

s.t. $F : X \times I \to Y$ is continuous.
$(x, t) \mapsto f_t(x)$

$f_{0,1} : X \to Y$ are homotopic. $f_0 \simeq f_1$.

every map is continuous in the following

A <u>deformation retraction</u> of X onto $A \subset X$

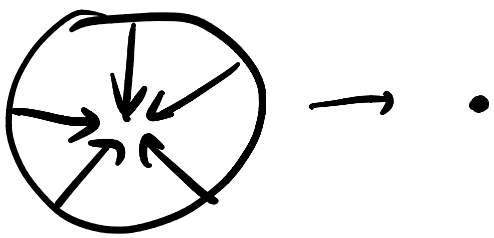
is a homotopy from $f_0 = id : X \to X$

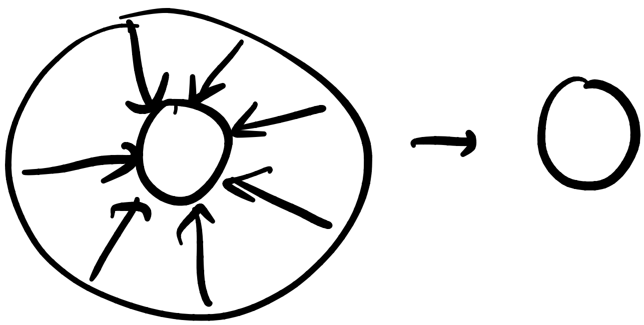
to $f_1 = r : X \to X$

a retraction

$r(X) = A \subset X$

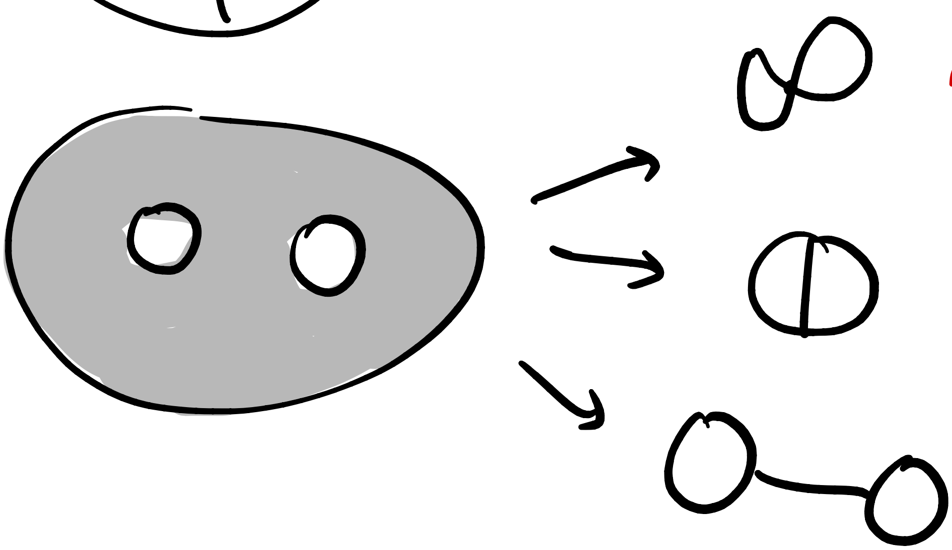
$r^2 = r$. like (a projector)





$\simeq$ is an eq. relation.

X deformation retracts to B is not.



<u>Def:</u> $X \simeq Y$ X is homotopy equivalent to Y

if $\exists$ $f : X \to Y$ and $g : Y \to X$ s.t. $f \circ g \simeq \mathbb{1}$

$g \circ f \simeq \mathbb{1}$.

eg: if X def. retracts to $A \subset X$

via $f : X \times I \to X$

w/ $r : X \to A$

$i : A \hookrightarrow X$

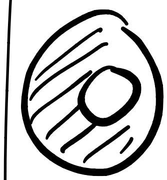
then $r \circ i = \mathbb{1}$ on A

and $i \circ r \simeq \mathbb{1}$ by f.

$\Rightarrow X \simeq A$.

$\bullet \ \bullet \xrightarrow{r} \bullet$

is not a def. retraction

 $\xrightarrow{r} \bullet$

$X \simeq Y \iff \exists Z$ which deformation retracts to X or Y.

If $X \simeq pt$, "X is contractible".

"same"

eg:

category	notion of same
• spaces w/ metrics $\gamma_{\mu\nu}(x)$	isometry
• smooth manifolds	invertible diffeomorphism
• top. mflds	homeomorphism

Homotopy equivalence $\neq$ homeomorphic.

eg: Ball & point (# of dims is a homeomorphism inv't)

eg: $L_{1/7}$ and $L_{2/7}$ $\left(\equiv S^3/\Gamma_{\dots}\right)$

are homotopy equivalent but not homeomorphic.

$\boxed{\text{"combinatorial torsion" distinguishes them.}}$

same homology $\not\Leftrightarrow$ homotopy equivalence

eg $L_{2/5}$ and $L_{3/5}$ have the same $H^\bullet(L,\mathbb{Z})$ but are not homotopy equivalent.

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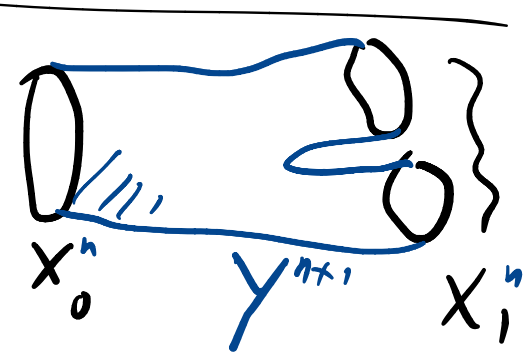
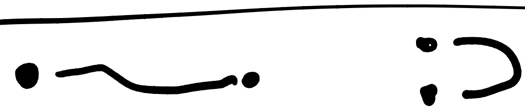
another equivalence relation:

$X_0 \overset{B}{\simeq} X_1$ if $\exists$ a manifold Y s.t. $X_0 - X_1 = \partial Y$.

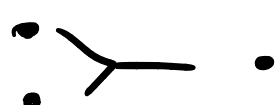


"Bordism"

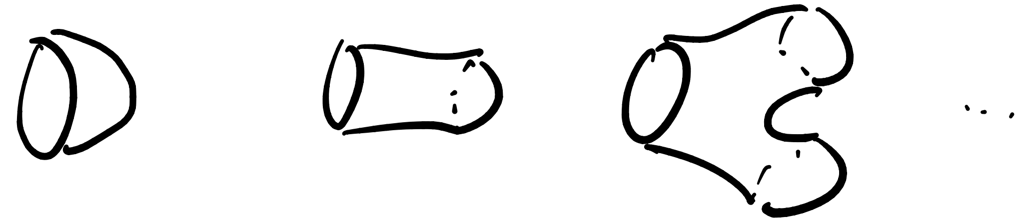
eg: if $\partial Y = \emptyset$ bordism betw $\emptyset$ and $\emptyset$.

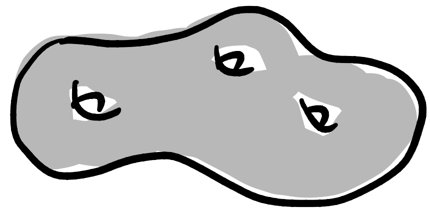
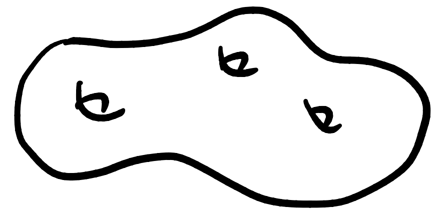


n pts $\overset{B}{\simeq}$ (n+2k) pts.



not a mfld.

In 1d: 

In 2d: ∂  $=$  $- \ \emptyset$

for any closed R.S.

Bordism doesn't preserve $\chi = 2-2g$.

Some surfaces are not the bdy of anything

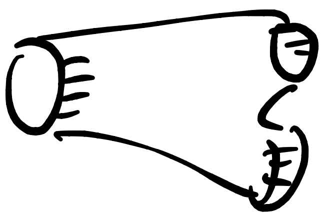
(e.g. Klein bottle)

Atiyah-Segal approach to TQFT:

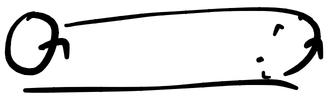
path integral

n-mfld w/o bdy $\longrightarrow$ # Z

n-mfld w/ bdy $\longrightarrow$ $\Psi\begin{bmatrix}\text{bdy}\\ \text{data}\end{bmatrix} \in \mathcal{H}_{\text{bdy}}$

 $\longrightarrow$ $\mathcal{H}_L \otimes \mathcal{H}_R^* \otimes \mathcal{H}_R^*$

$\simeq$ operators: $\mathcal{H}_L \to \mathcal{H}_R \otimes \mathcal{H}_R$

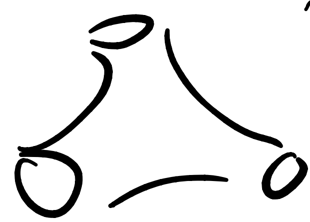


L R

$\longrightarrow$ $\mathcal{H}_L \otimes \mathcal{H}_R^*$

operators: $\mathcal{H}_L \to \mathcal{H}_R$

eg: Chern-Simons thy:
[Swingle Walter ...]

entanglement of Ψ()

eg: [irrational CFT data $\longrightarrow$
"Gaffnian" non-unitary? [S. Simon et al]
or gapless?