Last time: Poincaré duality.

Relabel: $\Delta_p \equiv \Delta^\vee_{d-p}$

$X \leftrightarrow Z$

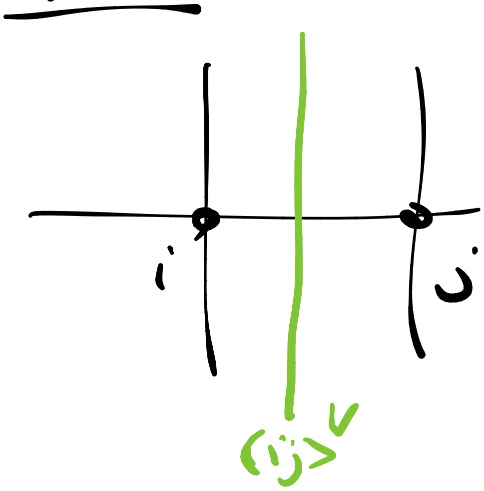
physically trivial. $H_p(X, \mathbb{Z}_2) \cong H_{d-p}(X, \mathbb{Z}_2)$.

vs: Kramers-Wannier-Wegner duality

$p=0$ from T.C. $\quad H_{clock} = -\Gamma \sum_{\langle ij \rangle \in \Delta_1} \underbrace{X_i X_j^\dagger}_{\text{plaquette}} - g \sum_{i \in \Delta_0} Z_i$ + h.c.

Let $\sigma^z_{\ell = \langle ij \rangle} = X_i X_j^\dagger$

$\ell \in \Delta_1 = \Delta^\vee_{d-1}$



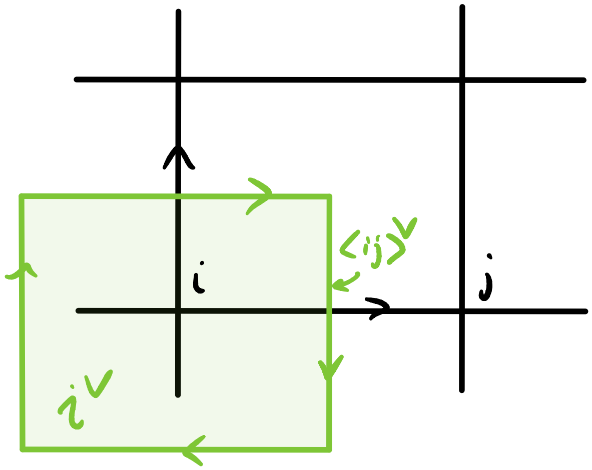
Constraints on σ^z: (1)

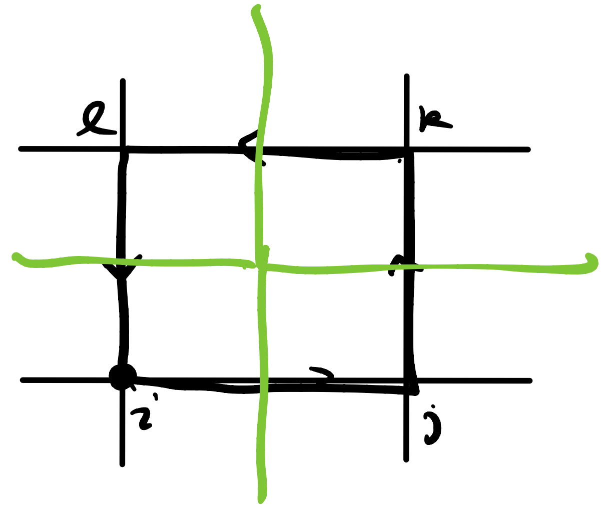
$p=0$ $d=2$: $\prod_{\ell \in \partial p} \sigma^z_\ell = 1 \quad \forall \underline{p} \in \Delta_2 = \underbrace{\Delta^\vee_{d-2}}$

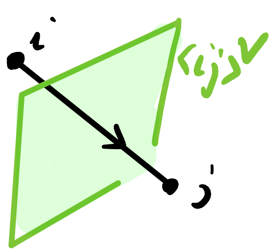
$p=0$ $d=3$: $\prod_{\ell \in \partial p} \sigma^z_\ell = 1$ $\qquad \partial p = v^\vee(p^\vee)$.

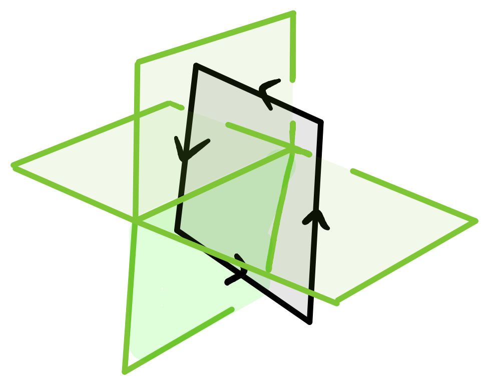
$\boxed{\begin{matrix} p=0 \\ d=2 \end{matrix}}$

$=$





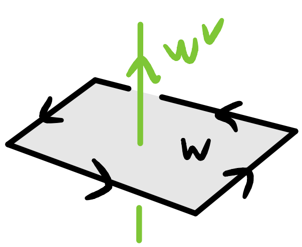


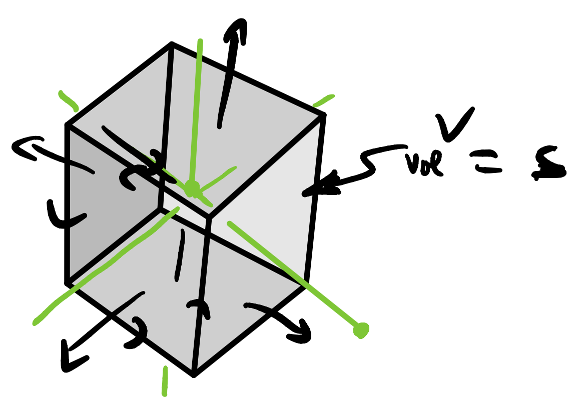


$\boxed{p=0,\ d=3}$

$$\sigma^z_w = \prod_{\ell\in\partial w} X_\ell$$

$\boxed{p=1\ \ d=3}$

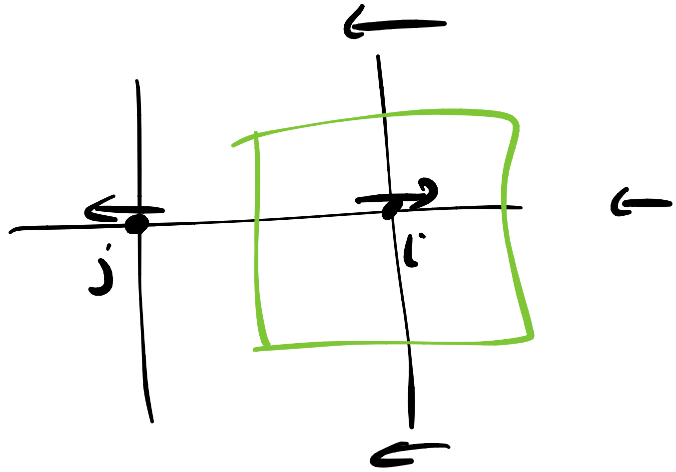




$$\prod_{\ell\in v(s=vol^\vee)} \sigma^z_\ell = 1.$$

②

A)



$$\sigma^z_{ij} = X_i^\dagger X_j$$

$$X_i = e^{2\pi i k_i / N}$$

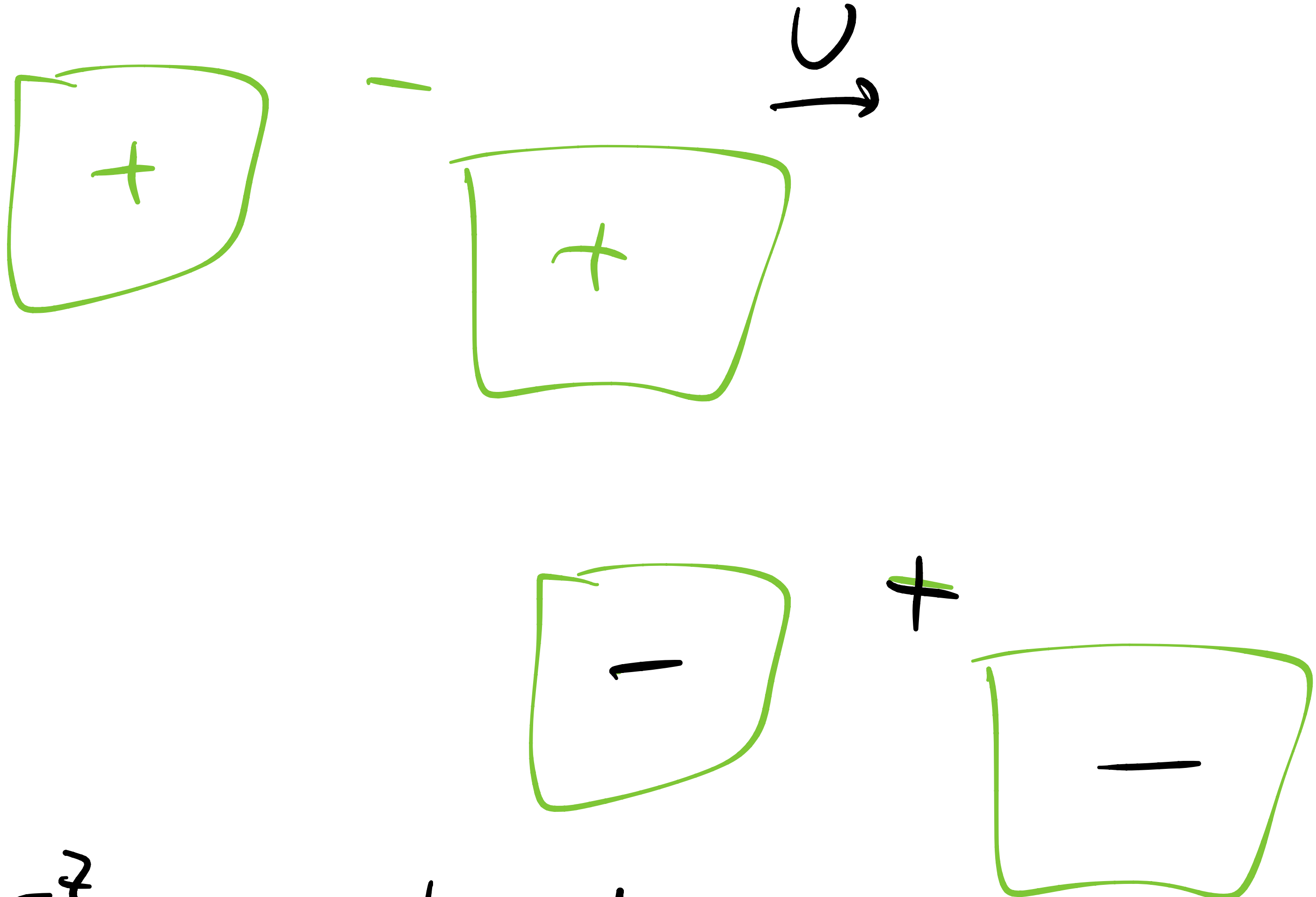
$$= e^{\frac{2\pi i}{N}(k_i - k_j)}$$

$\Longrightarrow$ contractible

$\rightarrow$ only get $|gs\rangle = \sum_{\substack{\text{CONTRACTIBLE} \\ \text{CURVES} \\ C_0}} |C_0\rangle$

B) take a config of $|x_i\rangle$ on Δ.

$[H, U] = 0 \qquad U = \prod_i Z_i . \qquad (X_i \to -X_i)$



σ^z_{ij} are <u>unchanged.</u>

$\Rightarrow$ N degenerate g.s. of ferromagnet are invisible to the dual vars.

<u>KWW duality erases the topological data.</u>

$$H = -\Gamma \sum_{\sigma \in \Delta_{p+1}} \underbrace{\prod_{\ell \in \partial\sigma} X_\ell} - g \sum_{\ell \in \Delta_p} Z_\ell$$

$$\rightsquigarrow \underbrace{\prod_{\ell \in V(w)} Z_\ell = 1} \quad \forall\, w \in \Delta_{p-1} .$$

$$\sigma^z_\sigma = \prod_{\ell \in \partial\sigma} X_\ell . \qquad \text{satisfy} \quad \left[\begin{array}{l} 1 = \prod_{\sigma \in V(w)} \sigma^z_\sigma \\ \forall\, w \in \Delta_{p+2} = \Delta^\vee_{d-p-2} . \end{array} \right.$$

$$\sigma \in \Delta_{p+1} = \Delta^\vee_{d-p-1}$$

p cells $\longleftrightarrow$ $(d-p-1)$ cells

---

p-form gauge theory in continuum :

A_p a p-form.

d = exterior derivative

$\underbrace{dA}$

d : p-forms $\longrightarrow$ $(p+1)$ forms

$$\underbrace{(dA)_{i_1 \cdots i_{p+1}} = (p+1)!\, \partial_{[i_1} A_{i_2 \cdots i_{p+1}]} .}$$

$$(*\omega_q)_{i_{q+1}\cdots i_d} \equiv \frac{\epsilon_{i_1\cdots i_d}\,\omega^{i_1\cdots i_q}}{q!}\sqrt{\gamma}$$

$* : q\text{-form} \to \underline{D-q}$ form

$(\underline{D=d+1})$

---

why care about dA_p? field strength

eg $\underline{E\&M}$: A_1. $dA = F$

$A \to A + d\lambda$. $F \to F + d^2\lambda = F$.

$\boxed{d^2 = 0}$.

F is gauge invariant.

$$\underbrace{dA_p}_{p+1} = * \underbrace{d\tilde{A}_{d-p-1}}$$

$$D-(p+1) = d+1-(p+1)$$
$$= d-p.$$

$\Rightarrow \tilde{A}$ is a $\underline{d-p-1}$ form.

1+1d: 0-form g.t. $\longleftrightarrow$ 0-form g.t. (KW)

2+1d: 0-form g.t. $\longleftrightarrow$ 1-form g.t.

($G = U(1)$ XY model $\longleftrightarrow$ abelian higgs)

3+1 d: 1-form $\longleftrightarrow$ 1 form

E $\longleftrightarrow$ B

$G = \mathbb{Z}_2$: deconfined phase | confined phase on Δ

confined phase on $\Delta^\vee$ $\longrightarrow$ g/Γ deconfined phase on $\Delta^\vee$.

$$H = -\Gamma \sum_{\ell \in \Delta_{p+1}} \sigma^z_\ell - g \sum \prod_{w \in \partial \sigma^\vee} \sigma_w^x \qquad \sigma^\vee \in \Delta_{d-p}$$

topologically boring, physically nontrivial.

$$\boxed{\eta \wedge * \omega \equiv (\eta, \omega)\, \underline{\underline{\text{vol}}}} \quad \forall \eta.$$

def of $*\omega$

$A_p \wedge B_q$ is a $p+q$ ~~form~~.

$$(A_p \wedge B_q)_{i_1 \ldots i_{p+q}} = A_{i_1 \ldots i_q} B_{i_{q+1} \ldots i_{p+q}} \pm \ldots$$

$\neq$

(requires an orientation.)

$\wedge,\ d,\ *,$

2. Supersymmetry $\to$ cohomology, Morse theory

2.1 supersymmetric QM.

supersymmetry: A supercharge Q is an anticommuting symmetry generator. $[H, Q] = 0$

with susy alg: $[Q, H) = 0$. $\{Q, Q\} = 0$, $\{Q^\dagger, Q^\dagger\} = 0$.

$$\underbrace{\{Q^\dagger, Q\} = 2H.}$$

$$\Longrightarrow \langle\psi| \{Q^\dagger, Q\} |\psi\rangle = 2\langle\psi|H|\psi\rangle$$

$$= \| Q|\psi\rangle \|^2 + \| Q^\dagger |\psi\rangle \|^2 \geq 0$$

$$= 0 \Longleftrightarrow Q|\psi\rangle = 0 = Q^\dagger|\psi\rangle$$

$\forall |\psi\rangle$.

$H \geq 0$.

In D dims, w/ Poincaré symmetry

$$H = P_0$$

$\Longrightarrow$ supercharges are spinors.

central charges

$$\{Q^\dagger_\alpha, Q_\beta\} = 2\gamma^\mu_{\alpha\beta} P_\mu + \underbrace{C^I_{\alpha\beta}}_{\text{invariant tensors}} Z_I$$

(ignore for now)

$\sqrt{\text{transl}} \cdot \sqrt{\text{transl}} = \text{translation}$

$\underline{D = 0+1}$. $\quad H|n\rangle = E_n|n\rangle$. $\quad E \geq 0$.

$$\{Q^\dagger, Q\}|n\rangle = 2E_n|n\rangle.$$

$\boxed{\text{If } E_n > 0}$ let $a^\dagger = \dfrac{Q^\dagger}{\sqrt{2E_n}}$ $\quad a = \dfrac{Q}{\sqrt{2E_n}}$.

$$\{a, a^\dagger\} = 1, \quad a^2 = 0$$

$$\begin{cases} a|-\rangle = 0 & a^\dagger|-\rangle = |+\rangle \\ \underline{a|+\rangle = |-\rangle} & a^\dagger|+\rangle = 0. \end{cases}$$

$$a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad a^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

$$(-1)^F = 2a^\dagger a - 1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

fermion parity

Def : $\{(-1)^F, a\} = 0 = \{(-1)^F, a^\dagger\}$

<u>conclusion</u> <u>abt E>0</u> states: they come in B-F pairs.

If

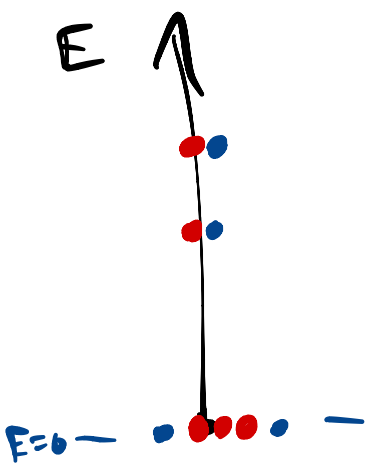
$\boxed{E=0}$ $\quad \{Q^\dagger, Q\}|E=0\rangle = 0$

has only 1d irreps :

$Q|E=0\rangle = 0 = Q^\dagger|E=0\rangle$.

on which $(-1)^F$ can be $+1$ or -1.

can be any # of $E=0$ states.



If there are <u>no</u> $E=0$ states

$\equiv$ <u>susy is spontaneously broken</u>.

$\Rightarrow$ gs has $E>0$

$Q|gs\rangle = |gs'\rangle$

<u>weird</u>: • can happen in finite volume.

• " " that as $L\to\infty$, susy is restored!

Maybe SUSY is a sym. of Nature.

If $\underbrace{Q|\text{vacuum}\rangle = 0}$ $\Rightarrow$ 1-particle states would be B-F pairs.

must be SSSB.

$\underline{\text{eg}}$: $\mathcal{H} = \underline{\mathcal{H}_{\substack{\text{particle}\\ \text{on } \mathbb{R}}} \otimes \mathcal{H}_\psi} = \mathcal{H}_{\text{particle in 1d w/ spin 1/2}}$.

$$= \text{span}\{|x\rangle\} \otimes \text{span}\{|-\rangle, \underbrace{\psi^+|-\rangle}_{=|+\rangle}\}$$

$[x,p] = i$

$\{\psi, \psi^+\} = 1$, $\psi^2 = 0$.

$\underline{\text{Let}}$: $\begin{cases} Q = \psi(p - iW'(x)) \\ Q^+ = \psi^+(p + iW'(x)) \end{cases}$

$$\begin{pmatrix} [\psi, p] = 0 \\ [\psi, x] = 0 \\ \vdots \end{pmatrix}$$

$\psi^2 = 0 \to \{Q, Q\} = 0 = \{Q^+, Q^+\}$

$[H, Q] = 0$

$[H, Q^+] = 0$.

$$\frac{1}{2}\{Q^+, Q\} \equiv H = \frac{p^2}{2} + \frac{(W'(x))^2}{2} - \underbrace{\frac{[\psi^+,\psi]}{2}W''(x)}.$$

K.E. potential $V = (W')^2$ | Zeeman term

$\psi = \sigma^-$, $\psi^\dagger = \sigma^+$

$\Rightarrow [\psi, \psi^\dagger] = -\sigma^3$

$\downarrow$

$+\sigma^3 \underline{\frac{W''(x)}{2}}$

$h_z = \frac{W''(x)}{2}$.

<u>easy to find susy g.s</u>:

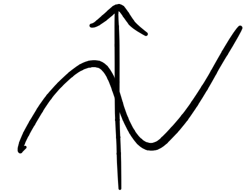
instead of $H|\Psi\rangle = E|\Psi\rangle$ solve $\underline{\underline{Q|\Psi\rangle}} = 0 = Q^\dagger|\Psi\rangle$

$$|\Psi\rangle = \int dx\, |x\rangle \otimes \left[\Psi_+(x)|+\rangle + \underline{\underline{\Psi_-(x)|-\rangle}}\right]$$

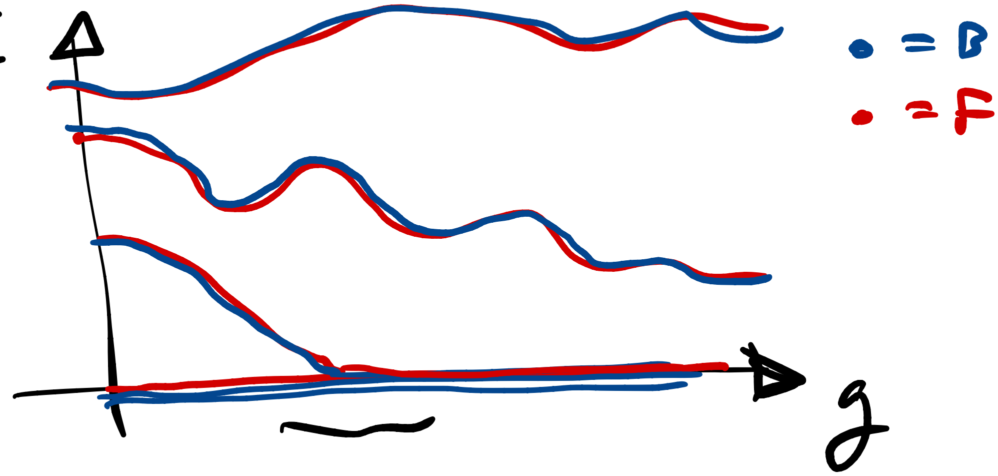
$Q|\psi\rangle = 0 \Rightarrow (p + iW'(x))\Psi_+(x) = 0$.

$(\psi^\dagger|+\rangle = |-\rangle.)$ $\qquad \partial_x \Psi_+ = W'(x)\Psi_+$

$p = -i\partial_x$ $\qquad \Rightarrow \Psi_{\pm}(x) = c_{\pm} e^{\pm W(x)}$

(SSSB happens if neither $\overline{e^W \text{ nor } e^{-W}}$ is normalizable eg )

Witten Index:



$$n_B^{E=0} - n_F^{E=0} = \underline{\mathrm{tr}(-1)^F} \quad \text{is an invariant!}$$

$$\mathrm{tr}_{\mathcal{H}}(-1)^F = \mathrm{tr}_{\mathcal{H}}(-1)^F e^{-\beta H}$$

states w/ $E \neq 0$ don't contribute

ind. of β.

$$\mathrm{tr}\, e^{-\beta H} = \int_{\substack{\phi(\tau)=\phi(\tau+\beta) \\ \psi(\tau)=-\psi(\tau+\beta)}} D\phi\, D\psi\, e^{-S(\phi,\psi)}$$

$$\mathrm{tr}\,(-1)^F e^{-\beta H} = \int_{\substack{\phi(\tau)=\phi(\tau+\beta) \\ \psi(\tau)=+\psi(\tau+\beta)}} D\phi\, D\psi\, e^{-S(\phi,\psi)}$$

P.B.C. for fermions.

"Index"

$$\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F = \mathcal{H}_{even} \oplus \mathcal{H}_{odd}$$

$(-1)^F = 1$ $\quad$ $(-1)^F = -1$

$$Q + Q^\dagger = \begin{pmatrix} 0 & M^\dagger \\ M & 0 \end{pmatrix}$$

odd $E=0$ states satisfy $M^\dagger \psi_F = 0$.

even $E=0$ " " $M \psi_B = 0$.

def:

$$\mathrm{ind}(M) \equiv \dim\ker(M) - \dim\mathrm{coker}(M)$$

$$\equiv \#\left\{\begin{matrix}\text{solns of} \\ M\psi = 0\end{matrix}\right\} - \#\left\{\begin{matrix}\text{solns of} \\ M^\dagger\psi = 0\end{matrix}\right\}$$

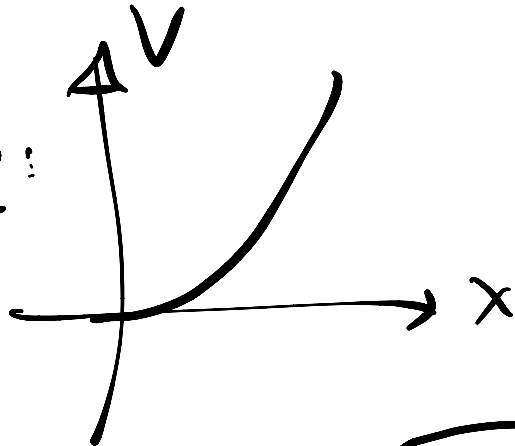
$$= \mathrm{tr}\,(-1)^F.$$

If $\mathrm{tr}(-1)^F \neq 0$ then susy is <u>not</u> broken.

watch out: $W'(x) = mx - gx^2$.

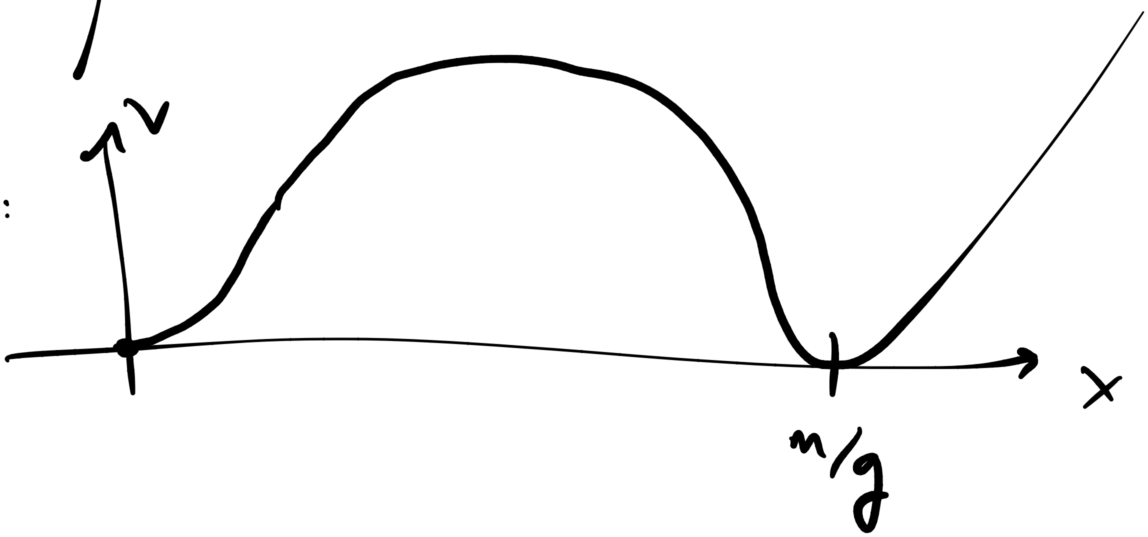
$V = (W')^2$

$\underline{g=0}$:



$V(x) \sim x^2$ as $x \to \infty$

$\underline{g>0}$:



$V(x) \sim g^2 x^4$ as $\cancel{x} \to \infty$

can change

$\mathrm{tr}(-1)^F$.

analog of duality on forms

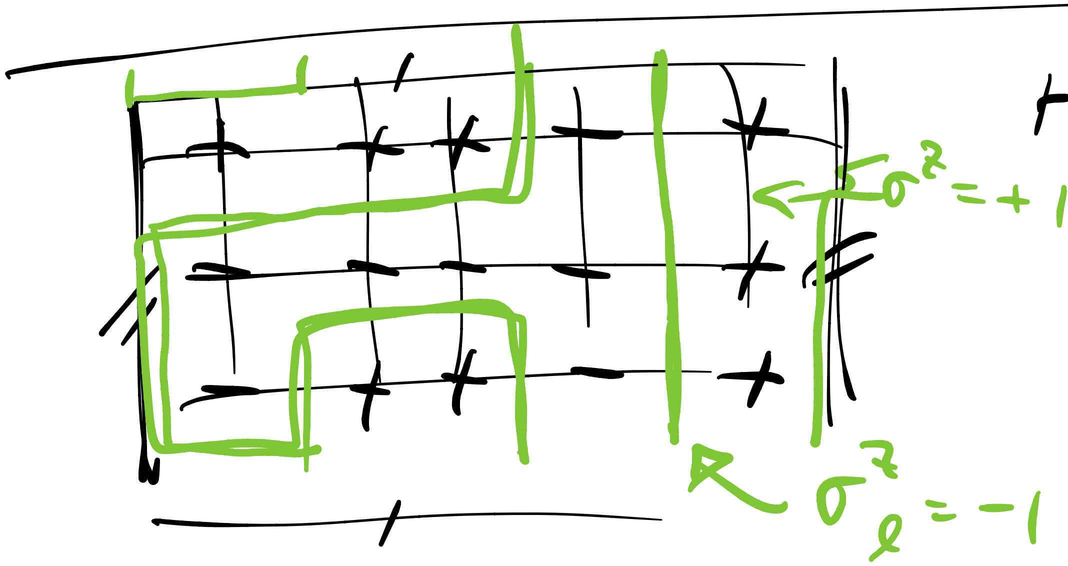
is $*$

$$\begin{cases} d: \Omega^p \to \Omega^{p+1} \\ d^\dagger: \Omega^p \to \Omega^{p-1} \end{cases}$$

$$(\alpha,\beta) = \int \alpha_p \wedge * \beta_p$$

$$(\alpha, d^\dagger \beta) \equiv (d\alpha, \beta)$$

$$\underline{d^\dagger = (-1)^x * d *}.$$



$$H = -\sum_{\langle ij \rangle} X_i X_j$$

winding # around y $= 2 = 0 \bmod 2$

around x $= 0$.

$$\Delta_p \leftrightarrow \check{\Delta}_{d-p-1}$$

$p=1, d=3$: $\underline{E \leftrightarrow B}$.

$$M = U d V$$

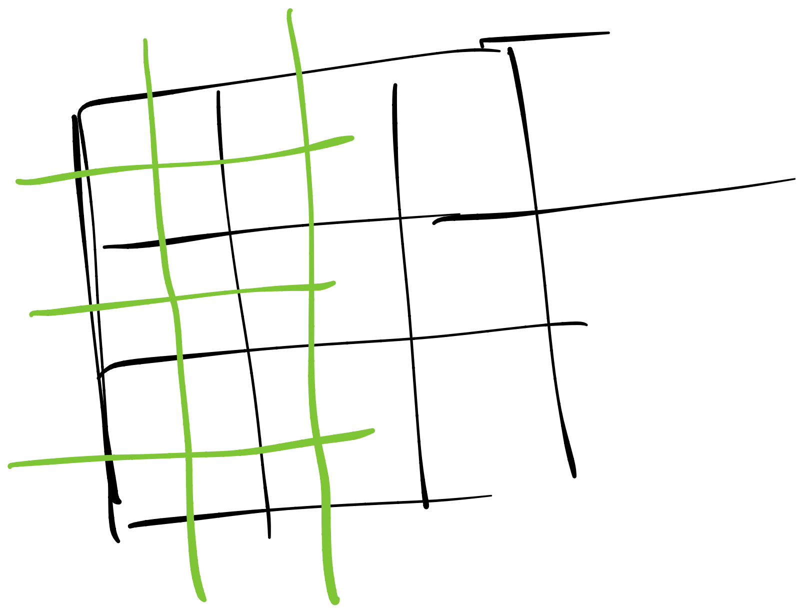
$$M^\dagger = V^\dagger d U$$

$$d = \begin{pmatrix} m_1 & & & & & & \\ & m_2 & & & & & \\ & & \ddots & & & & \\ & & & m_r & & & \\ & & & & 0 & & \\ & & & & & 0 & \\ & & & & & & 0 \end{pmatrix}$$

$$m_i > 0$$

$$W(x) \to W(x) + \mu x \qquad (W')^2$$

$$H \to H + \mu^2 + 2\mu W'(x)$$



rough $\leftrightarrow$ smooth

$e \leftrightarrow m$

① $\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F = \mathcal{H} \otimes \mathrm{Rep}\begin{pmatrix}\{\psi, \psi^\dagger\} = 1 \\ \psi^2 = 0\end{pmatrix}$

for any $|\Psi\rangle_B$ w $\psi|\Psi\rangle = 0$.

$\exists$ $\underbrace{\psi^\dagger |\Psi\rangle}_{F}$ $\Rightarrow \dim \mathcal{H}_B = \dim \mathcal{H}_F$.

$$Q + Q^\dagger = \begin{pmatrix} 0 & M \\ M^\dagger & 0 \end{pmatrix}$$

$\Rightarrow M$ is square $\Rightarrow$ $\mathrm{tr}(-1)^F = 0$.

② compute $\mathrm{tr}(-1)^F$ from ρ_{ball} ? ← ?