- no class Monday
- think about paper topics!
- defects in ordered media $\longrightarrow$ next quarter:

Physics 239 $\cong$ 211C:

Phases of quantum matter

(SPTs, LSM theorems & anomalies,

frustrated magnetism, topological insulators

...)

$$H^\bullet(M) \cong \{\text{harmonic forms on } M\}$$

used Hodge decomposition: any form is

$$\omega = d\alpha + d^\dagger\beta + \underset{\text{harmonic}}{\gamma}$$

$\underline{[Q,H]=0.}$ $\Rightarrow$ each coh. class has rep which is an eigenstate of H.

$$H|\psi\rangle = E|\psi\rangle,\ E>0. \qquad K \equiv \frac{Q^\dagger}{2E}$$

$$1|\psi\rangle = \{Q, \tfrac{Q^\dagger}{2E}\}|\psi\rangle \underset{Q|\psi\rangle=0}{=} QK|\psi\rangle \Rightarrow |\psi\rangle \text{ is exact.}$$

any $[\omega] \in H^0 \longrightarrow$ harmonic form.

Conversely: if $H|\psi\rangle = 0$

$$\Rightarrow 0 = \langle\psi|H|\psi\rangle = \underbrace{\| Q|\psi\rangle\|^2}_{\geq 0} + \underbrace{\| Q^\dagger|\psi\rangle\|^2}_{\geq 0}$$

$$\Rightarrow \underbrace{Q|\psi\rangle = 0} = Q^\dagger|\psi\rangle.$$

$\Rightarrow [|\psi\rangle] \in H^0$ is nontrivial $\quad |\psi\rangle \neq Q|\alpha\rangle$

suppose $|\psi\rangle = Q|\alpha\rangle \Rightarrow H|\psi\rangle = 0$

$\Rightarrow H|\alpha\rangle = 0.$

$[H,Q]$

$\Rightarrow Q|\alpha\rangle = Q^\dagger|\alpha\rangle = 0$

$\Rightarrow \underline{|\psi\rangle = Q|\alpha\rangle = 0.}$

$\blacksquare$

2.3 Susy QM & Morse theory

method #2: Choose a f'n $h : \mathcal{M} \to \mathbb{R}$

$$\Delta S = - \int d^D x \left(\frac{\gamma^{ij}}{2} \partial_{\phi^i} h \, \partial_{\phi^j} h + \underbrace{\partial_{\phi^i} \partial_{\phi^j} h}_{m_{ij}} \bar{\Psi}^i \Psi^j \right)$$

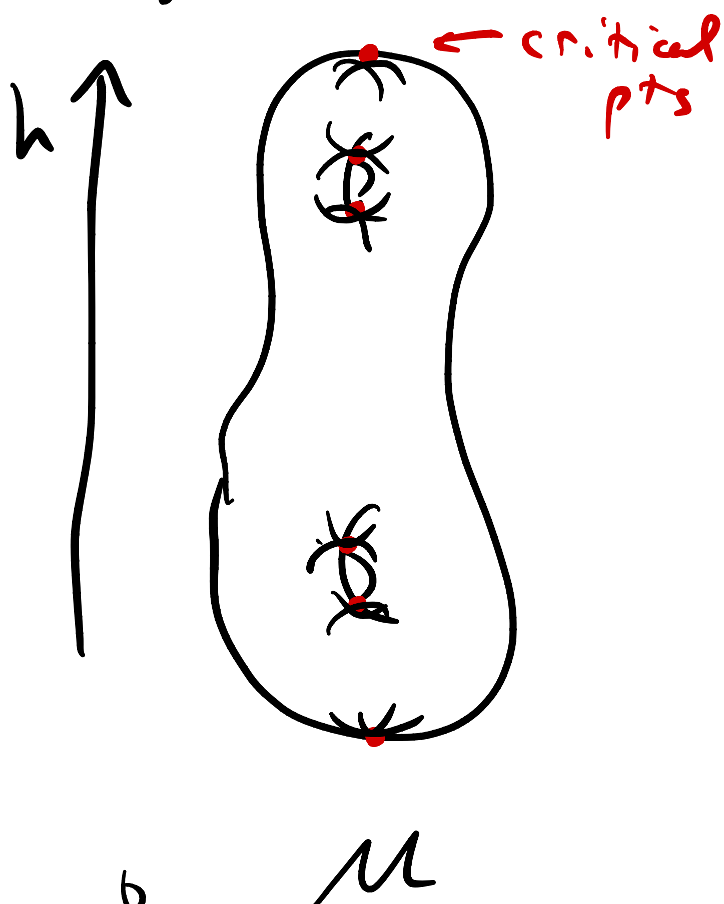
$$\left(\Delta L = \int d\theta \, d\bar{\theta} \; h(\Phi^i(t, \theta, \bar{\theta})) \right)$$

$\Longrightarrow$ supersymmetric.

$$V(\phi) = \frac{1}{2} |\vec{\nabla} h|^2 \qquad m_{ij}(\phi) = \partial_{\phi^i} \partial_{\phi^j} h .$$

lifts degeneracy

$\longrightarrow$ critical pts of h.



Assume: critical pts of h

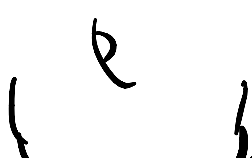
$(\partial_{\phi^i} h = 0 \ \forall i)$

are isolated pts

with $\partial_i \partial_j h$ has no

zero eigenvalues.

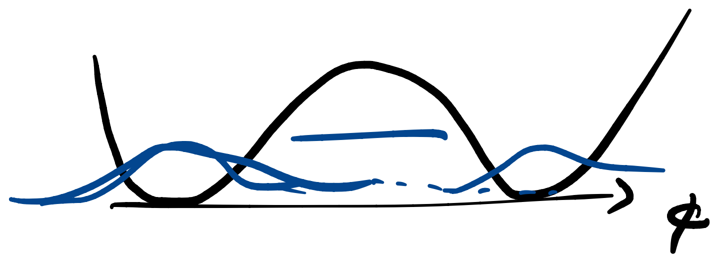
"MORSE FUNCTION"

not: 

← non-generic.

$$\{p^A,\ A=1..k\} = \{\text{points } p\in\mathcal{M} \text{ s.t. } \partial_{\phi^i} h = 0 \ \forall i\}$$

$=$ classical $E=0$ groundstates.



$\boxed{Q}$: which are bosonic & which are fermionic?

$$[(-1)^F, \mathcal{O}_B] = 0 = \{(-1)^F, \mathcal{O}_F\}$$

convention: $(-1)^F|0\rangle = |0\rangle$.

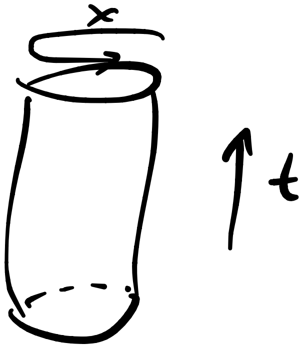
note: ΔS preserves $F = \delta_{ij}\bar{\psi}^i\psi^j$.

(inv't under $\psi \to e^{i\alpha}\psi$, $\bar{\psi} \to e^{-i\alpha}\bar{\psi}$)

Toy model: single massive majorana fermion in $D=2$

$$S[\psi] = \frac{1}{2}\int d^2x \left(\bar{\psi}\, i\partial\!\!\!/\, \psi - m\bar{\psi}\psi \right)$$

Take $\gamma^0 = \sigma^2 \Rightarrow \psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$ $\quad \psi_{1,2} = \psi_{1,2}^*$. $\quad m \in \mathbb{R}\setminus\{0\}$



PBC: $\psi_{i=1,2}(x+L) = \psi(x)$

$$\psi_i(x) = \sum_n e^{i\frac{2\pi x}{L}n}\,\psi_{in}$$

$n \in \mathbb{Z}$

$$= \underbrace{\sigma_{i=1,2}}_{\equiv \psi_{0i}} + \sum_{n\neq 0} e^{\frac{2\pi i n x}{L}}\psi_{in}$$

↑ zero-momentum mode.

canonical quantization $\Longrightarrow \{\sigma_i, \sigma_j\} = 2\delta_{ij}$ $\quad i=1,2.$

other modes have $k = \frac{2\pi n}{L}$, $E = \sqrt{k^2+m^2} > 0$ $\forall n>0.$
$n>0$

$\rightarrow$ empty in the g.s.

$$c = \frac{\sigma_1 + i\sigma_2}{2} \qquad c^\dagger = \frac{\sigma_1 - i\sigma_2}{2} \rightarrow \{c, c^\dagger\} = 1.$$

is represented by $\begin{cases} c|0\rangle = 0 \\ c^\dagger|0\rangle = |1\rangle. \end{cases}$

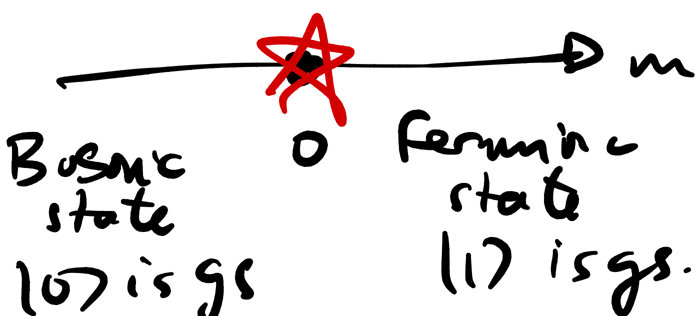
acting on $|0\rangle, |1\rangle$, $\sigma^{1,2}$ are the Pauli Matrices.

Plug in $\psi_i = \sigma_i + \dots$ into H

$$\to H = -im\,\sigma_1\sigma_2 = m\,\sigma_3 = \underline{\underline{m(-1)^F}}$$

PHASE DIAGRAM:

$$\begin{cases} (-1)^F|0\rangle = |0\rangle \\ (-1)^F|1\rangle = -|1\rangle . \end{cases}$$



Q: vary $c = \psi_1 + i\psi_2$

$$\underline{\underline{c(x+L) = e^{i\alpha}c(x)}} \quad ?$$

Generalize step 1: n such modes w/ masses $m_1 \dots m_n$.

gs when $m_1, \dots m_n > 0$ is bosonic.

Then $(-1)^F|gs\rangle = (-1)^{\#\text{ of negative } m_i}|gs\rangle$.

step 2: $\quad L \ni \quad m_{ij} \, \bar{\Psi}_i \Psi_j$

assume $\underline{m_{ij} = m_{ji}}$.

$$\text{by} \begin{cases} \Psi_i \to R_{ij} \Psi_j \\ \bar{\Psi}_i \to \tilde{R}_{ij} \bar{\Psi}_j \end{cases}$$

$$m_{ij} = \left(R \ \mathrm{diag}(m_1 \ldots m_n) \tilde{R} \right)_{ij}$$

$\longrightarrow$ previous case

---

step 3: fluctuation around $\underline{P_A}$.

$$m_{ij}^A = \left. \frac{\partial^2 h}{\partial \phi_i \partial \phi_j} \right|_{P_A}$$

let $\boxed{n^A \equiv \# \text{ of negative evals of } m_{ij}^A}$

($\equiv$ Morse index of p^A.)

$$(-1)^F \left| \begin{matrix} \text{classical vac} \\ \text{associated w/ } p^A \end{matrix} \right\rangle = (-1)^{n^A} |p^A\rangle .$$

$\equiv |p^A\rangle$

Pairing up due tunneling preserves $\mathrm{tr}(-1)^F$

$$\boxed{\mathrm{tr}(-1)^F = \sum_A (-1)^{n_A}}.$$

$\Rightarrow$ ind of h.

$$= \underbrace{\chi(M)}_{\text{ind. of } h}.$$

(similarly if we pick h to inv't under K then Lef can also be computed.)

$$S[\phi] = \int \underbrace{\gamma_{ij}(\phi)}_{\text{can pick R'n normal coords}} \partial_\mu \phi^i \partial^\mu \phi^j$$

Near p w/ $\phi = 0$:

$$\gamma_{ij}(\phi) = \delta_{ij} + R^{(p)}_{ijkl} \phi^i \phi^j + O(\phi^3)$$

$$= \int \Big[\underbrace{\partial_\mu \phi^i \partial^\mu \phi_i}_{} + \underbrace{R^{(p)}_{ijkl}}_{\#} \partial\phi^i \partial\phi^j \phi^k \phi^l + O(\dots) \Big]$$

coupling constants.

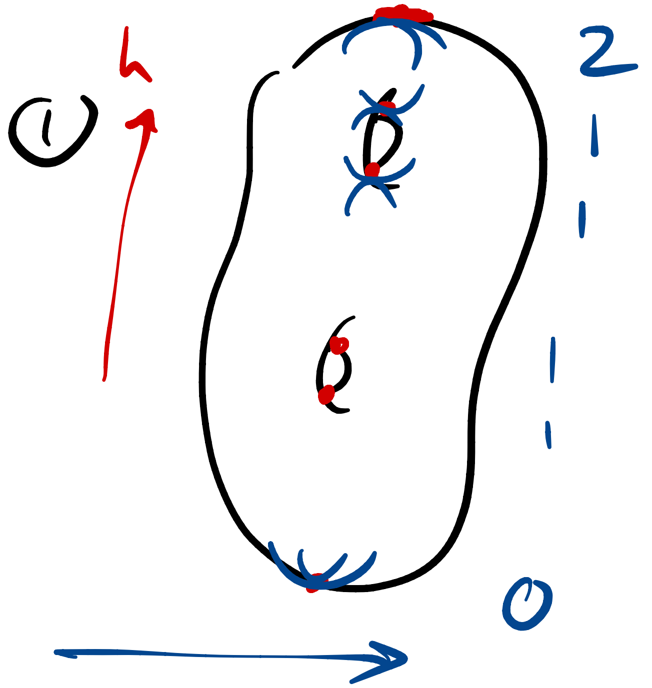
$$\underbrace{m_{ij}(\phi)}_{= \left.\frac{\partial^2 h}{\partial\phi_i \partial\phi_j}\right|_{\phi_i(t,x)}} \bar{\psi}_i \psi_j \in \mathcal{L}$$

$$V(\phi) = \frac{|\partial h|^2}{2}$$

$$\simeq 0 + (\phi - \phi^*)^i (\phi - \phi^*)^j m_{ij}(\phi^*) + O(\phi - \phi^*)^3$$

Examples :

① $\mathcal{M} = \Sigma_2$ genus 2.

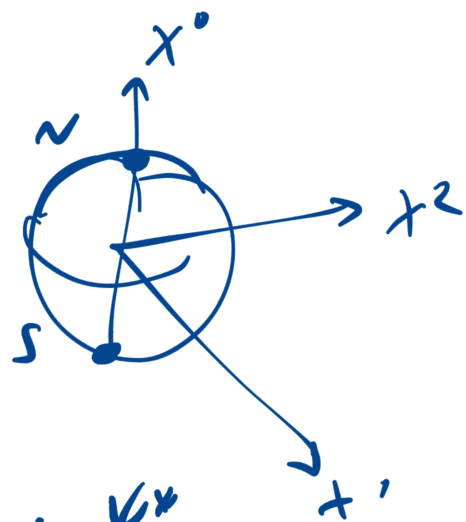


$$\Rightarrow tr(-1)^F = (-1)^2 + 4(-1)^1 + (-1)^0 = -2$$

$$= 2-2g\Big|_{g=2}.$$

② $\underline{\mathcal{M} = S^n} = \left\{ x_i \,\middle|\, \sum_{i=0}^{n} x_i^2 = 1 \right\}$

$\leadsto \quad h(x) = x_0 .$



claim : $\underline{tr(-1)^F = 1 + (-1)^n}$.

$K : x_n \to -x_n \quad \psi_n \to -\psi_n , \quad \psi_n^* \to -\psi_n^*$

$\Rightarrow \underline{tr(-1)^F K = 1 - (-1)^n}$. $\Rightarrow$ no $\underline{SSB}$ for any n.

③ $\mathcal{M} = \mathbb{C}P^N$. $h = |z_0|$. $\to \underline{tr(-1)^F = N+1}$.

Let $Q \equiv$ s-percharge for NLSM w/o h.

$$Q_t \equiv e^{-th} Q e^{th}$$ is the supercharge after $\underline{\Delta S}$.

$$\{Q_t, Q_t^\dagger\} = 2H_t$$

$$\underline{\underline{d_t}} \equiv \underline{\underline{e^{-th} d e^{+th}}}$$ is the action of Q_t on forms.

$\Rightarrow d_t, d$ have the same cohomology ind. of t.

$\Rightarrow$ can compute $H^\bullet(M)$ by taking $\underline{\underline{t}}$ large.

Near p^A

$$H_t \cong \frac{1}{2} \sum_i \Big(\underbrace{-\partial_{\phi_i}^2 + t^2 m_i \phi_i^2}_{= H_i} + \overbrace{t\, m_i [a_i^\dagger, a_i]}^{\equiv K_i} \Big) + \mathcal{O}(\phi^3 \ldots)$$

$m_i \equiv$ evals of $\partial_{\phi_i} \partial_{\phi_j} h \Big|_{p^A}$.

smaller as t grows

has $E_0 = \underline{\underline{t}} \, (\underline{\underline{A}} + \mathcal{O}(1/t))$

$$E_{(N,k)} \equiv t \sum_i \left(|m_i|(1+2N_i) + \underline{\underline{m_i k_i}} \right)$$

$N_i = 0, 1, 2 ..$ S+10 levels

$k_i = 0, 1$ fermion #.

gs requires $\underline{N_i = 0}$. and $\begin{pmatrix} k_i = 1 \\ \Leftrightarrow m_i < 0 \end{pmatrix}$

$\Rightarrow E_{(N,k)} = 0.$

is a q-form if $\underline{\sum_i k_i = q}$.

Ass. w/ each critical pt p^A is a $\underline{\text{unique}}$
approx (perturbative) groundstate $|\alpha_A\rangle$ w/ $E = 0$
is a q-form if $n_A = q$.

$\underline{\text{only}}$ these states can be true susy g.s.

$\Rightarrow$ weak Morse inequalities:

$$b^p(M) \leq m^p \equiv \left(\begin{array}{l}\#\text{ of critical pts}\\ \text{of } h \text{ w/ Morse index}\\ p.\end{array}\right)$$

$b^p(M)$ ↑ true g.s. $\qquad$ m^p ↑ approx g.s.

Tunneling.

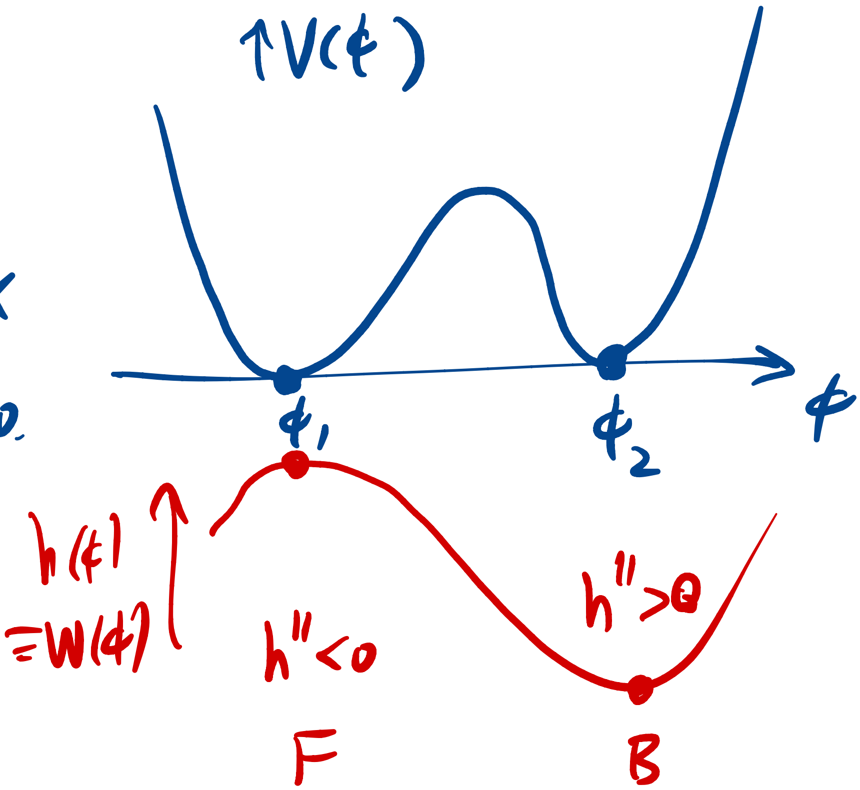
Let $X^p \equiv \text{span}\{|\alpha_p\rangle \text{ of Morse index } p\}$

$\delta: X^p \to X^{p+1}$ w/ $\delta^2 = 0$

cohomology of $\delta = H^p(M)$.

Consider: $M = \mathbb{R}$

with $W(\phi) = h(\phi) = gx^3 - x$



$a|\psi\rangle = 0, \; a^\dagger|\psi\rangle = 0$

$\langle\phi|\psi_+\rangle = |\downarrow\rangle \otimes e^{-W(\phi)}$

$\langle\phi|\psi_-\rangle = |\uparrow\rangle \otimes e^{+W(\phi)}$

neither is norm'ble. $\Rightarrow$ SUSY BROKEN

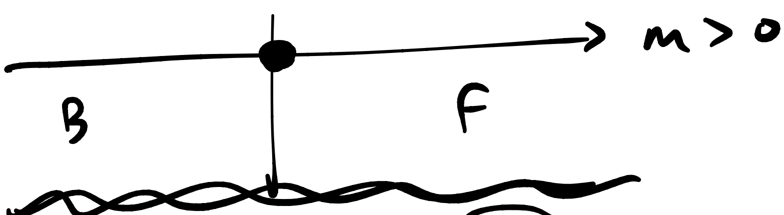
$$S[\psi] = \int d^2x \left[\bar{\psi} \gamma \psi - m \bar{\psi}\psi \right]$$

$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ $\quad \psi_{i=1,2} = \psi_i^{\dagger}.$ $\quad \gamma^0 = \sigma^2.$

$c \equiv \psi_1 + i\psi_2$ $\quad \boxed{c(x+L) = e^{i\alpha} c(x).}$

$\bar{\psi}\psi = i\psi_1\psi_2.$

$$T: \left\{ \begin{array}{l} i \to -i \\ \psi_1 \to \psi_1 \\ \psi_2 \to -\psi_2 \\ c \to c. \end{array} \right.$$



$$\Delta S = m' \bar{\psi}\gamma^5\psi$$

$$\gamma^5 = \gamma^0\gamma^1$$

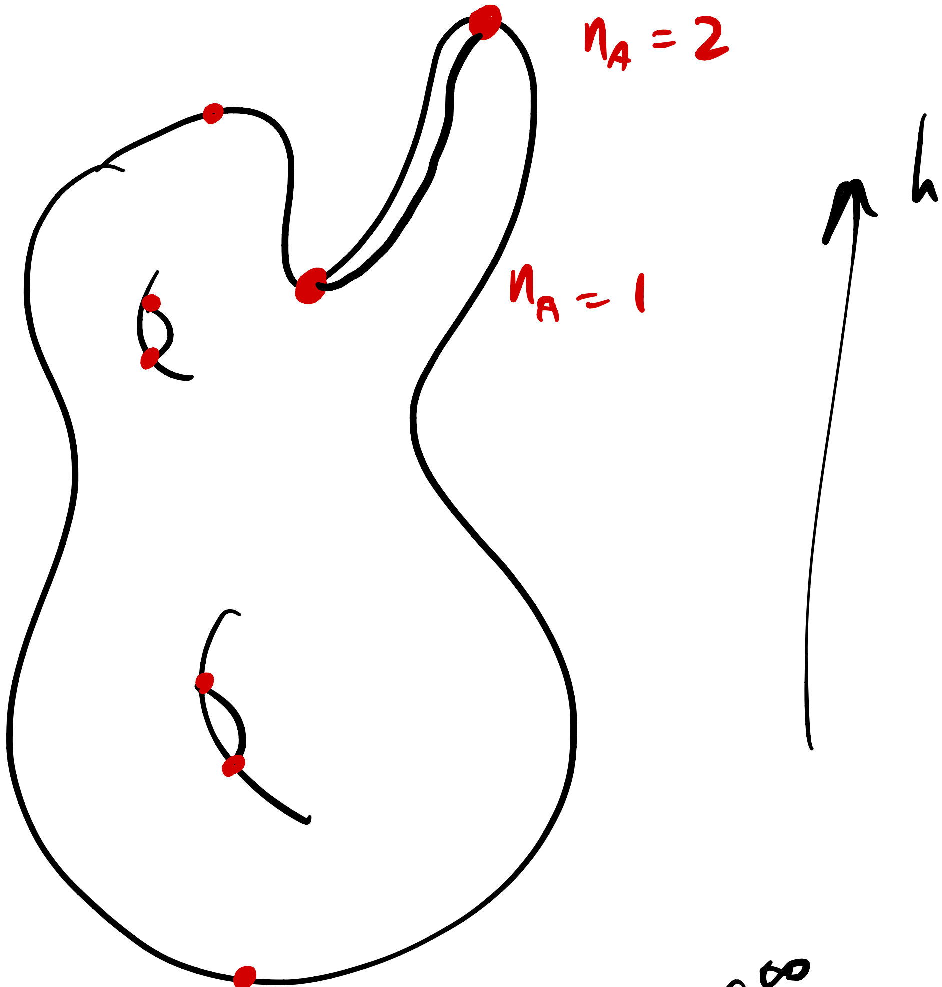
$\gamma^0 = \sigma^2$

$\gamma^1 \stackrel{?}{=} \sigma^3$

$\gamma^1 = \sigma^1$

$$= m'(\psi_1 \ \psi_2)\, \sigma^1 \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$= 0.$$



$$\mathcal{U} = \left\{ \text{space of } C^\infty \text{ gauge fields on } \boxed{X_3} \right\}$$

$$h[A] = \underline{S_{cs}[A]}.$$

$\rightarrow$ Floer homology of X_3.

Witten, Top. Quantum Field Theory, CMP