Last time:

$$\Omega_p = \mathrm{span}_A \{ \sigma \in \Delta_p \}$$

$\equiv$ $p$-cells

$$0 \to \Omega_d \to \cdots \to \Omega_{p+1} \xrightarrow{\partial_{p+1}} \Omega_p \xrightarrow{\partial_p} \Omega_{p-1} \xrightarrow{\partial_{p-1}} \cdots \to \Omega_0 \to 0$$

$\partial \sigma_p =$ $(p-1)$-chains in the body of $\sigma_p$

(w/ orientation and multiplicity)

$\partial^2 = 0$. chain complex

$$\mathrm{Im}\,\partial_{p+1} \subset \mathrm{Ker}\,\partial_p$$

def: $[c] = [c + \partial \sigma]$

$\sigma$ is a $p+1$-chain.

$c$: $p$-cycle $\equiv \partial c = 0$.

$$H_p(\Omega) \equiv \mathrm{Ker}(\partial_p : \Omega_p \to \Omega_{p-1}) \,/\, \mathrm{Im}(\partial_{p+1} : \Omega_{p+1} \to \Omega_p)$$

$\equiv$ cycles mod boundaries.

Comments: • $H_p(\Omega)$ is a abelian group

$$[c] + [c'] \equiv [c + c'] .$$

• $H_p(\Omega)$ is a v.s. over $A$

w/ dim. $\equiv \underline{\underline{b_p(\Omega)}} \equiv$ $p^{th}$ betti #

claim: $H_p(\Omega) \equiv H_p(X, A)$ is ind. of cellulation of $X$.

---

## 1.2 p-form $\mathbb{Z}_N$ toric code

(so far: 1-form $\mathbb{Z}_2$ toric code)

Consider a cell complex $\Delta$.

put a $\mathcal{H}_N \equiv \text{span}\{|n\rangle \; n = 0 \ldots N-1\}$

on each p-cell $\sigma_p \in \Delta_p$.

"qudit"

$$\omega \equiv e^{2\pi i/N}$$

$$\begin{cases} Z \equiv \sum_{n=1}^{N} |n\rangle\langle n| \omega^n = \begin{pmatrix} \omega & & & \\ & \omega^2 & & \\ & & \ddots & \\ & & & \omega^{N-1} \end{pmatrix} & \text{clock} \\ X \equiv \sum_{n=1}^{N} |n\rangle\langle n+1| = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & \ddots \\ 1 & & & \end{pmatrix} & \text{shift} \end{cases}$$

$$\left( |n+N\rangle \equiv |n\rangle . \right)$$

$$XZ = \omega Z X .$$

$$\begin{cases} X = X^\dagger \text{ only for } N=2. \\ Z = Z^\dagger \end{cases}$$

But $\forall N$

$$XX^\dagger = \mathbb{1}$$

$$ZZ^\dagger = \mathbb{1} .$$

$$X^N = \mathbb{1} = Z^N .$$

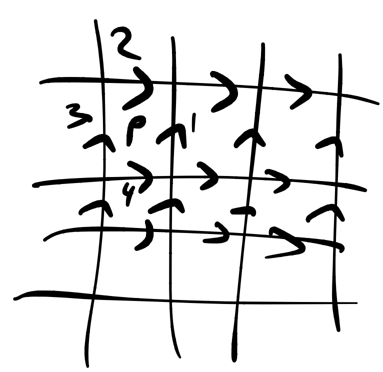
<u>New</u> <u>ingredient for $N>2$:</u>

[ choose an <u>orientation</u> of each p-cell.

$$\xrightarrow[\sigma]{} \qquad \xleftarrow[-\sigma]{}$$

$$\underline{Z_{-\sigma} \equiv Z_\sigma^\dagger = Z_\sigma^{-1}} .$$

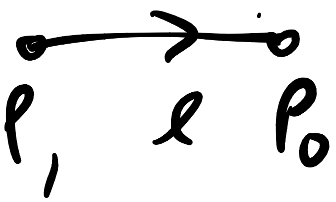
<u>eg:</u>



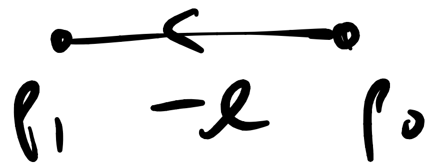

$$\underline{\partial p = l_1 - l_2 - l_3 + l_4}$$

$\sigma \equiv$ p-cell

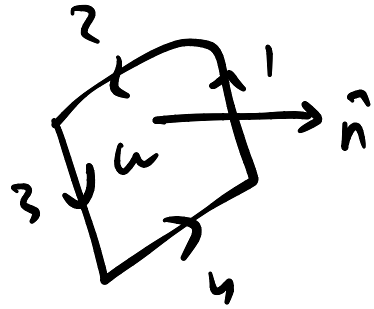
$-\sigma \equiv$ p-cell w/ opposite orientation.




$$\partial \ell = p_0 - p_1$$




$$\partial(-\ell) = -p_0 + p_1$$



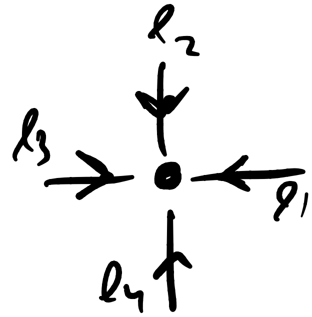

$$\partial \omega = \ell_1 + \ell_2 + \ell_3 + \ell_4$$

## vicinity map:

$$\Omega_{p+1} \underset{\nu}{\overset{\partial}{\rightleftarrows}} \Omega_p \underset{\nu}{\overset{\partial}{\rightleftarrows}} \Omega_{p-1}$$


$$\nu : \Omega_p \to \Omega_{p+1}$$

$$\sigma \mapsto \nu(\sigma) = \sum_{\substack{\mu \in \Delta_{p+1}, \\ \partial\mu = k\sigma + \dots}} k\mu$$




eg $\nu(s) = \sum_i \ell_i$

Recall: inner product on $\Omega_p = \{\sigma, \sigma \in \Delta_p\}$

on a basis $\langle \sigma, \sigma' \rangle = \begin{cases} 1 & \text{if } \sigma = \sigma' \\ -1 & \text{if } \sigma = -\sigma' \\ 0 & \text{else} \end{cases}$

good def of $\nabla \equiv \partial^+$ adjoint

$$\langle \sigma, \nabla \sigma' \rangle \equiv \langle \partial \sigma, \sigma' \rangle .$$

$$H_{TC} = -J_{p-1} \sum_{s \in \Delta_{p-1}} \left( A_s + A_s^+ \right)$$

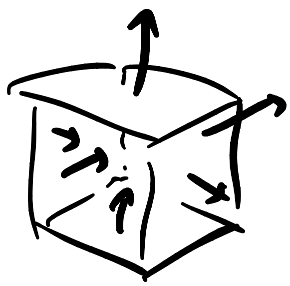
$$- J_{p+1} \sum_{\mu \in \Delta_{p+1}} \left( B_p + B_p^+ \right)$$

$$- T_p \sum_{\sigma \in \Delta_p} Z_\sigma + \ldots$$

$$\begin{cases} A_s \equiv \prod_{\sigma \in V(s) \subset \Delta_p} Z_\sigma \\ B_\mu \equiv \prod_{\sigma \in \partial \mu} X_\sigma . \end{cases}$$

eg: $d=3$ cubic lattice $\quad$ $\underline{p=2}$.

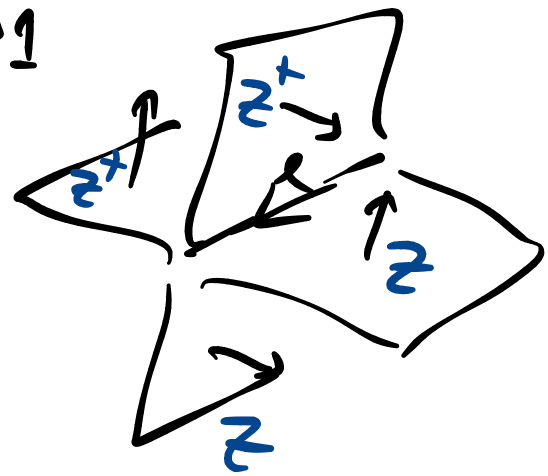
dofs on faces.



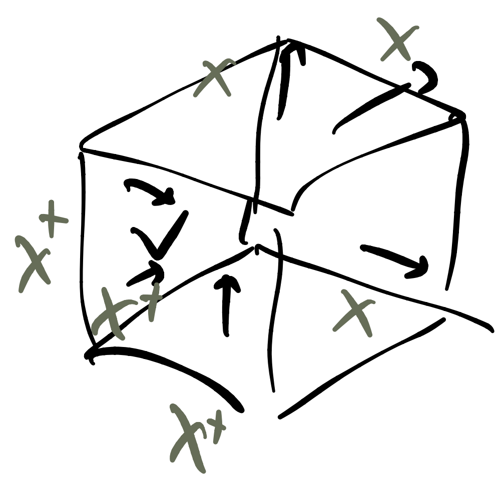
orient in $+\hat{x}, +\hat{y}, +\hat{z}$.

for each link $\ell \in \Delta_1$

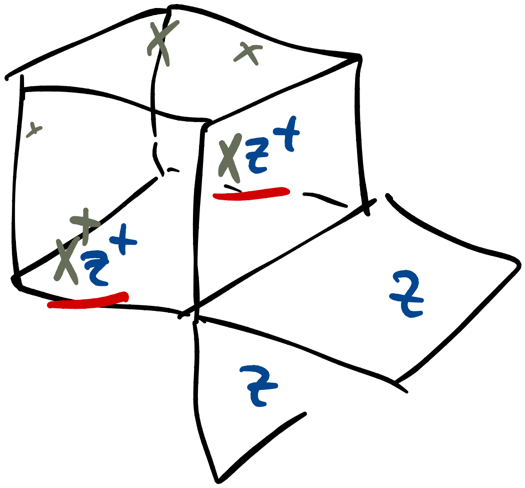
$$A_\ell = \prod_{\substack{4 \text{ faces} \\ \sigma}} Z_\sigma$$




$$B_v = \prod_{\substack{6 \text{ faces} \\ \text{in } \partial v}} X_\sigma$$




$\underline{\text{claim:}}$ $[A_\ell, B_v] = 0 \quad \forall\, \ell, v.$




$$\underline{X\otimes X^\dagger} \quad \underline{Z^\dagger \otimes Z^\dagger}$$

$$= \underbrace{\omega\,\omega^{-1}}_{=1}\; Z^\dagger \otimes Z^\dagger \; X \otimes X^\dagger .$$

$$\begin{cases} X^\dagger Z^\dagger = \omega Z^\dagger X^\dagger \\ X Z^\dagger = \omega^\dagger Z^\dagger X \end{cases}$$

$$B_\mu A_s = \prod_{\sigma\in\partial\mu} X_\sigma \prod_{\sigma'\in V(s)} Z_{\sigma'}$$

$$= \underline{\prod_{\sigma\in\partial\mu}}\ \underline{\prod_{\sigma'\in V(s)}}\ \underline{\omega^{\underline{\langle\sigma,\sigma'\rangle}}}\ A_s B_\mu$$

$$= \omega^{\sum_{\sigma\in\partial\mu}\sum_{\sigma'\in V(s)} \langle\sigma,\sigma'\rangle}\ A_s B_\mu$$

$$= \omega^{\langle \partial\mu,\, V(s)\rangle}\ A_s B_\mu$$

$$\overset{\text{def of } V}{=} \omega^{\langle \cancel{\partial^2}\mu,\, s\rangle}\ A_s B_\mu \qquad \cancel{\partial^2} \to 0, \quad \partial^2 = 0$$

$$= A_s B_\mu .$$

■

suppose $J_{p-1} \gg J_{p+1}$

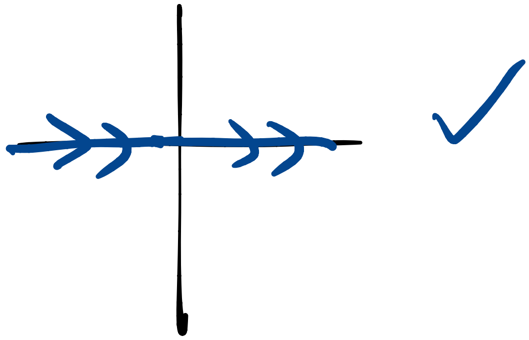
impose $A_s = 1 \quad \forall s.$

eg: $N=3,\ p=1$

$|\,-\!\!\!-\,\rangle \equiv |n=0\rangle$

$|\rightarrow\rangle = |n=1\rangle$

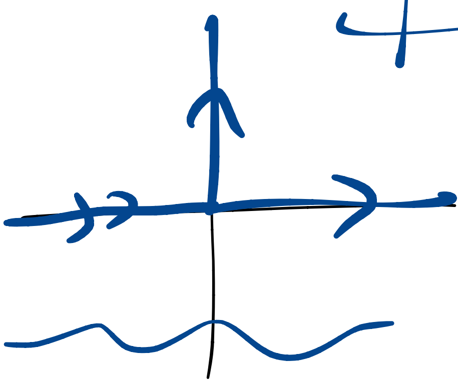
$|\twoheadrightarrow\rangle = |n=2\rangle$



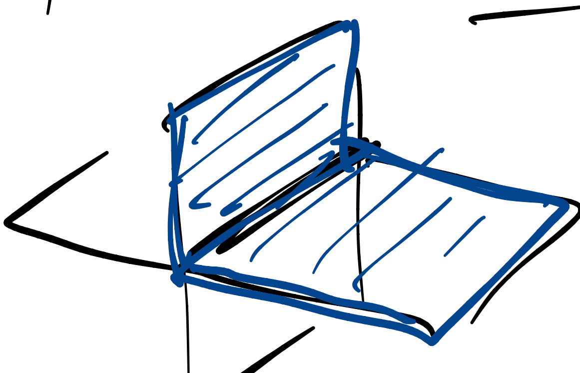
$$1 \overset{!}{=} \prod_{\ell \text{ going into } s} Z_\ell \iff \boxed{\sum_{\sigma \text{ going into } s} n_\sigma = 0 \mod N.}$$

Gauss Law.

$\equiv$ closed string-nets



for $p>1$: closed p-brane nets.



$B_\mu$ | closed p-branes $\rangle$ = | closed p-branes' $\rangle$

$$\boxed{B_\mu |c\rangle = |c + \partial\mu\rangle}$$

like a kinetic term for p-branes
and a creation/annihilation op.

$B_\mu |g.s.\rangle = |g.s.\rangle \ \forall \mu$

$\Rightarrow \Psi(C) \equiv \langle c |$ groundstate $\rangle$

has $\Psi(C) = \Psi(C + \partial\mu)$

$\begin{cases} A_s = 1 \text{ is the condition that } c \text{ is a cycle} \\ B_\mu = 1 \quad \text{" " eq. relation modulo } \mathrm{Im}\,\partial_{p+1}. \end{cases}$

a basis of g.s. $\longleftrightarrow$ $\underline{H_p(\Delta, \mathbb{Z}_N)}$.

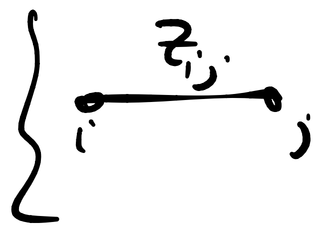
---

Rel'n to p-form $\mathbb{Z}_N$ gauge theory:

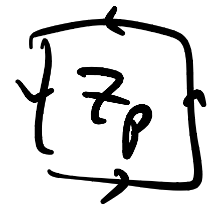
add electric matter living on the p-cells.

$\Phi_\ell = \Phi^\dagger_{-\ell}$.

gauge inv:




$$\begin{cases} \Phi_\ell \mapsto \omega_\ell \, \Phi_\ell \\ Z_\sigma \mapsto \prod_{\ell \in \partial\sigma} \omega^\ell \, Z_\sigma \end{cases}$$




$$\Delta H = g \sum_{\sigma \in \Delta_p} \Big( \prod_{\ell \in \partial p} \Phi_\ell \, Z_\sigma \Big)$$ is gauge inv't

Unitary gauge: $\Phi_\ell = 1$.

gives back $p$-form T.C $+ \Delta H$.

$\boxed{p=0}$ dof on 0-cells.

$(\partial_0 : \Omega_0 \to \Omega_{-1}) = 0. \Rightarrow$ no star op.

$\partial_1 : \Omega_1 \to \Omega_0$

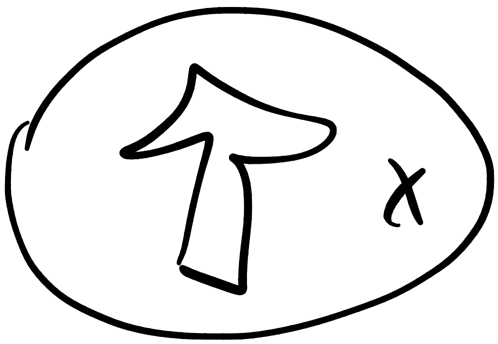
$$\Rightarrow H = -\sum_{\ell \in \Delta_1} B_\ell + h.c. = -\sum_\ell \prod_{s \in \ell} Z_s + h.c.$$

$$= -\sum_{\langle ij \rangle} \left( Z_i Z_j^\dagger + h.c. \right)$$

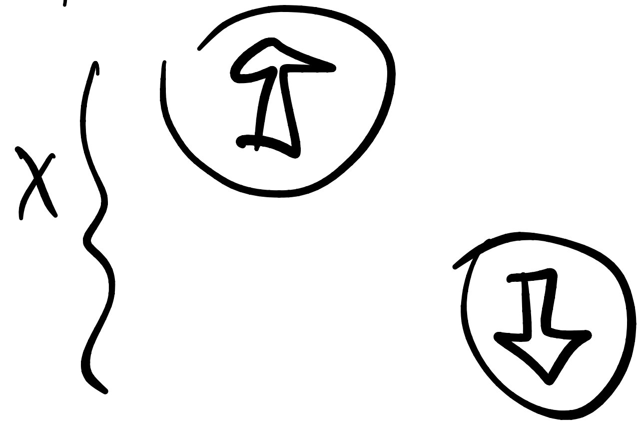
A $\mathbb{Z}_N$ ferromagnet.

$$|gs\rangle = \bigotimes_s |n\rangle_s \qquad n = 0 .. N-1$$

$H_0(X, \mathbb{Z}_N)$ is a top.



$N = |A|$ groundstates if $X$ is connected.



if $X$ has $h$ components $\rightarrow$ $|A|^h$ groundstates

$$\dim_{\mathbb{Z}_N} H_0(X, \mathbb{Z}_N) \equiv b_0(X) = \#\text{ of connected components of } X.$$

$$b_0(X) \equiv \dim_F H_0(X, F)$$

ind of $F$, a field.

(eg $\mathbb{Z}_N$ w $N$ prime)

Inner product on chains:

$$\sigma, \sigma' \in \Delta_p$$

$$\langle \sigma, \sigma' \rangle \equiv \begin{cases} 1 & \text{if } \sigma = \sigma' \\ -1 & \text{if } \sigma = -\sigma' \\ 0 & \text{else.} \end{cases}$$

---

## Some Examples:

$\bullet\, p$

Simplest possible: one 0-cell

$$0 \xrightarrow{\partial} A \xrightarrow{\partial} 0$$

$$\Rightarrow H_0(pt, A) = A.$$

$$\Rightarrow \begin{cases} b_0(pt) = 1. \\ b_{n \neq 0}(pt) = 0. \end{cases}$$

$p_0 \quad p_1$

$\bullet \quad \bullet$

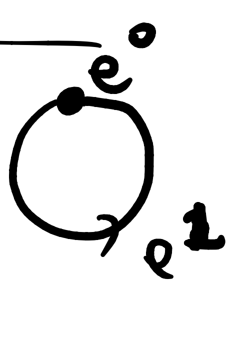
$$0 \to A^2 \to 0$$

$$H_0(k \text{ pts}, A) = A^k$$

$$b_0(k \text{ pts}) = k \ldots$$

---

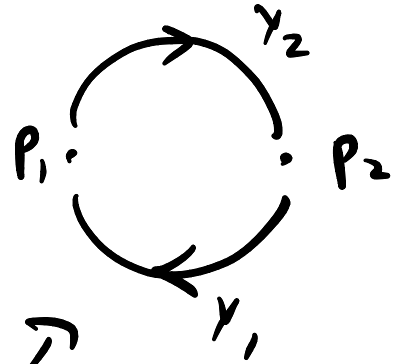
circle.




$$0 \to A \xrightarrow{0} A \to 0.$$

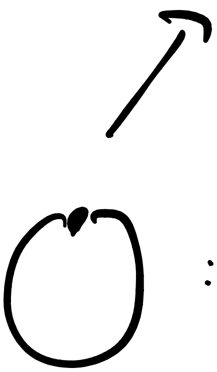
$$\partial e^1 = e^0 - e^0 = 0 \quad \Rightarrow \begin{cases} H_0(S^1, A) = A \\ H_1(S^1, A) = A. \end{cases}$$

$\underline{\Delta'}$ :




$$\underline{\partial\gamma_1 = p_1 - p_2 = -\partial\gamma_2}$$

$$0 \to \underline{\underline{A^2}} \xrightarrow{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}} \underline{A^2} \to 0$$



$: \underline{\Delta}$

$$\ker \partial = \langle \gamma_1 + \gamma_2 \rangle$$

$$\Rightarrow H_1(\Delta', A) = A$$

$$\mathrm{Im}(\partial) \Rightarrow [p_1] = [p_2]$$

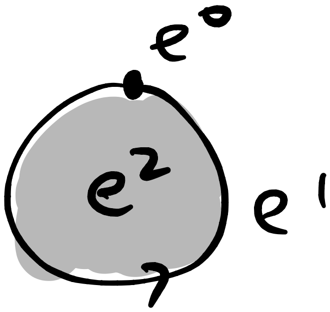
$$\Rightarrow H_0(\Delta', A) = A.$$

$\Delta \to \Delta'$ was like adding:

$$\underbrace{0 \to \underline{\underline{A}} \xrightarrow{1} \underline{\underline{A}} \to 0}_{\mathrm{Im} = \ker}$$

no homology.

**2-ball** : (disk)

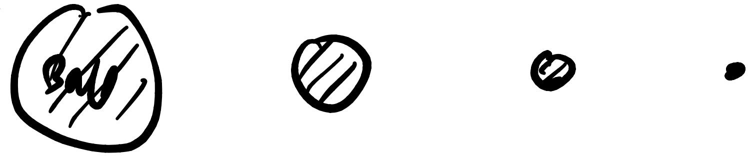



$$\begin{cases} \partial e^2 = e^1 \Rightarrow \text{kills } H_1 \\ \partial e^1 = e^0 - e^0 = 0. \end{cases}$$

$$0 \to \underbrace{A \xrightarrow{1} A}_{} \xrightarrow{0} A \to 0$$

same as $\bullet$ $\to \begin{cases} H_0(\text{Ball}, A) = A. \quad H_{i>0}(\text{Ball}) = 0. \end{cases}$

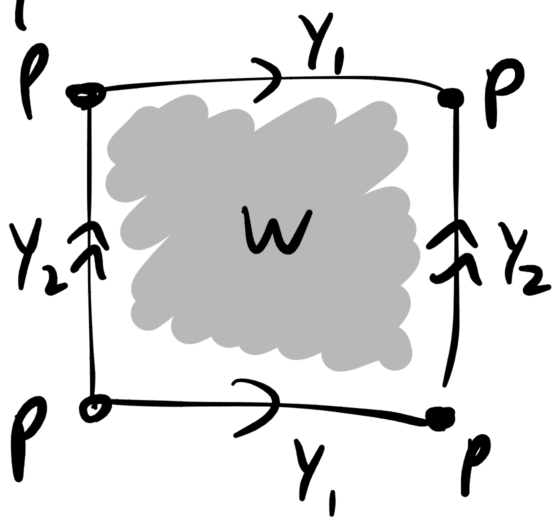
why:



$\exists$ family of continuous maps $\equiv$ homotopy

2-Torus.

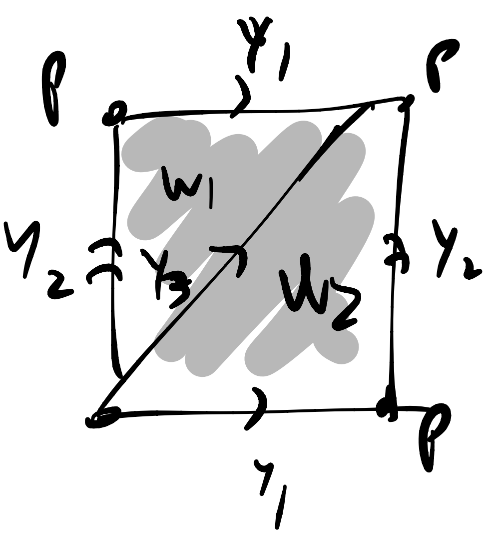
$T^2 = S^1 \times S^1$.




=


$$\begin{cases} \partial W = \gamma_2 + \gamma_1 - \gamma_2 - \gamma_1 = 0. \\ \partial \gamma_2 = p - p = 0 = \partial \gamma_1 \end{cases}$$

$$\begin{array}{ccccccccc} 0 & \to & A & \xrightarrow{0} & A^2 & \xrightarrow{0} & A & \to & 0 \\ & & \| & & \| & & \| & & \\ & & H_2(T^2) & & H_1(T^2) & & H_0(T^2). & & \end{array}$$

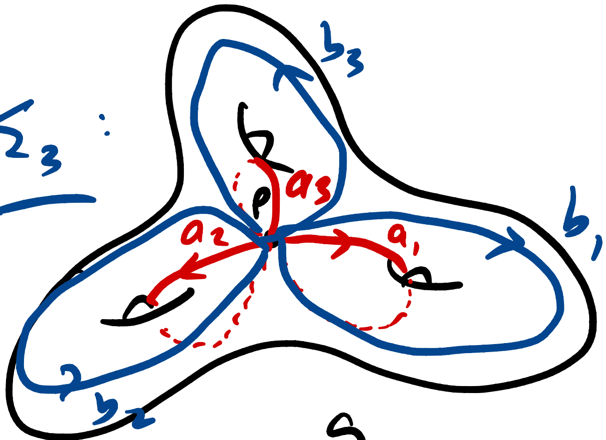


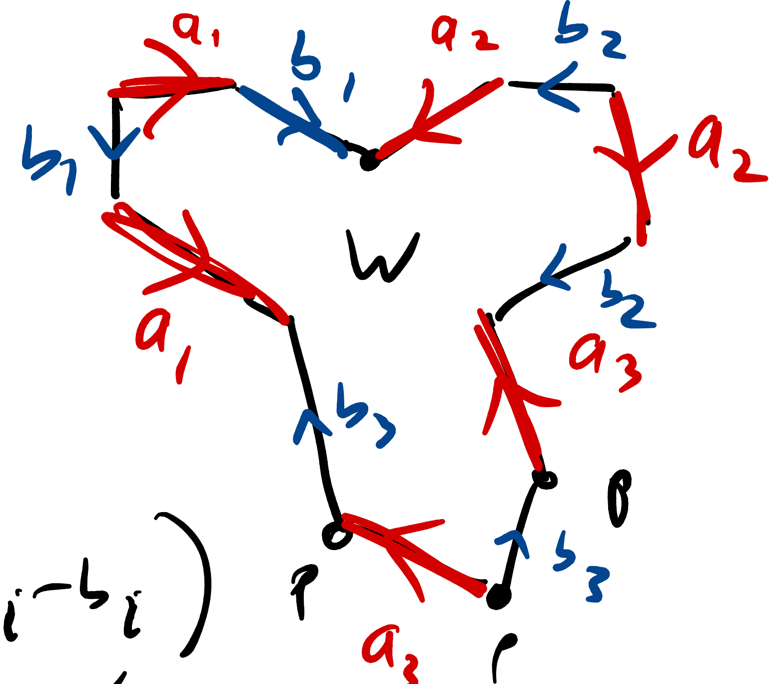

$$0 \to A^2 \xrightarrow{\partial_2} A^3 \xrightarrow{0} A \to 0$$

$$\partial_2 = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$ has rank 1

$\Rightarrow$ same homology.

$\Sigma_3$ :




$$\partial W = \sum_{i=1}^{g} \underbrace{(a_i + b_i - a_i - b_i)}_{=0}.$$

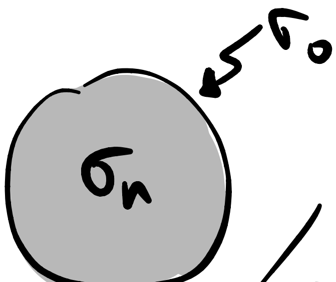
$$0 \to A \xrightarrow{0} A^{2g} \xrightarrow{0} A \to 0$$

$$H_0(\Sigma_g) = A \qquad H_1(\Sigma_g) = A^{2g} \qquad H_2(\Sigma_g) = A.$$

Spheres: Make a $S^n$, $n \geq 1$

$$= B_n / \partial B_n$$

all pts in the bdy are equivalent $= 1$ pt.




$/\sim$

$$0 \to A \to 0 \to 0 \cdots 0 \to A \to 0$$

$$\uparrow \qquad\qquad \uparrow$$

$$\Omega_n \qquad\qquad \Omega_0$$

$$H_i(S^n, A) = \begin{cases} A & \text{if } i = n \text{ or } 0 \\ 0 & \text{else.} \end{cases}$$

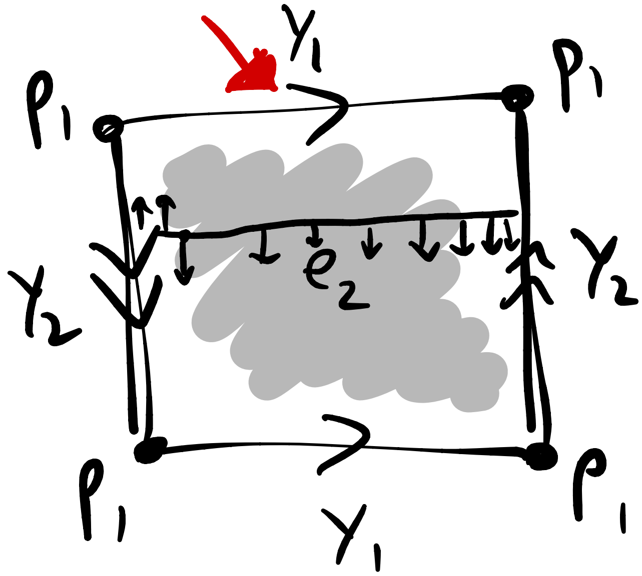
Poincaré duality for $X_d$ compact,

$$b_p(X_d) = b_{d-p}(X_d).$$

---

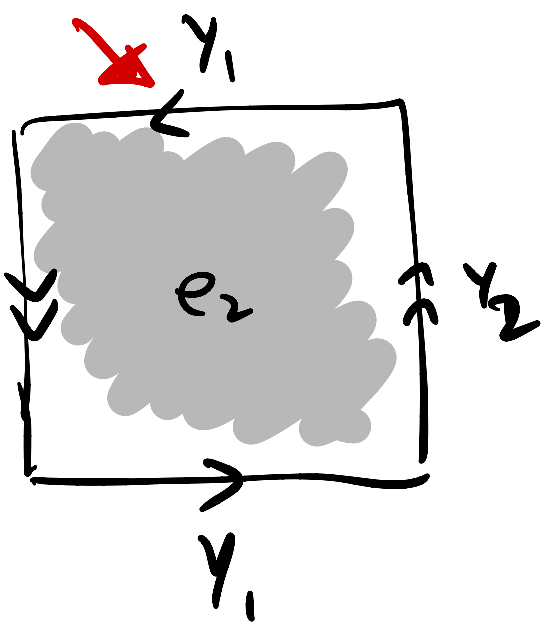
**eg:**

**Klein Bottle:**

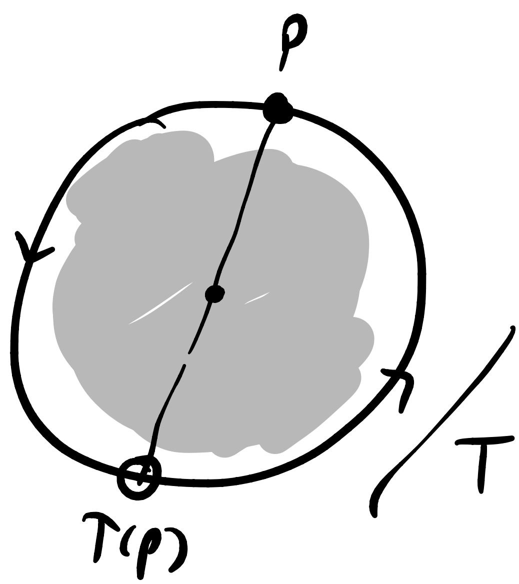



NOT ORIENTABLE

**$\mathbb{RP}^2$:**




$\cong$




each generator $\eta$ of $H_1(X,\mathbb{Z})$ $\longleftrightarrow$ a string operator $W(\eta)$

$$|gs_\eta\rangle = W(\eta)|gs_0\rangle$$

$$\sum_{\text{contractable curves } c} |c\rangle \equiv |gs_0\rangle$$