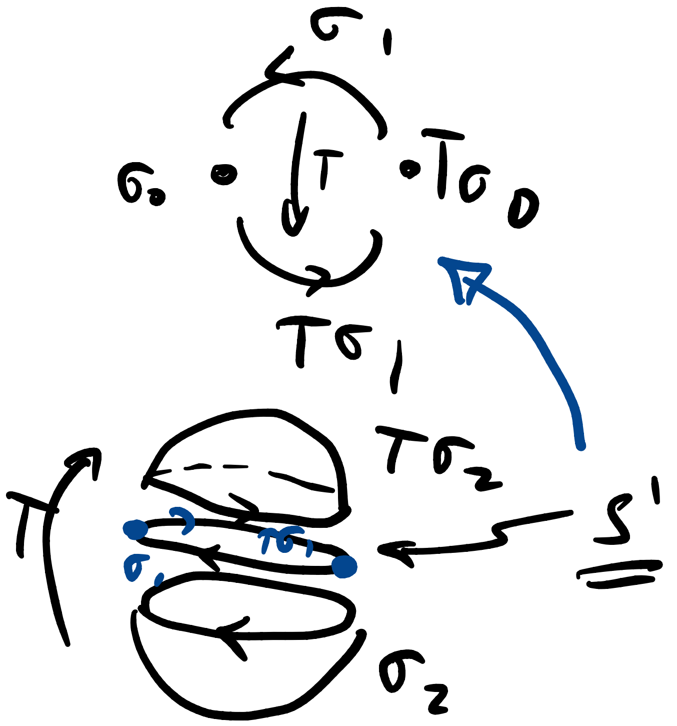
Examples of cell complexes and their homology

(cont'd)

Spheres. $S^0 = \{ \underset{\sigma_0}{\cdot} \quad \underset{T\sigma_0}{\cdot} \} = \{x^2=1\}$

$T \equiv$ antipodal map



$\partial\sigma_1 = \sigma_0 - T\sigma_0 .$

$\underline{\underline{\partial\sigma_2 = \sigma_1 + T\sigma_1}}$

$\partial T\sigma_2 = -\partial\sigma_2$

$\underline{\underline{\partial^2 = 0}}$

each rank 1.

$$0 \to A^2 \xrightarrow{\begin{pmatrix}1&1\\-1&-1\end{pmatrix}} A^2 \xrightarrow{\begin{pmatrix}1&-1\\1&1\end{pmatrix}} \underbrace{A^2 \xrightarrow{\begin{pmatrix}1&1\\-1&-1\end{pmatrix}} A^2}_{\text{repeats}} \xrightarrow{\begin{pmatrix}1&-1\\1&1\end{pmatrix}} \underline{\underline{A^2}} \to 0$$

$\uparrow$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $\uparrow$

$\underline{H_\bullet}$: $H_n(S^n) = A$. $\quad H_i(S^n) = 0 \cdots$ $\quad H_0(S^n) = A$

$i = 1 \ldots n-1$

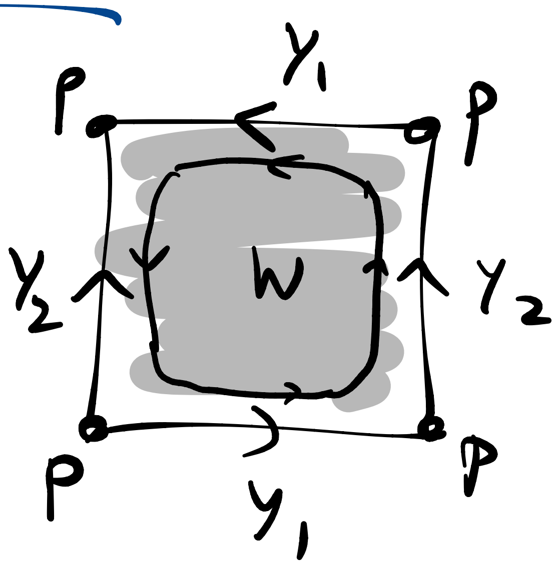
An example w/ torsion homology.

$$0 \to A \xrightarrow{(2,0)} \underline{\underline{A^2}} \xrightarrow{\binom{0}{0}} A \to 0$$

$$\partial_2 w = y_2 + y_1 - y_2 + y_1$$

$$= 2y_1$$

$$\partial_1 y_1 = p - p = 0 = \partial_1 y_2 .$$



Klein bottle.
(non-orientable)

A matters.

- $A = \mathbb{Z}_2$. $\underline{\times 2 = \times 0}$.

$$0 \to \mathbb{Z}_2 \xrightarrow{(0,0)} \mathbb{Z}_2^2 \xrightarrow{\binom{0}{0}} \mathbb{Z}_2 \to 0$$

$$H_0(K, \mathbb{Z}_2) = \mathbb{Z}_2$$
$$H_1(K, \mathbb{Z}_2) = \mathbb{Z}_2^2$$
$$H_2(K, \mathbb{Z}_2) = \mathbb{Z}_2 .$$

- $A = \mathbb{Z}_3$ $\quad 2y_1 = -y_1 \bmod 3 \Rightarrow \underline{\partial_2 \text{ has no kernel.}}$

$$\Rightarrow H_2(K, \mathbb{Z}_3) = 0, \quad H_1(K, \mathbb{Z}_3) = \mathbb{Z}_3$$
$$H_0(K, \mathbb{Z}_3) = \mathbb{Z}_3 .$$

$\underline{A = \mathbb{Z}}$. $0 \to \underline{\underline{\mathbb{Z}}} \xrightarrow{(2,0)} \underset{\langle y_1, y_2 \rangle}{\mathbb{Z}^2} \xrightarrow{\binom{0}{0}} \mathbb{Z} \to 0$

$H_2(K,\mathbb{Z}) = 0$ $H_1(K,\mathbb{Z}) = \langle y_1, y_2 \mid 2y_1 = 0 \rangle$

$\cong \mathbb{Z}_2 \times \mathbb{Z}$ ← torsion homology

$H_0(K,\mathbb{Z}) = \mathbb{Z}$.

$\underline{A = \mathbb{Z}_6}$: $3 \cdot 2 = 0 \bmod 6$

Torsion subgroup of $G \equiv TG = \{g \in G \mid \underset{n \geq 1}{ng = 0}\}$

$$\left\{ \begin{array}{l} H_2(K,\mathbb{Z}_6) = \mathbb{Z}_2 = \langle 3w \rangle \\ H_1(K,\mathbb{Z}_6) = (\mathbb{Z}_6/2\mathbb{Z}_6 = \mathbb{Z}_3) \times \mathbb{Z}_6 \quad = \langle y_1, y_2 \rangle \\ H_0(K,\mathbb{Z}_6) = \mathbb{Z}_6 = \langle p \rangle . \end{array} \right.$$

- $H_0(K,\mathbb{Z}_n) \neq H_0(K,\mathbb{Z}) \bmod n$
- torsion homology does not require non-orientable

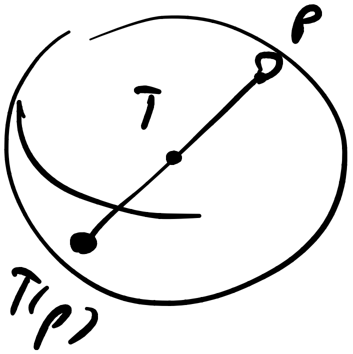
$\underline{\mathbb{RP}^n}$ $\equiv$ space of lines through $\vec{0} \in \mathbb{R}^{n+1}$.

$= \{\vec{v} \in \mathbb{R}^{n+1}\} / (\vec{v} \sim \vec{v}\lambda)$, $\lambda \in \mathbb{R} \setminus \{0\}$ $\quad$ (eg $\mathbb{RP}^3 = SO(3)$)

$\equiv$ $\quad$ pick a gauge (λ) where $|\vec{v}| = 1$.

$= S^n / (\hat{v} \sim -\hat{v})$

$=$



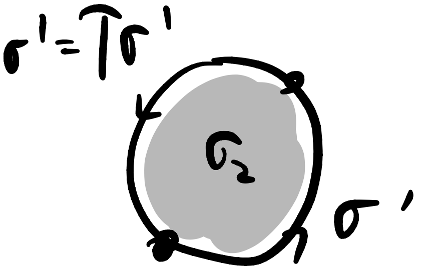
$/ T$

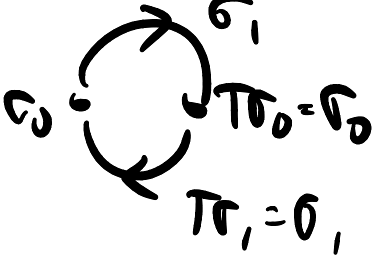
$= B_n \Big/ \left(\begin{matrix} \hat{v} \sim -\hat{v} \text{ on} \\ \partial B_n = S^{n-1} \end{matrix} \right)$

$\nearrow$ northern hemisphere

$\partial B_n / \sim \; = S^{n-1} / T = \mathbb{RP}^{n-1}$

Iterative cell decomp :





$$\partial\sigma_1 = \sigma_0 - T\sigma_0 = \sigma_0 - \sigma_0 = 0.$$

$$\partial\sigma_2 = \sigma_1 + T\sigma_1 = 2\sigma_1$$

$$\partial\sigma_3 = \sigma_2 - \underbrace{T\sigma_2}_{=\sigma_2} = \sigma_2 - \sigma_2 = 0.$$

$$0 \to A \xrightarrow{2} \underset{3}{A} \xrightarrow{0} \underset{2}{A} \xrightarrow{2} \underset{1}{A} \xrightarrow{0} \underset{0}{A} \to 0$$

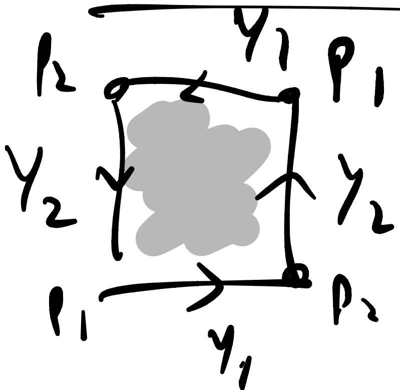
$\underline{A = \mathbb{Z}}$. $\underline{\mathbb{RP}^3}$.
$$\begin{cases} H_3(\mathbb{RP}^3, \mathbb{Z}) = \langle \sigma_3 \rangle = \underline{\underline{\mathbb{Z}}} \\ H_2(\quad) = 0 \\ H_1(\quad) = \langle \sigma_1 \mid 2\sigma_1 = 0 \rangle = \mathbb{Z}_2 \\ H_0(\quad) = \mathbb{Z}. \end{cases}$$

$$H_i(\mathbb{RP}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}_2 & i \text{ odd } < n \\ \mathbb{Z} & i = n \quad n \text{ odd} \\ 0 & \text{else} \end{cases}$$

$$H_n(\mathbb{RP}^n, \mathbb{Z}) = 0 \quad n \text{ even}$$

$\Longleftrightarrow$ $\mathbb{RP}^n$ n even is non-orientable.

<u>Macaulay2</u>



$\underline{\mathbb{CP}^n} = \{$ complex lines through $\vec{0} \in \mathbb{C}^{n+1}\}$

$= \{\vec{z}\}/(\vec{z} \sim \lambda\vec{z}) \quad \lambda \in \mathbb{C}\setminus\{0\}$

choose gauge w/ $|\vec{z}| = 1$.

$= \underline{\underline{S^{2n+1}}}/(\vec{z} \sim \lambda z), \ |\lambda| = 1$]

consider the subspace w/ $z^{N+1} \neq 0$

choose λ to make $z^{N+1} > 0$

$$\Big\{ \underset{\substack{\uparrow \\ N+1 \text{ vector}}}{\vec{z}} = (\underset{\substack{\uparrow \\ N \text{ vector}}}{\vec{w}}, \underbrace{\sqrt{1-|\vec{w}|^2}}_{\in\, z^{N+1}}) \qquad |\vec{w}|^2 \leq 1. \Big\} = B_{2n}$$

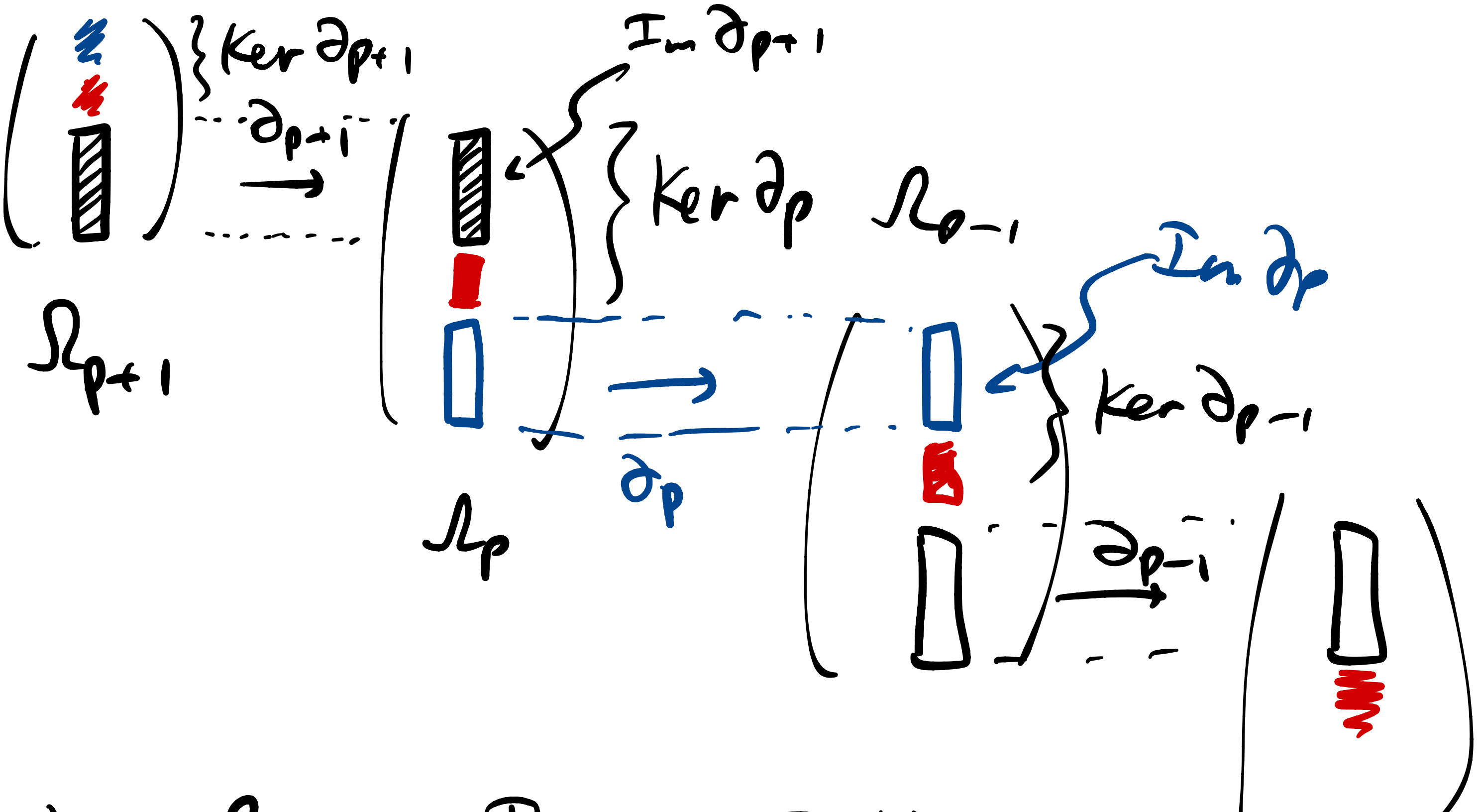
$$\partial B_{2n} = \Big(\{z^{N+1} = 0 \rightarrow |w| = 1\} = S^{2n-1}\Big) \Big/ \vec{w} \sim \lambda\vec{w}$$

$= \mathbb{CP}^{n-1}$.

cells $= \sigma_0 \cup \sigma_2 \cup \sigma_4 \cup \ldots \cup \sigma_{2n}$

$\partial = 0$.

$$H_n(\mathbb{CP}^n, A) = \begin{cases} A & n \text{ even} \\ 0 & n \text{ odd} \end{cases}.$$



Euler Poincaré Thm: $I_p(\Delta) \equiv$ # of p-cells in Δ

$$\chi(\Delta) \equiv \sum_{p=0}^{d} (-1)^p I_p = \sum_{p=0}^{d} (-1)^p b_p$$

$b_p = \dim_A H_p(\Delta, A)$

(take A to be a field)

Pf:

$$-\dim \mathrm{Im}\,\partial_{d+1} \quad = 0$$

$$\left.\dim \ker \partial_d + \dim \mathrm{Im}\,\partial_d\right\} I_d$$

$$-\dim \ker \partial_{d-1} - \dim \mathrm{Im}\,\partial_{d-1}$$

$$+\dim \ker \partial_{d-2} + \dim \mathrm{Im}\,\partial_{d-2}$$

$$\vdots$$

$$\dim H_p = \dim \ker \partial_p - \dim \mathrm{Im}\,\partial_{p+1}$$

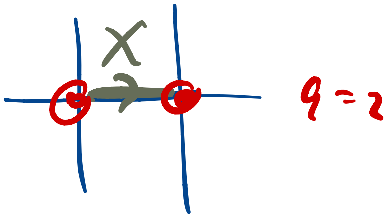
∎

1.4 Higgsing & changing A & exact sequences

T.C. w/ $\underline{A = A_1} \supset A_2$ nontrivial subgroup.

eg: $A_1 = \mathbb{Z}_{pq}$ $\quad \underline{\underline{A_2 = \mathbb{Z}_p = \langle g^q \rangle}}$

$= \langle g \mid g^{pq} = 1 \rangle$



$$\underline{\Delta H = -\underset{\uparrow}{h} \sum_\ell X_\ell^q}$$

proliferate charge in A_2 but not in A_1/A_2

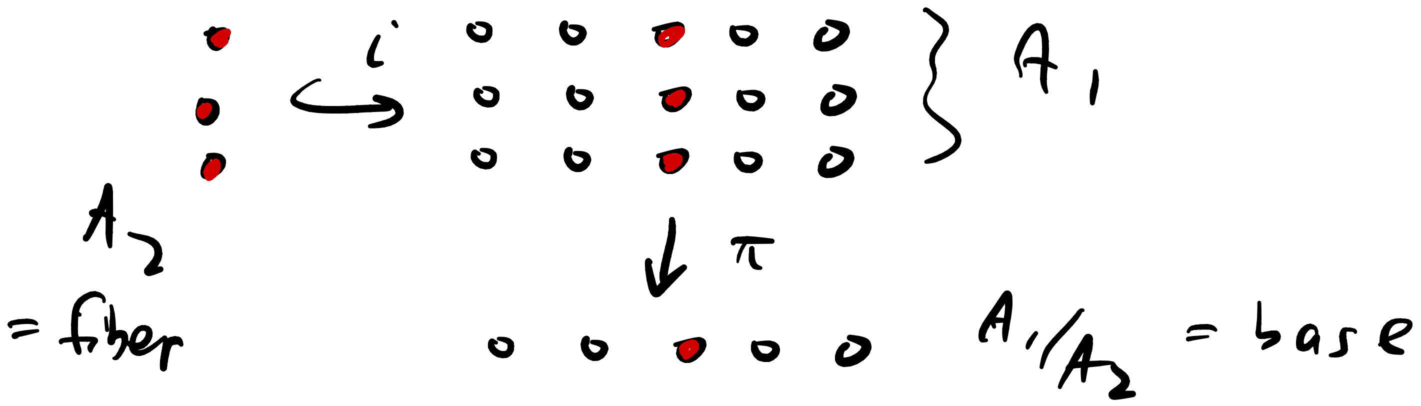
in the example,

we're left w/ TC w/ $\underline{A = \mathbb{Z}_q}$.

$$0 \to A_2 \xhookrightarrow{i} A_1 \xrightarrow{\pi} A_1/A_2 \to 0$$

is an exact sequence ("$\underline{\underline{\text{short}}}$ exact seq.")

$\equiv$ Im(prev map) = Ker(next map) ie $\underline{\underline{\text{no homology}}}$.



$\Longrightarrow$ short exact seq. of <u>chain maps</u>

$$0 \to \Omega_\bullet(A_2) \xhookrightarrow{i} \Omega_\bullet(A_1) \xrightarrow{\pi} \Omega_\bullet(A_1/A_2) \to 0$$

$\left([i, \partial] = 0 = [\pi, \partial] \Leftrightarrow i, \pi \text{ are chain maps} \right)$

<u>Fact</u>: Given such a short exact seq. of chain maps $\exists$ long exact seq. in homology

$$\begin{tikzcd}
H_p(A_2) \arrow[r, "i_*"] & H_p(A_1) \arrow[r, "\pi_*"] & H_p(A_1/A_2) \arrow[dll, "\partial_*"'] \\
H_{p-1}(A_2) \arrow[r] & H_{p-1}(A_1) \arrow[r] & H_{p-1}(A_1/A_2) \arrow[dll, "\partial_*"'] \\
\dots & &
\end{tikzcd}$$

"connecting homomorphism"

(or "Bockstein")

$$0 \to A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{\pi} C_\bullet \to 0$$ chain maps

i.e.

$$\begin{tikzcd}
 & \cdots \arrow[d] & b \arrow[r] \arrow[d] & \pi(b)=c \arrow[d] & \\
0 \arrow[r] & A_p \arrow[r, "i"] \arrow[d, "\partial"] & B_p \arrow[r, "\pi"] \arrow[d, "\partial"] \arrow[rd, phantom, "\textcircled{c}"] & C_p \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & A_{p-1} \arrow[r, "i"] & B_{p-1} \arrow[r, "\pi"] & C_{p-1} \arrow[r] & 0 \\
 & a \arrow[r] \arrow[d, "\partial"] & i(a)=\partial b \arrow[r] \arrow[d, "\partial"] & \pi(\partial b)=0 \arrow[d, "\partial"] & \\
0 \arrow[r] & A_{p-2} \arrow[r] \arrow[d, "\partial a"] & B_{p-2} \arrow[r] \arrow[d] & C_{p-2} \arrow[r] \arrow[d] & 0 \\
 & {} \arrow[r, "i"] & 0 = i\partial a & {} &
\end{tikzcd}$$

① Rows are exact.

② everyone commutes

goal: construct $\partial_*[c] = [a] \in H_{p-1}(A)$

start w/ $\underline{\underline{c}} \in \ker \partial_p \subset C_p$.

$\partial c = 0$.

$\partial b \in \ker(\pi) = \mathrm{Im}(i)$

$= i a$ w/ $\partial a = 0$

$\Rightarrow [a] = \partial_*[c]$.

$$\partial_*[c] = [\, i^{-1} \partial \pi^{-1} c \,].$$

To show

$$\xrightarrow{\pi_*} H_{p+1}(C) \xrightarrow{\partial_*} \underline{H_p(A)} \xrightarrow{i_*} \underline{H_p(B)} \xrightarrow{\pi_*} \underline{H_p(C)} \xrightarrow{\partial_*} \cdots$$

is exact.

to see exactness at $H_\bullet(C)$:

suppose $\partial_*[c] = 0$ for $c \in C_p$.

$c = \pi(b)$ $\quad b \in B_p$

$\partial b = ia$, $\underline{[a] = \partial_*[c] = 0}$.

$\Rightarrow \underline{\underline{a = \partial a'}}$

$\partial b = ia = i\partial a'$

$= \partial i a'$

$\Rightarrow \partial(b - ia') = 0.$

$\pi(b - ia') = c$

$\Rightarrow$

$\underline{[c] = \pi_*[b - ia']}$.

$$\begin{tikzcd}
 & \cdots \arrow[d] & b \arrow[r] \arrow[d] & \pi(b) = c \arrow[d] & \\
0 \arrow[r] & a' \; A_p \arrow[r, "i"] \arrow[dd, "\partial"] \arrow[ddr, phantom, "\textcircled{c}"] & B_p \arrow[r, "\pi"] \arrow[dd, "\partial"] \arrow[ddr, phantom, "\textcircled{c}"] & C_p \arrow[r] \arrow[dd, "\partial"] & 0 \\
 & & & & \\
0 \arrow[r] & A_{p-1} \arrow[r, "i"] \arrow[ddd, "\partial"] & B_{p-1} \arrow[r, "\pi"] \arrow[ddd, "\partial"] & C_{p-1} \arrow[r] \arrow[ddd, "\partial"] & 0 \\
 & \partial a' = a \arrow[r] & i(a) = \partial b \arrow[r] & \pi(\partial b) = 0 & \\
 & & & & \\
0 \arrow[r] & A_{p-2} \arrow[r] \arrow[d, "\partial a"] & B_{p-2} \arrow[r] \arrow[d] & C_{p-2} \arrow[r] \arrow[d] & 0 \\
 & {} \arrow[r, "i"] & 0 = i\partial a & {} &
\end{tikzcd}$$

check: $H_\bullet(K, \mathbb{Z}_{2,3,6})$ satisfy the long-exact seq. from

$$0 \to \mathbb{Z}_2 \xhookrightarrow{i} \mathbb{Z}_6 \xrightarrow{\pi} \left(\mathbb{Z}_3 = \mathbb{Z}_6/\mathbb{Z}_2\right) \to 0.$$

Universal Coeff Thm :

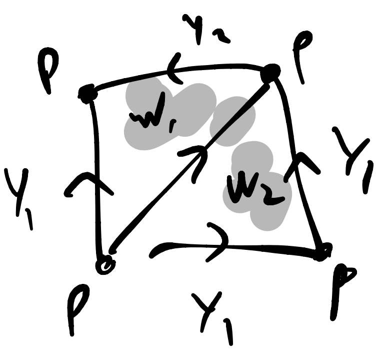
Any discrete abelian group is $A = \mathbb{Z}\times\mathbb{Z}\dots\times\mathbb{Z}\times\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2}$

$H(X, \mathbb{Z})$ determine $H(X, A)$ $\forall A$.

$\exists$ an exact seq:

$$0 \to \underset{\substack{\uparrow \\ \text{know}}}{H_p(X,\mathbb{Z})}\otimes A \to \underset{\substack{\uparrow \\ \text{want}}}{H_p(X,A)} \to \mathrm{Tor}(\underset{\substack{\uparrow \\ \text{know}}}{\underbrace{H_{p-1}(X,\mathbb{Z})}}, A) \to 0$$

$\mathrm{Tor}(B,A) \sim$ common zero-divisors of A & B.



0-form toric code:

$$H_0 = -\sum_{\substack{1\text{-cells}\\ \sigma}} \prod_{p \in \partial\sigma} X_p = 1.$$

$\longrightarrow$ 6 groundstates

2-form toric code:

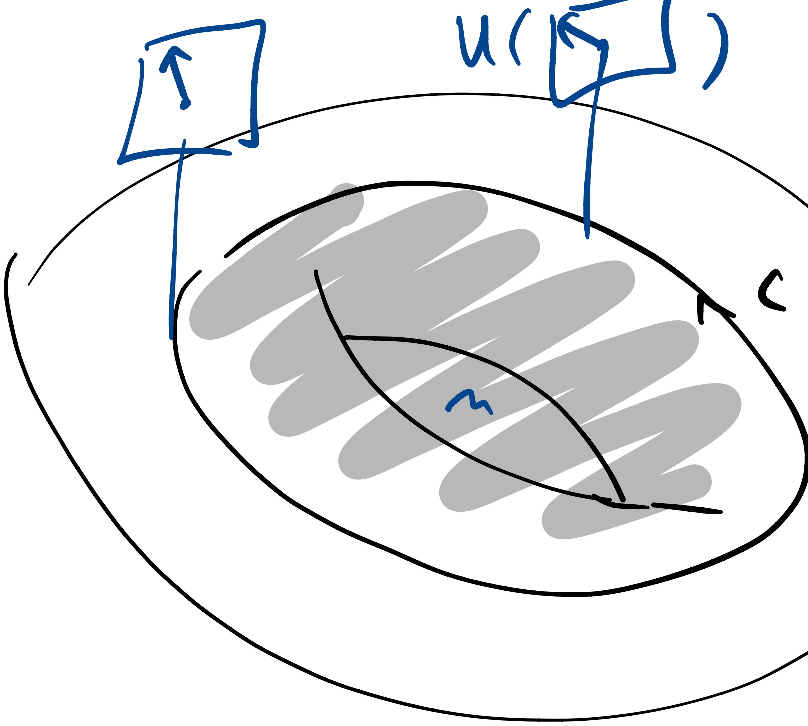
$$H_2 = -\sum_{\substack{1\text{-cells}\\ \sigma}} \prod_{w \in V(\sigma)} Z_w + h.c.$$

$$= -\underline{\underline{Z_{w_1} Z_{w_2}^{+}}} - \underline{\underline{Z_{w_1} Z_{w_2}}} + h.c.$$

$$Z_{w_1} = Z_{w_2} \qquad \underline{Z_{w_1}^2 = 1.}$$

$\longrightarrow$ 2 groundstates

✓



$$U = e^{i\oint_{c=} a}$$

$$= e^{i\int_M B}$$

$$W_c = \prod_{l\in C} X_l \text{ ``}= e^{i\oint_c a}\text{''}$$

$$W_c|\pi\rangle = -|\pi\rangle$$

$G = \mathbb{Z}_2$

vector bundle = fiber bundle where the fiber is a vector space

base = space

if $G = U(1)$. $W_c|\phi\rangle = e^{i\phi}|\phi\rangle$

if G $W_{op}|\ \rangle = \underbrace{U_{op}}_{\text{rep of } G}|\ \rangle$