Last time: • For some Δ, A matters in

$$H_p(\Delta, A)$$

• $H_p(\Delta, \mathbb{Z})$ determines $H_p(\Delta, A)$ $\forall A$ (abelian).

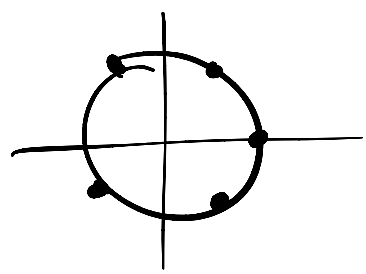
(Here Δ is a cell complex which is a decomposition of some manifold X.)

Toric Code w/ $A = \mathbb{Z}_N$ as $N \to \infty$?

$$\hat{Z} = \sum_{n=1}^{N} e^{\frac{2\pi i n}{N}} |n\rangle\langle n|$$

Think of the phase of Z as position.

$$\hat{X} = \text{translation op} = e^{i\hat{p}}$$

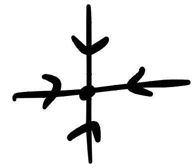


$\log Z$ discrete $\Rightarrow$ $p \in [0, 2\pi)$

$\log Z \simeq \log Z + 2\pi i$ $\Rightarrow$ p is discrete.

$\mathcal{H}_{N\to\infty} = \mathcal{H}_{\text{rotor}}$. $\quad |\theta\rangle \equiv \sum_n e^{in\theta} |n\rangle$

Think θ as the dir. of rotor.

star condition: $A_s|\rangle = |\rangle$ 

$$0 = \sum_{l\in v(s)} n_l \quad \cancel{\text{mod } N}$$

$$= \vec{\nabla}\cdot\vec{\tilde{E}}$$

$$\tilde{H} = +J\sum_{s\in\Delta_{p-1}} \Big(\sum_{\sigma\in v(s)} n_\sigma\Big)^2$$

$$- \sum_{\mu\in\Delta_{p+1}} \underbrace{\prod_{\sigma\in\partial\mu} e^{i\theta_\sigma}} + h.c.$$

$$[n_\sigma, e^{\pm i\theta_{\sigma'}}] = \pm e^{\pm i\theta_\sigma}\,\delta_{\sigma\sigma'}$$

$$= +J\sum_{s\in\Delta_{p-1}} (\nabla\cdot \tilde{E})^2 - \sum_{\mu\in\Delta_{p+1}} \underbrace{\cos(\nabla\times a)}_{\simeq\, 1 - b^2 + \dots}$$

$$\Delta H = -g\sum_{l\in\Delta_p} Z_l + h.c.$$

$$= -g\sum e^{in_l} + h.c. = -g\sum_l \cos n_l \simeq \dots + \frac{g}{2}\sum n_l^2 + \dots$$

$J \to \infty$ (gauss law exact)

$$\to H = \sum (gE^2 + B^2) = H_{\text{Maxwell}}$$

In 3+1 dims this a gapless topological phase.

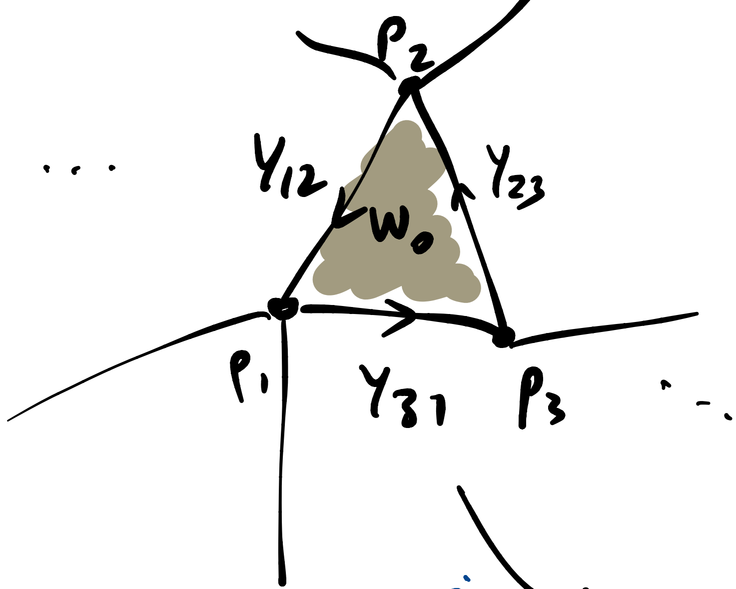
In 2+1 dims: If we can expand the cosines $\to$ gapless

If NOT: confines (gapped & non-topological)

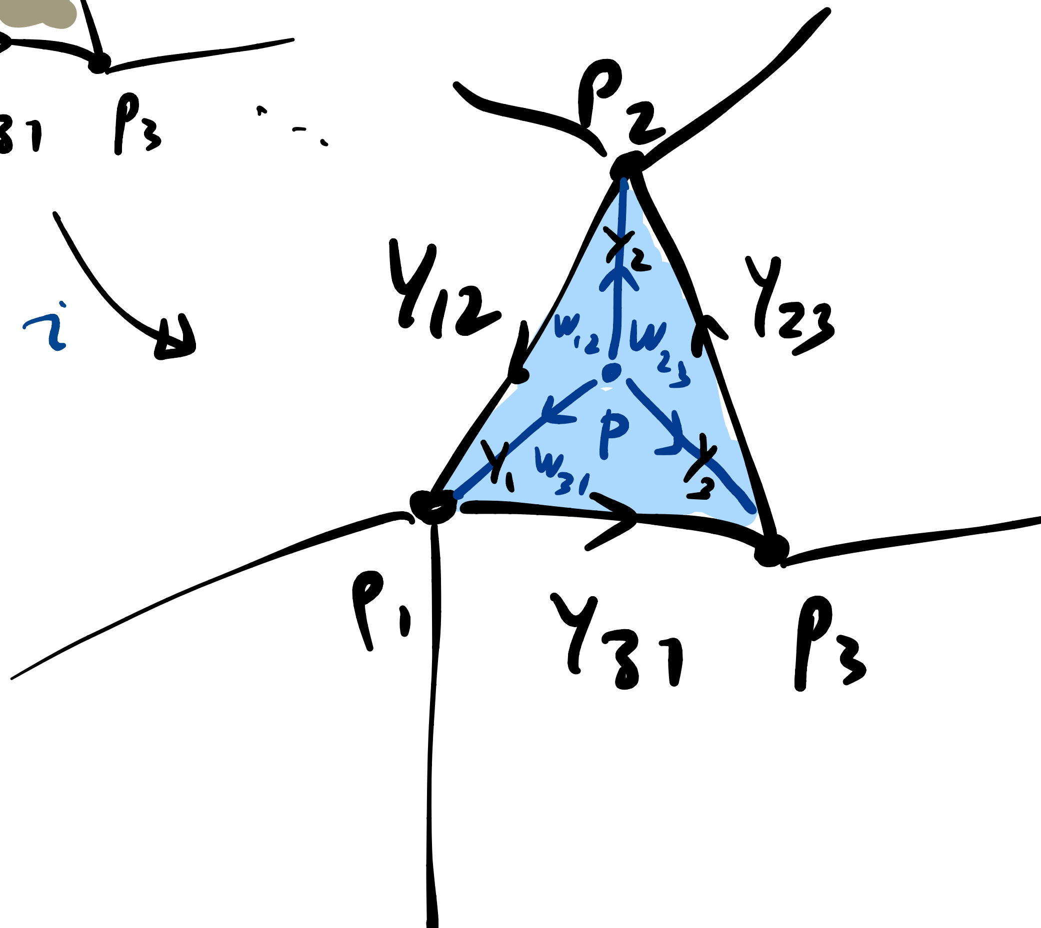
1.5 Independence of Cellulation

Given Ω_0 a cellulation of X

$\underline{\Omega}$: . . .



$\underline{\hat{\Omega}}$:



$i: \quad w_0 \longmapsto i(w_0) = w_{12} + w_{23} + w_{31}$

any other $\longmapsto$ itself

nobody $\longmapsto$ P or $Y_{1,2,3}$.

$$\Longrightarrow \quad 0 \longrightarrow \Omega_\bullet \xrightarrow{\ i\ } \hat{\Omega}_\bullet \xrightarrow{\ \pi\ } \Omega'_\bullet \longrightarrow 0$$

is a short exact seq.

$$\Omega'_q \equiv \hat{\Omega}_q / \Omega_q$$

claim: i, π are chain maps $\begin{cases} [i,\partial]=0 \\ [i,\partial]=0. \end{cases}$

$\Longrightarrow$ long exact seq. on $H_\bullet$:

$$\xrightarrow{\partial_*} H_p(\Omega) \xrightarrow{i_*} H_p(\hat{\Omega}) \xrightarrow{\pi_*} H_p(\Omega') \xrightarrow{\partial_*} H_{p-1}(\Omega) \xrightarrow{i_*} H_{p-1}(\hat{\Omega}) \xrightarrow{\pi_*} H_{p-1}(\hat{\Omega}) \longrightarrow \cdots$$

claim: $H_p(\Omega') = 0$.

$$\xrightarrow{\partial_* = 0} H_p(\Omega) \xrightarrow{i_*} H_p(\hat{\Omega}) \xrightarrow{\pi_*} 0 \xrightarrow{\partial_*} H_{p-1}(\Omega) \xrightarrow{i_*} H_{p-1}(\hat{\Omega}) \xrightarrow{\pi_*} 0 \xrightarrow{\partial_* = 0} \cdots \quad 0$$

$$\Longrightarrow H_p(\Omega) \overset{i_*}{\cong} H_p(\hat{\Omega}).$$

$\operatorname{Im}\phi = \ker(0) = B$.

If $0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{0} 0$ is exact $\Rightarrow A \overset{\phi}{\cong} B$.

$\ker\phi = 0$

<u>Pf of claim</u> : $\Omega' = \hat{\Omega}/\Omega$ contains only the added cells.

$\Omega_0' = \langle p \rangle$, $\Omega_1' = \langle y_i \rangle$

$$\Omega_2' = \langle w_{ij} \mid \underline{\underline{\sum_{\langle ij \rangle} w_{ij} = \underline{w_0 = 0}}} \rangle \ \text{mod } \Omega.$$

$\partial w_{ij} = -y_i + y_j + y_{ij} = -y_i + y_j$ mod Ω

$\partial y_i = p_i - p = p$ mod Ω

$$0 \to \underline{\underline{\mathbb{Z}}} \xrightarrow{(1,1,1)} \underline{\underline{\mathbb{Z}^3}} \xrightarrow{\partial_2} \underline{\underline{\mathbb{Z}^3}} \xrightarrow{\partial_1} \underline{\underline{\mathbb{Z}}} \to 0$$

$$\underbrace{\partial_2 = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}}_{\text{has rank 2}} \qquad \underbrace{\partial_1 = (1,1,1)}_{\text{rank 1}}$$

$\partial_1 \circ \partial_2 = 0.$

$p \in \operatorname{Im} \partial_1 \Rightarrow H_0(\Omega') = 0.$

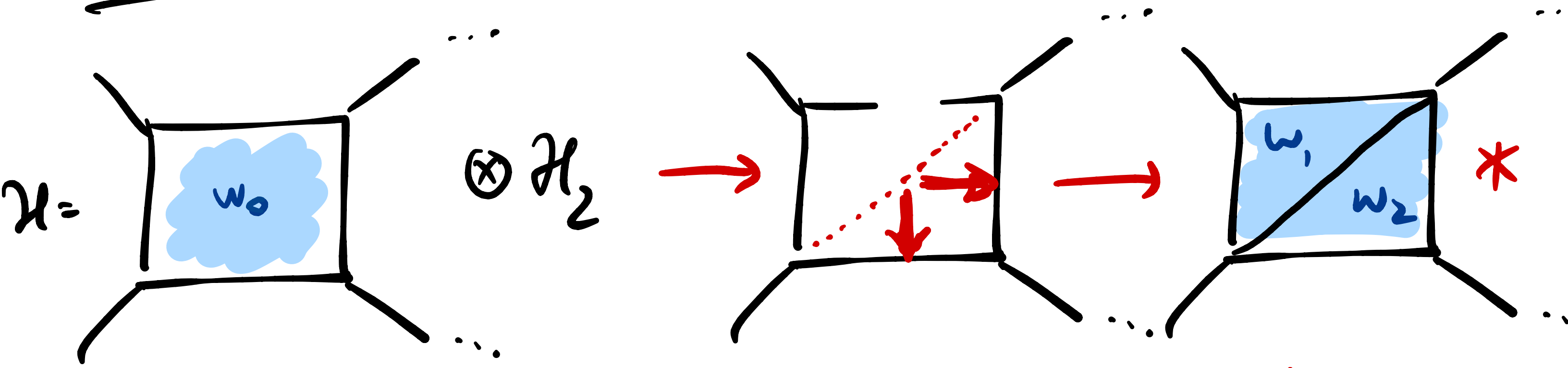
$\ker \partial_1 = \operatorname{Im} \partial_2 \to H_1(\Omega') = 0.$

$\ker \partial_2 = \langle w_{12} + w_{23} + w_{31} \rangle = \langle w_0 \rangle = 0$ mod Ω

$\Rightarrow H_\bullet(\Omega') = 0.$

idea: Ω' is a cellulation of a ball w/ the homology (H_0) removed.

A better way from physics:



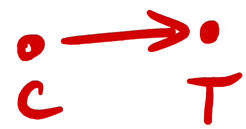
$$H_0 = H_{TC} \otimes \mathbb{1} - c\,\mathbb{1} \otimes X \longrightarrow U H_0 U^\dagger = H_1$$

gs of H_0 = $|gs_{\text{of } TC}\rangle \otimes |+\rangle$

claim: H_1 has the same gs as H_{TC} on $*$

control-NOT gate:

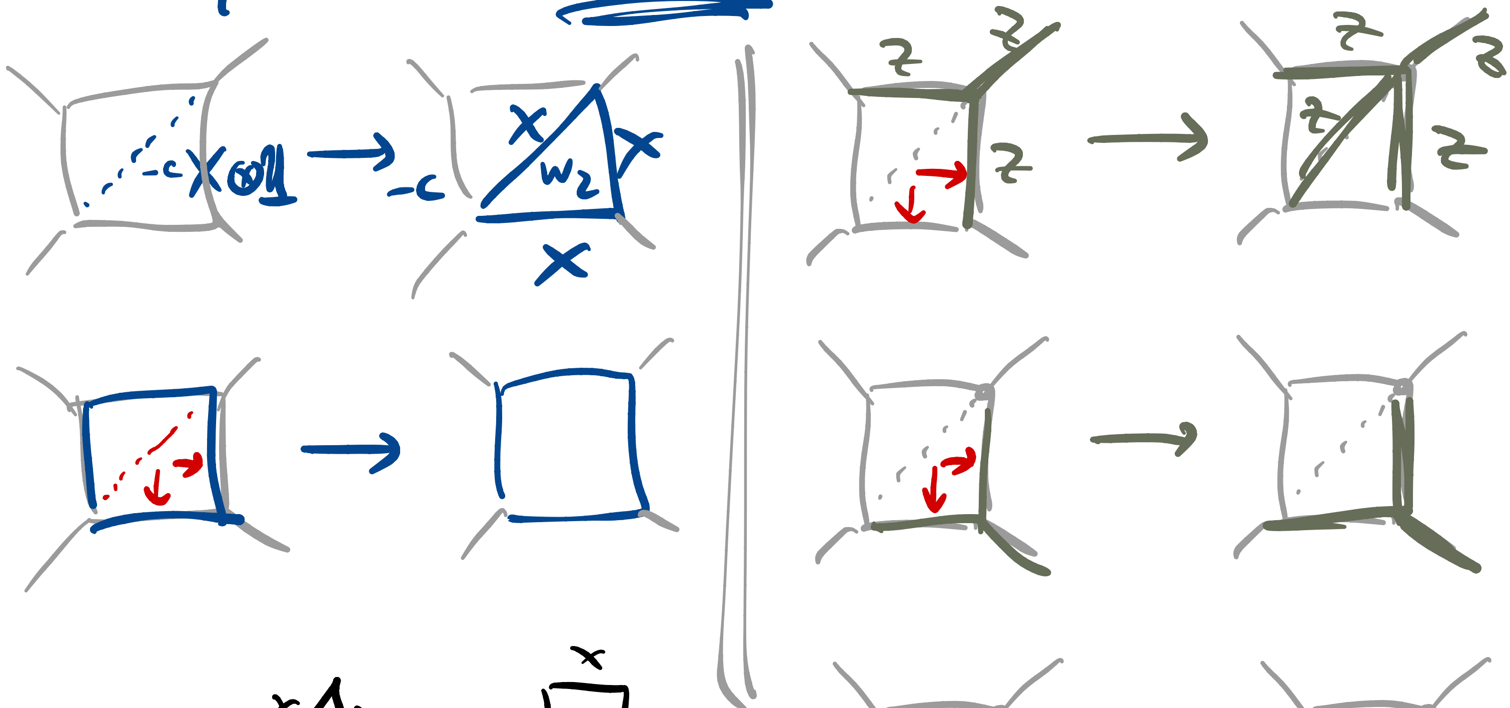
$$CX \equiv P_c(0) \otimes \mathbb{1}_T + P_c(1) \otimes X_T$$

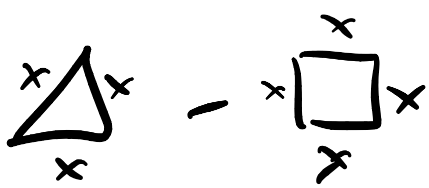


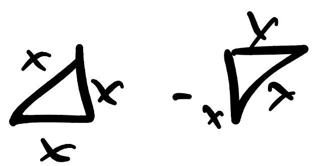
$$\begin{cases} P_c(0) = |0\rangle\langle 0|_c = \dfrac{1+Z_c}{2} \\ P_c(1) = |1\rangle\langle 1|_c = \dfrac{1-Z_c}{2} \end{cases}$$

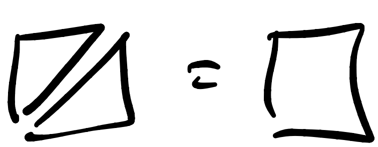
$$\begin{cases} \circlearrowleft \longleftrightarrow CX \circlearrowleft CX \\ \hline 1_c Z_T \longleftrightarrow Z_c Z_T \\ 1\ X \longleftrightarrow 1\ X \\ Z\ 1 \longleftrightarrow Z\ 1 \\ X_c 1_T \longleftrightarrow \underline{X_c X_T} \end{cases}$$

$$\begin{cases} CX^\dagger = CX \\ CX^2 = \mathbb{1} \\ CXCX^\dagger = \mathbb{1}. \end{cases}$$

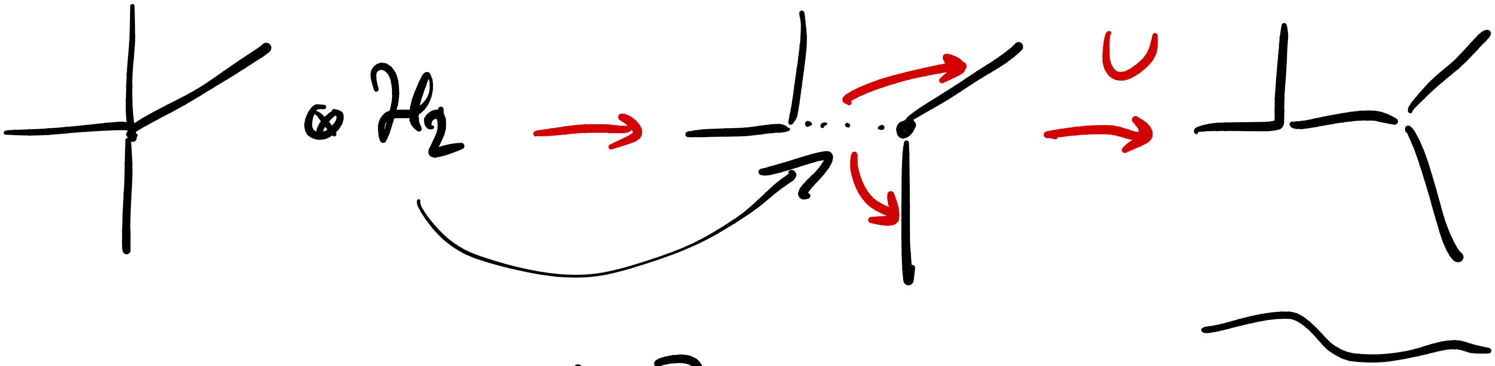


$H_1 = -$ 

vs $H_{TC} = -$ 



same groundstate subspace.



$$H_0 = H_{T_c} \otimes \mathbb{1} - c\, \underline{\underline{\mathbb{1} \otimes Z}}$$

$$U H_0 U^\dagger = H_1 \quad \ldots$$

<u>entanglement renormalization.</u>

<u>ex:</u> <u>do it for $\mathbb{Z}_N$.</u>

<u>claim:</u> any 2 cell decompositions of X are related by a sequence of the 2 moves.

$$H \to H + \sum_i g_i \mathcal{O}_i \qquad g \text{ small enough.}$$

vs: $H \to H + \text{big}\, \mathcal{O}_0$

making a hole.

1.6 Gapped boundaries and Relative Homology

special gapped boundary conditions

smooth b.c.

Rough bdy:

$$B_{123} = X_1 X_2 X_3$$

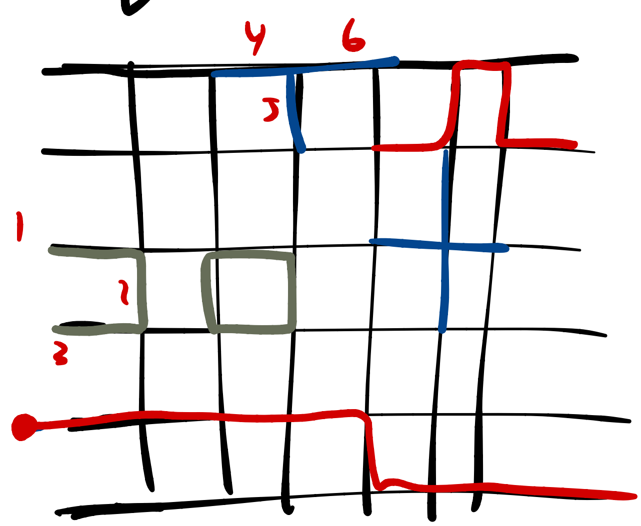
plaquettes are broken

smooth bdy:

$$A_{456} = Z_4 Z_5 Z_6$$

stars are broken

Rough b.c.



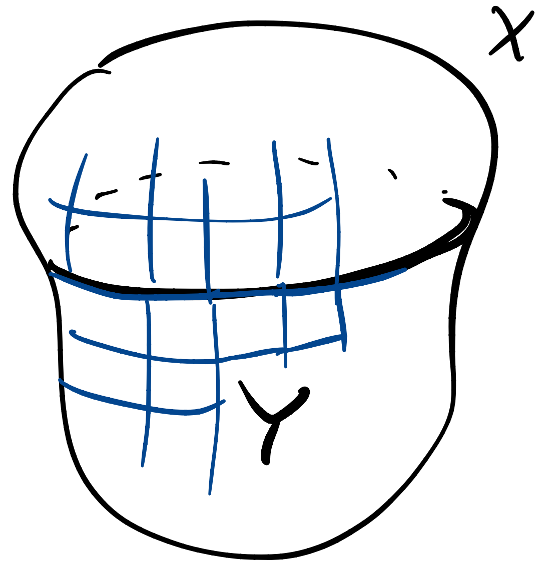
CLAIM: all ops commute

$$|gs, \text{smooth}\rangle = \sum_{\substack{\text{closed} \\ \text{strings } c}} |c\rangle$$

$$|gs, \text{rough}\rangle = \sum \left| \begin{matrix}\text{strings are allowed} \\ \text{to end on the rough bdy}\end{matrix} \right\rangle$$

Relative homology: $X \supset Y \quad \underline{\underline{Y = \overline{Y}}}$

$$0 \to \Omega_\bullet^{Y} \xrightarrow{i} \Omega_\bullet^{X} \xrightarrow{\pi} \Omega_\bullet^{X/Y} \to 0$$



i is inclusion

$$\underline{\underline{\Omega_p^{X/Y} \equiv \Omega_p^{X} / \Omega_p^{Y}.}}$$

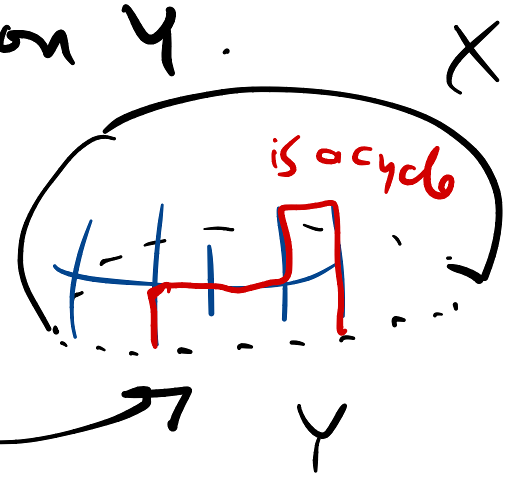
Homology of $\Omega_\bullet^{X/Y}$

$\partial Y \subset Y.$

$$\equiv H_\bullet(X, Y, A)$$

$$\equiv H_\bullet(X/Y, A) \equiv \text{Homology of } X \text{ Relative to } Y.$$

$=$ space of gs of TC w/ Rough boundary conditions on Y.



long exact seq:

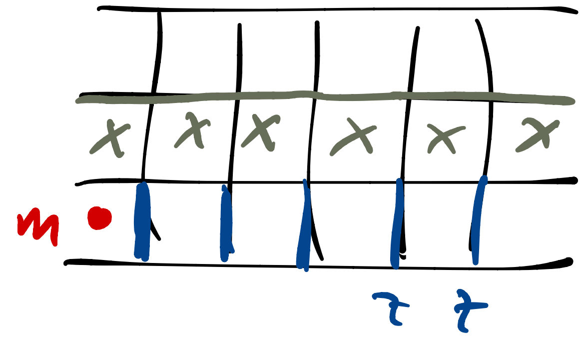
$$\cdots \to H_{p+1}(X/Y) \xrightarrow{\partial_*} H_p(Y) \xrightarrow{i_*} H_p(X) \xrightarrow{\pi_*} H_p(X/Y) \xrightarrow{\partial_*} H_{p-1}(Y) \to \cdots$$

Physical picture of b.c.'s

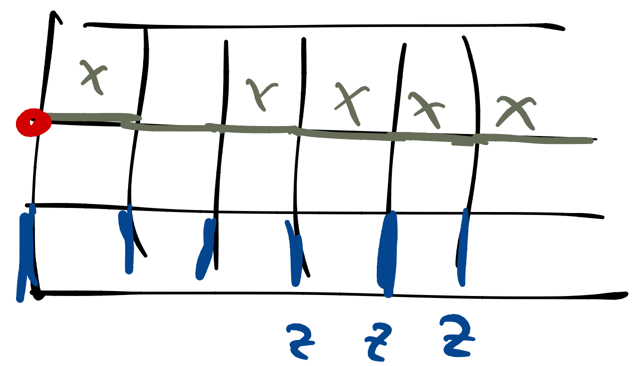
e particle is absorbed by a rough bdy.

m particle gets stuck at rough bdy.

rough



smooth



smooth bdy reverses roles of e & m.

$e \longleftrightarrow m$

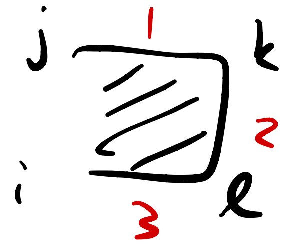
Def: An object o is condensed in state $|\psi\rangle$ if its creation op $\mathcal{O}_o$ has an expect ... $\langle\psi| \mathcal{O}_o |\psi\rangle \neq 0$.

e particle is condensed at a rough bdy.

m " " " " smooth bdy.

creation op for e: X_{ij} $\qquad \Delta H = -\Gamma_{bdy} X_{ij}$

$\xrightarrow{\Gamma_{big} \gg \dots} \left(|+\rangle_{ij} \otimes \dots\right), \quad X_{ij}|+\rangle_{ij} = |+\rangle_{ij}.$



$$B_{ijkl} = X_{ij} X_{jk} X_{kl} X_{li}$$

$$\xrightarrow{X_{ij} > 0} \# \, X_{jk} X_{kl} X_{li}$$

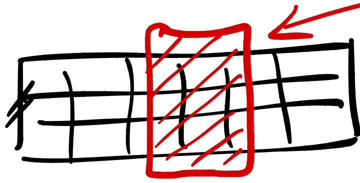
to go from $|\Psi_X\rangle \longrightarrow |\Psi_{X \cup Y \cup w}\rangle$
rough bc.

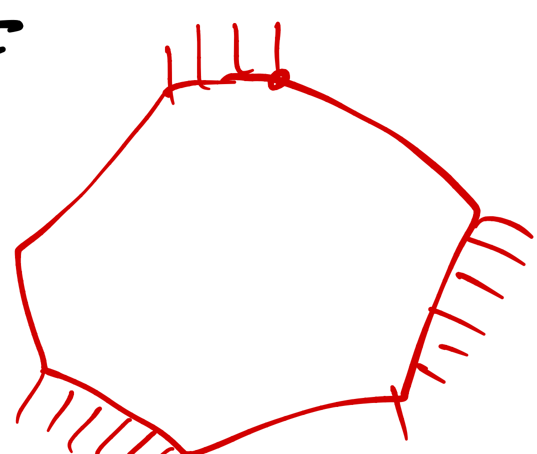
$$\Delta H = -\Gamma_{big} \sum_{g \in Y} X_g \, .$$

$\rightarrow$ Higgs phase in region Y.

$c^+ c^+$

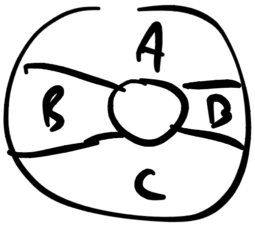
eg: $X =$ annulus $\quad Y =$





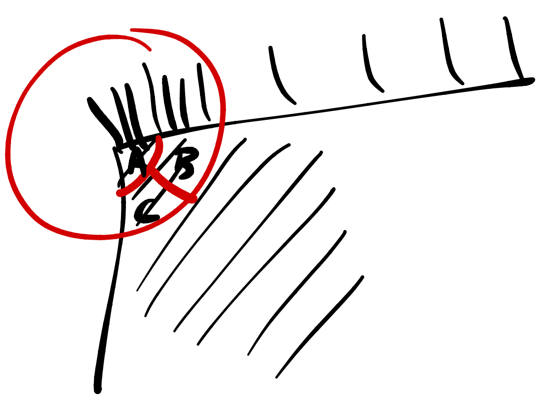
If TEE $\equiv I(A:C|B) > 0$ is an obstruction to making ρ_{ABC} from ρ_A, ρ_B, ρ_C

eg:



Maybe: gs on A, B, C w/ bcs.

$\overset{?}{\Longrightarrow}$ gs on ABC.



$$S_{AC} + S_{AB} - S_B - S_C$$

$$= 2 \log d_{\text{corner op.}}$$

$$C_g = \int D\mu(h)\ \underline{h g h^{-1}}$$

$\in$ group algebra

$$C_g C_k = \int D\mu(h_1) \int D\mu(h_2)\ \underline{h_1 g h_1^{-1} h_2 k h_2^{-1}}$$

$$= \int D\mu(h)\ \underline{N_{gk}^{h}}\ C_h$$

$$R_a \otimes R_b = \bigoplus_c \underline{M_{ab}^{c}} R_c$$

$$\left\{ \begin{array}{l} M_{ab}^{c} = \displaystyle\sum_\alpha \frac{\chi_a^\alpha \chi_b^\alpha \bar{\chi}_c^\alpha}{\chi_1^\alpha} \\[2ex] \underline{N_{\alpha\beta}^{\gamma}} = \displaystyle\sum_a \frac{\chi_a^\alpha \chi_a^\beta \bar{\chi}_a^\gamma}{\chi_1^\gamma} \end{array} \right.$$