Recall: Given Q, $Q^2 = 0 \longrightarrow H^\bullet(Q)$.

why is the cohomology of Q topological?

Ops. of the form $V = \{Q, \lambda\}$

(for some λ) acts trivially on

Q-cohomology. $\left(\delta_{susy}\mathcal{O} = [Q, \mathcal{O}]_\pm\right)$

$$\mathrm{tr}(-1)^F = \mathrm{tr}(-1)^F e^{-\beta H} = \int_{PBC} D\phi D\psi\, e^{-S}$$

change $S \to S + \int V$

$$\delta_V\left(\mathrm{tr}(-1)^F\right) = \int_{PBC} D\phi D\psi\, e^{-S} \int V$$

$$= \underbrace{\int e^{-S} \int \{Q, \lambda\}}_{} = \mathrm{tr}(-1)^F \{Q, \lambda\}$$

2 POVs: (1) operator: only states w/ $\begin{cases} Q|\psi\rangle = 0 \\ Q^\dagger|\psi\rangle = 0 \end{cases}$ contribute.

$\Rightarrow = 0.$

(2) $= \delta_{susy}\left(\int e^{-S} \lambda\right) = 0.$ (Ward id.)

General point: Given a symm. of $S[\phi]$
& $\int D\phi D\psi \ldots$

$$\langle \delta g \rangle = \int D\phi D\psi \, e^{-S[\phi]} \delta g$$

$$= \delta \left(\int D\phi D\psi \, e^{-S[\phi]} g \right)$$

$$= 0. \qquad (\text{Ward id.})$$

---

more generally: Consider $\mathcal{O}$ s.t. $[Q, \mathcal{O}]_\pm = 0$

$$\delta \left(\mathrm{tr}\, (-1)^F \mathcal{O}_1 \ldots \mathcal{O} \right) = 0.$$

In Witten's twisted susy theories,

$$T_{\mu\nu}(x) = \{ Q, \lambda_{\mu\nu}(x) \}$$

$$\propto \frac{\delta S}{\delta g^{\mu\nu}(x)} \implies \text{insensitive to } \delta g_{\mu\nu}.$$

Differential forms interlude:

A p-form on a smooth manifold $\mathcal{M}$

is made from $A_{i_1 \dots i_p}(x)$

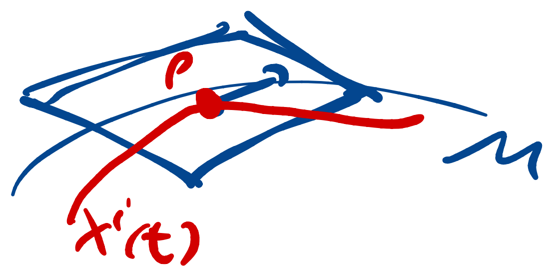
$$A = \frac{1}{p!} A_{i_1 \dots i_p}(x)\, dx^{i_1} \wedge \dots \wedge dx^{i_p}$$

dx^i are coord. differentials. ie cotangent vecs.

Recall: A tangent vector to $\mathcal{M}$ at p.

$v \in T_p\mathcal{M}$.

is $v = v^i \dfrac{\partial}{\partial x^i}$



$\forall f : \mathcal{M} \to \mathbb{R}$, $\forall$ path $x^i(t)$ through p

rate of change of f along path at p

is $$\left.\frac{df}{dt}\right|_p = \left.\frac{dx^i}{dt} \frac{\partial}{\partial x^i} f \right|_p .$$

$T_p\mathcal{M} = \mathrm{span}\left\{ \dfrac{\partial}{\partial x^i} \right\}$.

Cotangent vectors $\equiv$ elements of T_p^*M

eats vectors gives #s.

$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i{}_j.$$

$$dx^i \wedge dx^j \in T_p^*M \otimes T_p^*M$$

$$= -dx^j \wedge dx^i \in \Lambda^2 T_p^*M$$

~~fermions~~.

$\boxed{\wedge}$: completely AS product.
linear in each argument.)

smooth p-forms on M is a v.s. $\Omega^p(M)$ over $\mathbb{R}$

$$(A_p \wedge B_q)_{i_1 \cdots i_{p+q}} = \frac{(p+q)!}{p!\,q!} A_{[i_1 \cdots i_p} B_{i_{p+1} \cdots i_{p+q}]}$$

average over perms $(-1)^\sigma$.

$\boxed{d}$ $d : \Omega^p \to \Omega^{p+1}$ $\qquad d = dx^\nu \wedge \frac{\partial}{\partial x^\nu}$

$$(dA_p)_{i_1 \cdots i_{p+1}} = (p+1)\, \partial_{[i_1} A_{i_2 \cdots i_{p+1}]}.$$

$d^2 = 0$ because $[\partial_\mu, \partial_\nu] = 0$.

$\boxed{S:}$ $X_p \subset M$. $A_p \in \Omega^p(M)$

$$\int_{X_p} A_p \in \mathbb{R}.$$

Stokes Thm: $\displaystyle\int_{X_p} dA_{p-1} = \int_{\partial X_p} A_{p-1}$

(d is the adj of ∂ under the pairing $\underbrace{\Omega_p(M)}_{p\text{-cells}}$ and $\underbrace{\Omega^p(M)}_{p\text{-forms}}$)

so far no metric.

$\boxed{\star}$ $* : \Omega^p \to \Omega^{n-p}$ $n = \dim M$.

$$(*\omega_p)_{i_{p+1} \ldots i_n} = \frac{\sqrt{g}}{p!} \epsilon_{i_1 \ldots i_n} \omega^{i_1 \ldots i_p}$$

$(\eta, \omega) \equiv \int_M \eta \wedge * \omega$ inner prod. on $\Omega^p(M)$

$\eta \wedge * \omega = (\eta, \omega)\, \underline{\mathrm{vol}}$

Return to NLSM :

$$\underline{\mathcal{H}_{NLSM}} \qquad\qquad \underline{\Omega^p(M)}$$

$$A(\phi)|0\rangle \longleftrightarrow A \in \Omega^0(M)$$

$$A_i(\phi)\psi^{i*}|0\rangle \longleftrightarrow A_i d\phi^i \in \Omega^1(M)$$

$$|A_p\rangle = A_{i_1\ldots i_p}\psi^{i_1*}\ldots\psi^{i_p*}|0\rangle \longleftrightarrow A_{i_1\ldots i_p} d\phi^{i_1}\wedge\ldots\wedge d\phi^{i_p} \in \Omega^p(M)$$

$$\mathcal{H} = \bigoplus_{p=0}^{n} \Omega^p(M).$$

$$Q|A_p\rangle = |\underbrace{dA_p}_{p+1 \text{ form.}}\rangle$$

$Q^\dagger$ removes a fermi ψ^i, differentiates in the i direction.

$$Q^\dagger|A_p\rangle = \partial_{\phi^{i_p}} A_{i_1\ldots i_p}\psi^{*i_2}\ldots\psi^{*i_p}|0\rangle$$

$$= \underline{|d^\dagger A\rangle}.$$

$(\eta, d^\dagger \omega) \equiv (d\eta, \omega)$. $\forall \omega, \eta$.

$$d^\dagger = s * d *$$.

$s = \pm 1$. depends on p, n.

$*^2 = (-1)^{p(n-p)}$.

why? $0 = \int_M d(\omega_1 \wedge * \omega_2)$

if $\partial M = 0$

note: $(d^\dagger)^2 = 0$.

$$H_{NLSM} = \frac{1}{2}\{Q^\dagger, Q\} = \frac{1}{2}(d d^\dagger + d^\dagger d)$$

$$\equiv \Delta$$ Laplace-Beltrami op. on forms.

susy g.s.

$Q|\psi\rangle = Q^\dagger|\psi\rangle = 0$.

is a harmonic p-form.

$F|A^p\rangle = p|A^p\rangle = 0$

$\Delta A = 0$.

# of harmonic p-forms $= b^p(M)$. pth Betti # of M.

$$\underset{\substack{\text{NLSM} \\ \text{on } \mathcal{M}}}{\mathrm{tr}}\,(-1)^F = \sum_{p=0}^{n} (-1)^p\, b^p(\mathcal{M}) = \chi(\mathcal{M}).$$

---

if $\mathcal{M}$ has an <u>isometry</u> $\Rightarrow \exists\, K$ on $\mathcal{H}$ s.t. $\underline{[H, K] = 0}$.

$\rightarrow K$ acts on each Ω^p $\quad \left\{ \begin{array}{l} [K, Q] = 0. \\ [K, F] = 0. \end{array} \right.$

let $\quad b^{p}(\kappa) = \#\left\{ \begin{array}{l} \text{harmonic} \\ p\text{-forms w/} \\ \text{eval } \kappa \text{ of } K \end{array} \right\}$

$$K A_{i_1 \ldots i_p} = \kappa A_{i_1 \ldots i_p}.$$

$$\Rightarrow \mathrm{tr}\,(-1)^F K = \sum_{p=0}^{n} (-1)^p\, b^{p(\kappa)}\, \kappa \equiv \mathrm{Lef}(K).$$

Lefschetz index.

---

we know $Q^2 = 0$

$\Rightarrow \underline{d^2 = 0}. \;\Rightarrow\;$ im $d: \Omega^{p-1} \to \Omega^p$

$\subset$ ker $d: \Omega^p \to \Omega^{p+1}$

$$H^p(\mathcal{M}) = \frac{\ker d: \Omega^p \to \Omega^{p+1}}{\mathrm{im}\, d: \Omega^{p-1} \to \Omega^p}$$

de Rham cohomology of $\mathcal{M}$.

[Note: NLSM has a time-reversal sym
$\Longrightarrow$ ~~eigen~~ eigenfns are real.
if not, $(\omega, \eta) = \int \omega^* \wedge * \eta$]

$* : \Omega^p \to \Omega^{n-p}$.

$$\underbrace{\psi^{*1} \cdots \psi^{*p}}_{\text{filled}} |0\rangle \xrightarrow{*} \psi^{*p+1} \cdots \psi^{*n} |0\rangle$$

filled $\longleftrightarrow$ empty.

is a particle-hole transf.

The NLSM in $D=2$ has a discrete chiral sym

$\psi \to \gamma^5 \psi$. $\qquad \gamma^5 = \prod_{i=0}^{1} \gamma^i$

$= \gamma^0 \gamma^1$

In the Weyl basis: $Q = \begin{pmatrix} Q_+ \\ Q_- \end{pmatrix}$

$$Q_\pm \longrightarrow \pm Q_\pm .$$

$$\gamma_5 Q_\pm = \pm Q_\pm \gamma_5 .$$

ie $\{\gamma_5, Q_-\} = 0$ just like $\{(-1)^F, Q\} = 0$.

$\underset{\text{arg}}{\xrightarrow{\text{same}}}$ $E > 0$ come in pairs of opposite chirality ($\in \gamma_5$ eval)

$E = 0$ are singlets.

$\Rightarrow$ tr γ^5 is topological.

tr $\gamma^5 K$ " "

---

explicitly: ψ^i, ψ^{i*} are γ^0 eigenstates

$\gamma^0 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

But $\{\gamma^5, \gamma^0\} = 0$.

$\gamma^5 : \psi^{*i} \leftrightarrow \psi^i$, $|0\rangle \leftrightarrow \psi^{*1} \dots \psi^{*n} |0\rangle$

$$\gamma^5 \psi^{i_1 *} \dots \psi^{i_q *} |0\rangle = c\, \epsilon_{i_1 \dots i_n} \psi^{i_{q+1} *} \dots \psi^{i_n *} |0\rangle$$

$(\gamma^5)^2 = 1$.

to compute tr γ^5: Decompose $\Omega^q \oplus \Omega^{n-q}$ into eigenstates $\pm$ of γ^5.

$$\mathrm{tr}\,\gamma_5 = \sum_{q=0}^{\lfloor n/2 \rfloor} (b_+^q - b_-^q)$$

$b_\pm^q \equiv \#$ of $\pm$ harmonic forms

$= \mathrm{sign}(M)$ HIRZEBRUCH SIGNATURE.

$\underline{\mathrm{tr}\,\gamma^5 K} =$

$$\sum_{q=0}^{[n/2]} k_2 \left(b^q_{+k_2} - b^q_{-k}\right)$$

$[K, \gamma^5] = 0.$

$\Rightarrow \Omega^q \oplus \Omega^{n-q}$

$= \bigoplus_{\pm} \bigoplus_{k_2} \{$ eigenstate of γ^5 AND $K \}$

---

when $q = n - q$.

$\Omega^q_{\pm} =$ self-dual/ASD q-forms.

eg: $\underline{n = 4,\ q = 2}$ $\underline{n = 2,\ q = 1}$.

$\mathrm{tr}\,\gamma_5 \overset{?}{=} \int$ curvature

or

$\boxed{\mathrm{tr}\,(-1)^F =} \chi(M)$

$= \underbrace{\int_{PBC\ \Phi(\tau+\beta) = \Phi(\tau)} D\phi\, D\psi\, D\bar{\psi}\, e^{-S}}$

$\underline{\text{ind. of } \beta}$. take $\beta \to 0$

$= \underbrace{\sum_{\text{cnfgs w/ } \delta\psi = 0}} \cdots = \int d\phi^i\, d\psi^i\, d\bar{\psi}^i\, e^{-S}$

$$S[\phi, \psi \text{ constant}] = \int \cancel{(\partial\phi)^2}^{\,0} + \cancel{\bar\Psi D \Psi}^{\,0}$$

$$+ \underline{\underline{R_{ijkl}}}\, \psi^i \bar\psi^k \psi^j \bar\psi^l$$

$$\underline{\underline{\int d\psi\, \psi = 1.}} \quad \underline{\int d\psi\, 1 = 0.}$$

$\underline{\text{eg} \quad M = \mathbb{R}^3.}$ $\qquad \Omega^0 \cong \Omega^3(\mathbb{R}^3)$

$$f(x)\, dx \wedge dy \wedge dz = f(x)\, \mathrm{vol}.$$

$$\Omega^1(\mathbb{R}^3) \simeq \Omega^2(\mathbb{R}^3)$$

$f_i(x) dx^i$

$$f_x\, dy \wedge dz + f_y\, dz \wedge dx + f_z\, dx \wedge dy$$

$$= f_i \frac{\epsilon_{ijk}}{2} dx^j \wedge dx^k$$

$$df = \partial_i f\, dx^i \quad (\text{grad}), \qquad d(f dx \wedge dy \wedge dz) = 0.$$

$$d(f_i dx^i) = (\partial_y f_z - \partial_z f_y)\, dy \wedge dz + \text{cyclic}$$

curl

$$= \frac{1}{3!} \epsilon_{ijk} \partial_i f_j \epsilon_{ilm}\, dx^l \wedge dx^m.$$

$d(f_x\, dy \wedge dz + \dots) = \partial_i f_i \, dx \wedge dy \wedge dz$

div

$d(\text{0-form}) = \text{grad}$ $\quad d(\text{1-form}) = \text{curl}$

$d(\text{2-form}) = \text{div}.$

<u>eg</u>: given a fluid flow on X

is $\vec{u} = \vec{\nabla}\phi$?

<u>electrostatics</u>:

$\vec{E} = -\vec{\nabla}\phi$ on $X \setminus \{\text{charges}\}$.

<u>E&M on $\mathbb{R}^4$.</u>

$$F = dA = E_i\, dx^i \wedge dt + B_x\, dy \wedge dz + \dots$$

$$= E_i\, dx^i \wedge dt + B_i \frac{\epsilon_{ijk}}{2} dx^j \wedge dx^k$$

$$*F = -B_i\, dx^i \wedge dt + E_i \frac{\epsilon_{ijk}}{2} dx^j \wedge dx^k.$$

<u>Maxwell's</u> (away from charges): $\begin{cases} dF = 0 \Leftarrow d^2 = 0 \\ d{*}F = 0. \end{cases}$

$$0 = d{*}F \propto \frac{\delta S[A]}{\delta A} \leftarrow \underline{\partial^\mu F_{\mu\nu} = 0.}$$

$$\propto \underline{d^\dagger F}$$

$$S[A] = -\frac{1}{2e^2}\int F \wedge {*F}$$

$$= -\frac{1}{4e^2}\int F_{\mu\nu}F^{\mu\nu}\sqrt{g}\, d^Dx.$$

Maxwell in vacuum $\Rightarrow [F],\ [{*F}] \in H^2(M)$

$$M = \mathbb{R}^4 \setminus \{\text{charges}\}$$

<u>eg:</u> static charge at 0. $\quad \underline{\mathbb{R}_t = \{(\vec{0}, t)\}}$

$$F = q\frac{dr \wedge dt}{r^2} = d\left(-q\frac{dt}{r}\right) \quad \text{is exact.}$$

$$[F] = 0 \in H^2(M)$$

$$dr = \frac{x^i dx^i}{r}, \qquad {*}\,dx \wedge dt = dy \wedge dz.$$

$$\Rightarrow {*F} = q\,\frac{x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy}{r}$$

$[{*F}] \neq 0$ in $H^2(M)$.

pairing between $H^2(M)$ and $H_2(M)$

$$\int_{S^2} *F = 4\pi q \implies \text{prove that } [*F] \neq 0.$$

gaussian surface

charge $\in H^2(M)$.

Include charge in EOM: $\Delta S = \oint_C A$

worldline trajectory of charge

"minimal coupling".

Abelian p-form gauge theory

$F_2 = dA_1$ is inv't under $A_1 \to A_1 + d\lambda_0$

$F_{p+1} = dA_p$, $\delta A_p = d\lambda_{p-1}$.

is inv't under

$$S[A] = -\frac{1}{2g^2}\int \underbrace{F_{p+1} \wedge * F_{p+1}}$$

$$= (F_{p+1}, F_{p+1})$$

$$\equiv \| F_{p+1} \|^2 .$$

$$\propto -\int \frac{\sqrt{g}}{(p+1)!} F_{i_1 \dots i_{p+1}} F^{i_1 \dots i_{p+1}} .$$

for $p=0$, $\mathcal{L} = -\frac{1}{2}(\partial\phi)^2$. massless scalar.

$$0 = \frac{\delta S}{\delta A} = d * F , \quad (\& \; dF = 0 .)$$

<u>minimal coupling</u>: $\Delta S = \int_{X_p} A_p$

worldvolume $\to X_p$

of a charged <u>$(p-1)$ brane</u>.

<u>if $D = 2 \bmod 4$</u> we can let <u>$F_{D/2} = \pm * F_{D/2}$</u> .

in $D=2$ this is a chiral scalar.

Duality: $dA_p = * dA^\vee_{D-p}$

$$\left\{\begin{array}{l} dF = 0 \\ d*F = 0 \end{array}\right. \begin{array}{c} \longleftrightarrow \\ \longleftrightarrow \end{array} \left\{\begin{array}{l} d*F^\vee = 0 \\ dF^\vee = 0. \end{array}\right.$$

$$Z = \int [dA]\, e^{-\frac{1}{2g^2}\int dA\wedge * dA}$$

$$= \int [dF][dA^\vee]\quad e^{-\frac{1}{2g^2}\int F\wedge *F + i\int dF\wedge A^\vee}$$

$$\left(\int [dA^\vee]\ldots = \delta[dF] \right)$$

$$= \int [dA^\vee]\, e^{-\frac{g^2}{2}\int dA^\vee\wedge * dA^\vee}$$

$$\underline{\underline{g \longleftrightarrow 1/g.}}$$

eg: $p=0$, $D=2$.

$$L = -\frac{1}{2g^2}(\partial\phi)^2 \longleftrightarrow -\frac{g^2}{2}(\partial\phi^\vee)^2$$

$\phi \simeq \phi + 2\pi$.

$\tilde\phi = \frac{\phi}{g}$. has radius $1/g$ $\longleftrightarrow$ has radius g.

$\tilde\phi \simeq \tilde\phi + \frac{2\pi}{g}$.

• for $p=1$ can make Non-Abelian

$F \in$ Lie alg of G $\quad F_2 = dA_1 - iA_1 \wedge A_1 \equiv dA_1 - A_1^2$

$$\delta A_1 = d\lambda_0 - iA_1\lambda_0 - i\lambda_0 A_1$$

$p \neq 1$?

$$\left(p+p \overset{!}{=} 1+p . \right)$$

• $S_{CS}[A] = \# \int A \wedge F \wedge F \;\cdots$

— ind. of metric

— if $\sum \text{ranks} = D$ makes sense.

— is gauge inv't up to a bdy term.

— for $D = 2+1$

$\int A \wedge F$ — is gaussian

— is more relevant than $F \wedge * F$.

$*$.

Hodge duality on forms

ind. of metric on M.

<u>claim 1:</u> $H^p(M) \cong \{\text{harmonic forms on } M\}$

<u>claim 2:</u> $H^p(M) \cong H^{n-p}(M)$

if M is compact.

- If $\partial M = \phi$, any harmonic form γ is closed & co-closed

$d\gamma = 0 \qquad d^\dagger\gamma = 0.$

$$(\omega, \underbrace{\Delta\omega}_{=0}) \equiv \int \omega \wedge *(\Delta\omega) = \underbrace{\|d\omega\|^2}_{\geq 0} + \underbrace{\|d^\dagger\omega\|^2}_{\geq 0}$$

$\Delta \equiv dd^\dagger + d^\dagger d$

$= 0 \iff d\omega = 0$ AND $d^\dagger\omega = 0.$

<u>claim w/o proof</u>: Any

$$\Omega^p \ni \omega = d\alpha + d^\dagger\beta + \underset{\text{harmonic}}{\gamma}$$

$$\Omega^p \equiv \operatorname{im} d \oplus \operatorname{im} d^\dagger \oplus \{\text{harmonic}\}.$$

This decomp. is orthogonal:

eg $(d\alpha, d^\dagger\beta) = (d^2\alpha, \beta) = 0$.

$(d\alpha, \gamma) = 0$ ....

If ω is closed then

$$\|d^\dagger\beta\|^2 = (d^\dagger\beta, d^\dagger\beta) = (\omega, d^\dagger\beta)$$

$$= (\underbrace{d\omega}_{=0}, \beta) = 0.$$

if ω is closed.

$\Rightarrow \omega = d\alpha + \gamma$.

$\Rightarrow [\omega] = [\gamma]$

ie $[\omega]$ has a harmonic representative.

It minimizes $\|\omega\|^2$ within $[\omega]$.

$$= \int \omega \wedge * \omega \geq 0$$

$\Delta * = * \Delta$. $\Delta = \{d, s*d*\}$

$\Rightarrow$ $*$: harmonic forms $\longrightarrow$ harmonic forms

$\Rightarrow$ isomorphism $*: H^p(M) \to H^{n-p}(M)$.

Susy gs of NLSM are the harmonic reps of cohomology

since $Q|\psi\rangle = Q^\dagger|\psi\rangle = 0$

If M is non-compact

$$H^p(M) \otimes H_c^{n-p}(M) \to \mathbb{R}$$

$*$ n-p-forms w/ compact support

$$S = \int dA \wedge * dA + \oint_c A$$

$$0 = \frac{\delta S}{\delta A} = d * F + j$$

$d * F = 0$ on $\mathcal{M} \equiv$ space $\setminus$ charges.

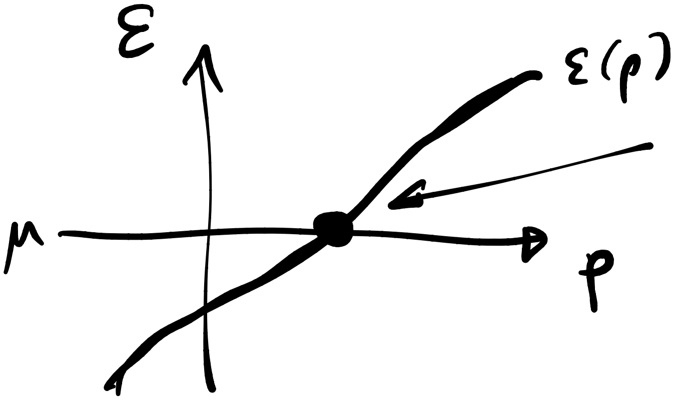
$$[*F] \in H^2(\mathcal{M})$$

$$F \overset{!}{=} \pm * F.$$

eg: $D=2$, $p=0$. $\quad \partial_\mu \phi \overset{!}{=} \pm \epsilon_{\mu\nu} \partial^\nu \phi$

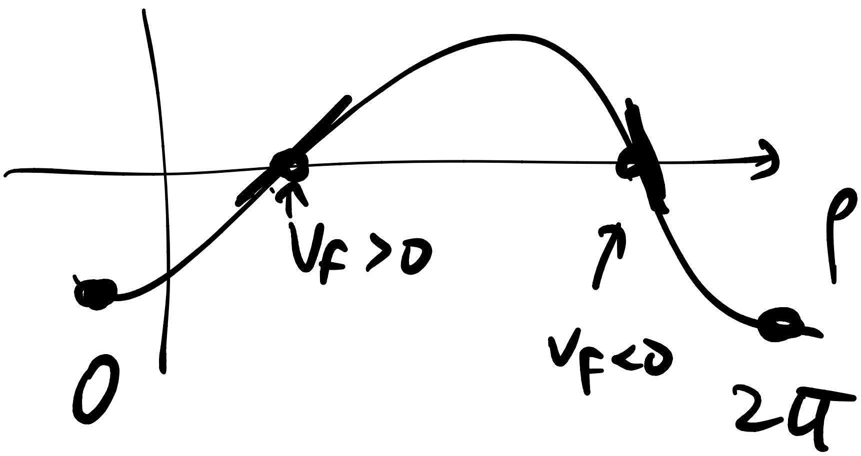
ie: $\partial_\pm \phi = 0.$

$$\Rightarrow \phi = \phi(t \mp x)$$



$$v_F = \frac{\partial \varepsilon}{\partial p} > 0.$$

vs in a lattice model:



[Nielsen-Ninomiya fermion doubling thm.]

A 2d CFT is labelled by (c_+, c_-).

$$\langle T_{\pm\pm}(x) T_{\pm\pm}(0) \rangle \sim \frac{c_\pm}{x^4}$$

<u>claim</u>: Nonzero $c_+ - c_-$ implies a <u>gravitational anomaly</u>.

(can't happen in a local lattice model.)