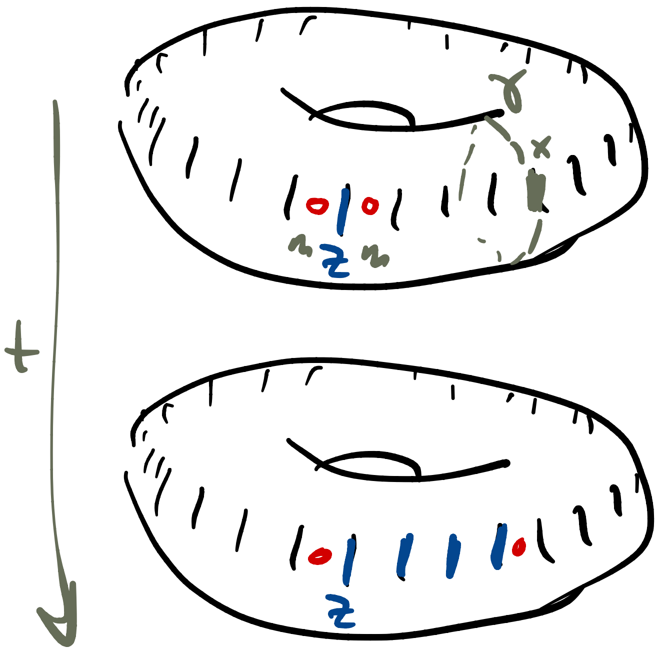
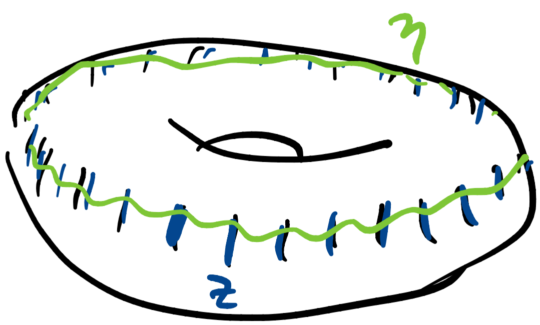
2d Toric code, continued

$$H_{TC} = -\sum_j A_j - \sum_p B_p$$



$$W(\gamma) = \prod_{\ell \in} X_\ell$$

$$W(\gamma)V(\eta) = -V(\eta)W(\gamma)$$



$$V(\eta) = \prod_{\ell \perp \eta} Z_\ell$$

GSD $\Longleftrightarrow$ braiding of top. excitations

Phase Diagram

$$\Delta H = -\sum_\ell \left(g X_\ell + h Z_\ell \right)$$

creation/annihilation op for e particles

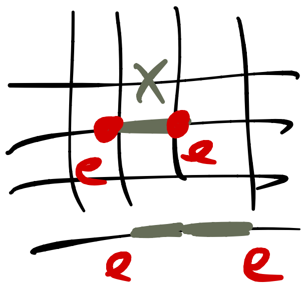
+ kinetic term for e particles

... for m particles

AND

a string tension

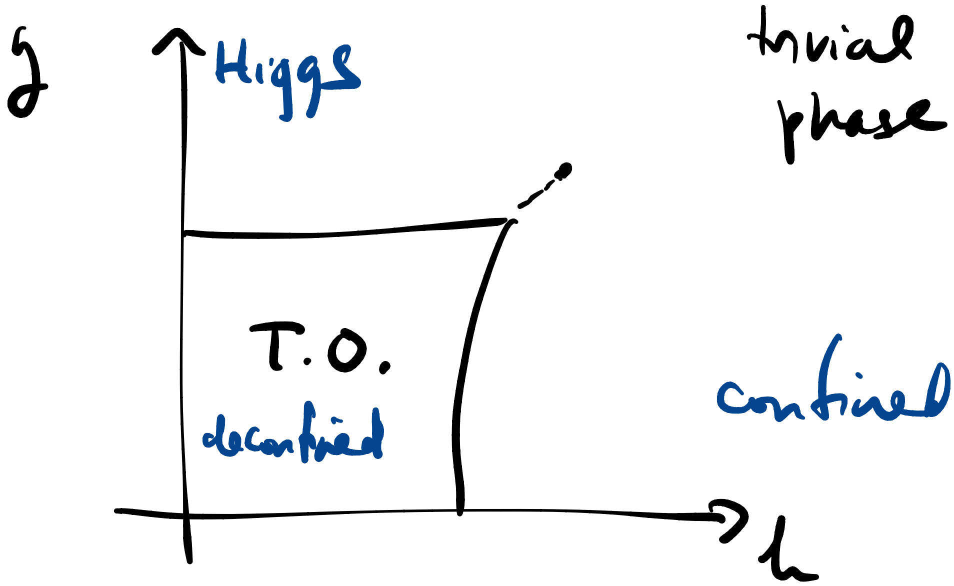
Energy/length of electric flux strings.



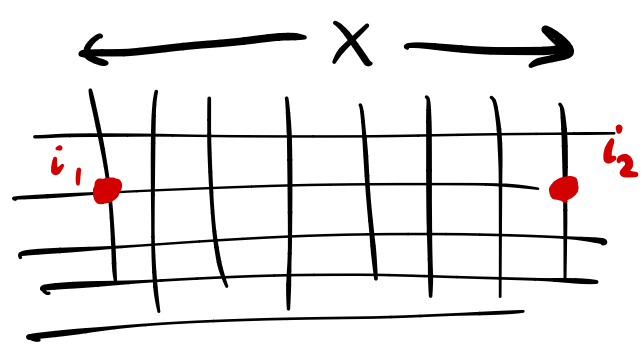
$|gs\rangle \rightarrow$

Large g : $\bigotimes_\ell |+\rangle_\ell$ trivial.

large h : $\bigotimes_\ell |0\rangle_\ell$ trivial.



Large h : no strings in g.s.



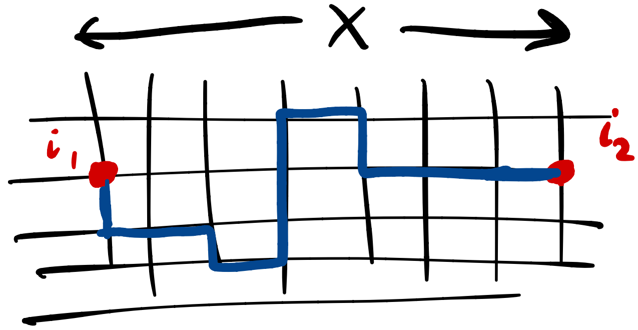
Insert external charge :

demand $-1 \stackrel{!}{=} A_{i_1} = A_{i_2}$.

Q: what is $V(x)$?

$$F = -\frac{\partial V}{\partial x}$$

$F(x) < \infty$ as $x \to \infty$ $\equiv$ deconfined.



$$H(h \to \infty) = -h \sum_{\ell} Z_\ell$$

$$= E_0 + 2h \,\text{length(string)}$$

$$\geqslant E_0 + 2hX$$

$$F = -\frac{\partial V}{\partial X} = 2h.$$

<u>Confinement</u>.

arises from condensation of m particles.

$(\langle X_\ell \rangle \neq 0.)$

<u>Large g</u>: condensation of e particle

= electric charges.

= <u>Higgs phase</u>

At $g=0$ e particles are <u>localized</u>

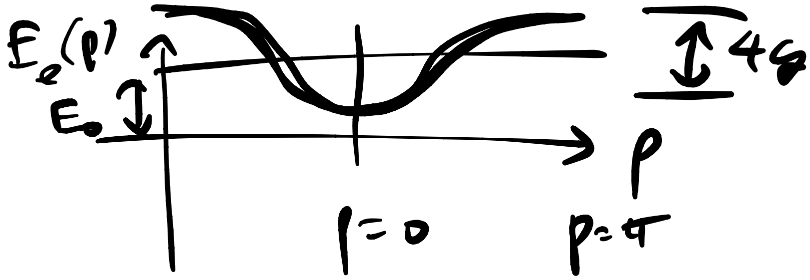
$E\left(\begin{array}{c}\text{state} \\ \text{w/ e} \\ \text{particle} \\ \text{at } x\end{array}\right) =$ ind of x.

or p of p.

$\Rightarrow$ <u>flat band</u>.

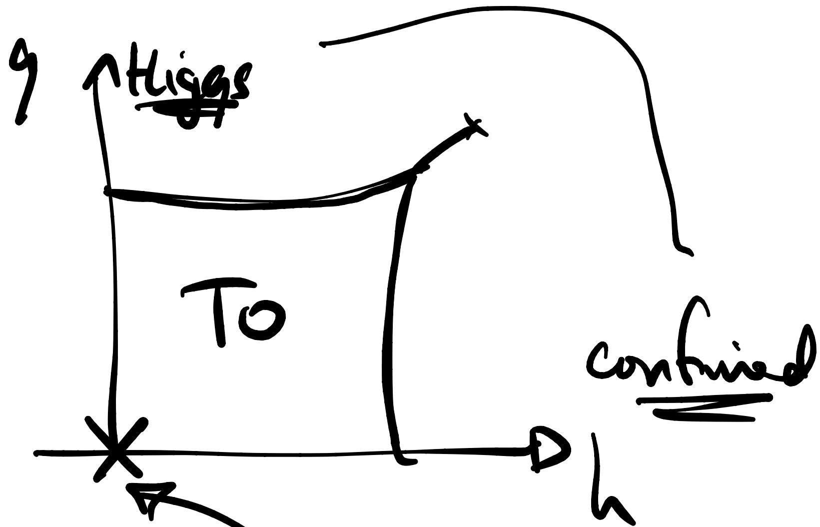
at finite g $\boxed{\text{ROUGHLY}}$

$$E_e(p) = \underbrace{E_0}_{=2} + (\cos p_x + \cos p_y)\,2g$$

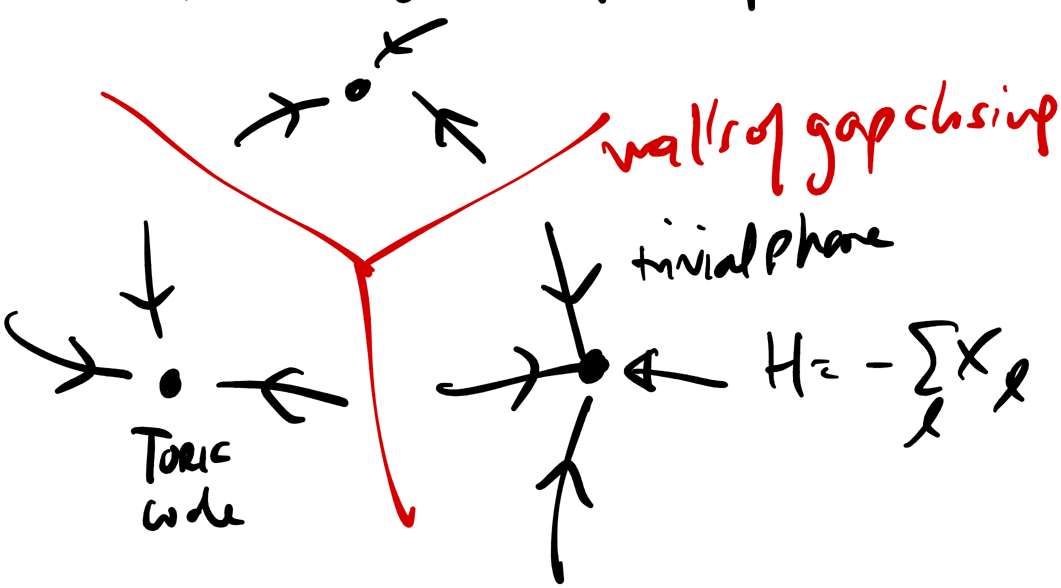


when $4g \gtrsim 1$

$\Rightarrow$ condense e particle.



RG comment: Toric Code is the RG fixed pt. in the deconfined phase



Toric code & gauge theory

TC $=$ $\mathbb{Z}_2$ gauge theory w/ charged bosonic **matter**.

a field ϕ_i

$\mathbb{Z}_2$-valued

gauge redundancy:

$$* \begin{cases} \phi_i \to s_i \phi_i \\ X_{ij} \to s_i X_{ij} s_j \end{cases}$$

$$s_i = \pm 1$$

$$H = -\sum_p B_p - \sum_\ell Z_\ell - g \sum_{ij} \phi_i X_{ij} \phi_j$$

$$- \sum_j A_j (-1)^{n_j}$$

$G_j \equiv \underline{A_j (-1)^{n_j}}$ generates the transf $*$

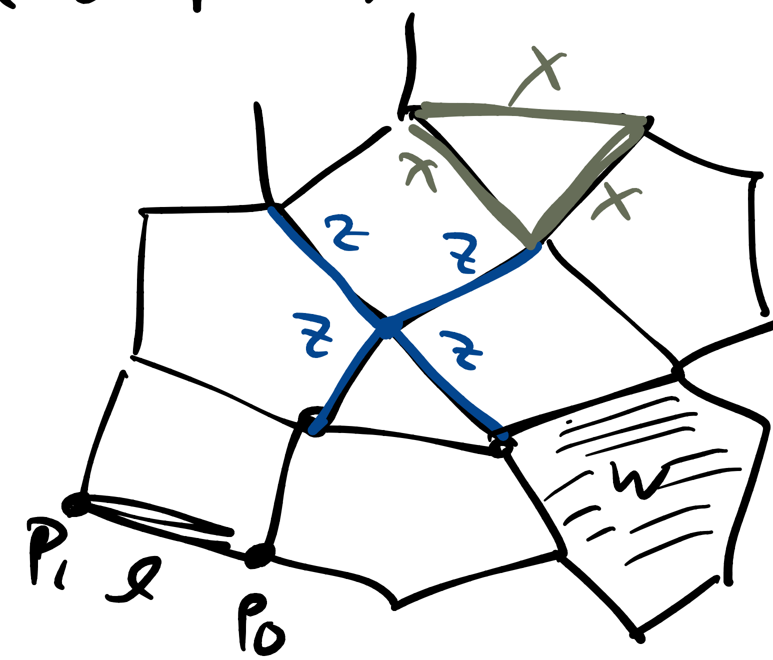
$$\left(\begin{gathered} (-1)^n \phi = -\phi (-1)^n \\ n = 0,1 = \#\text{ of particles} \end{gathered} \right)$$

$\left(G_j \overset{!}{=} 1 \text{ is like } \nabla \cdot E \overset{!}{=} 4\pi\rho \right)$

Set $\phi = 1 \longrightarrow$ T.C.

"unitary gauge".

General graphs (cell complex)



$$B_p = \prod_{\ell \in \partial p} X_\ell$$

$$A_j = \prod_{\ell \in v(j)} Z_j$$

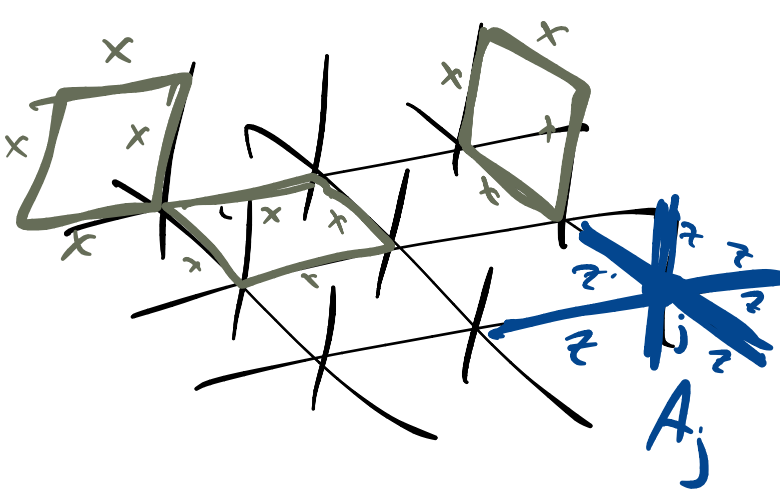
$$[A, B] = 0.$$

$$H = -\sum_j A_j - \sum_p B_p.$$

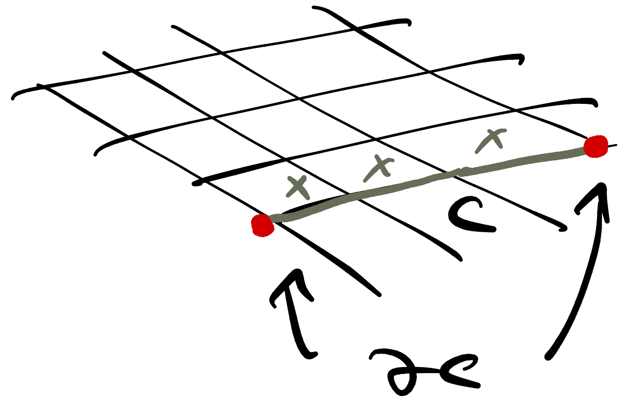
extra data: $\{\text{faces}\}$ w notion of boundary ∂p = collection of links.
$\quad\quad\quad\quad\quad\quad p \quad \rightarrow B_p$

$\{\text{links}\}$ w $\partial \ell = \{p_0, p_1\}$
$\quad\quad \ell \quad\quad \rightarrow A_j$

eg: 3d cubic lattice (toric code on)



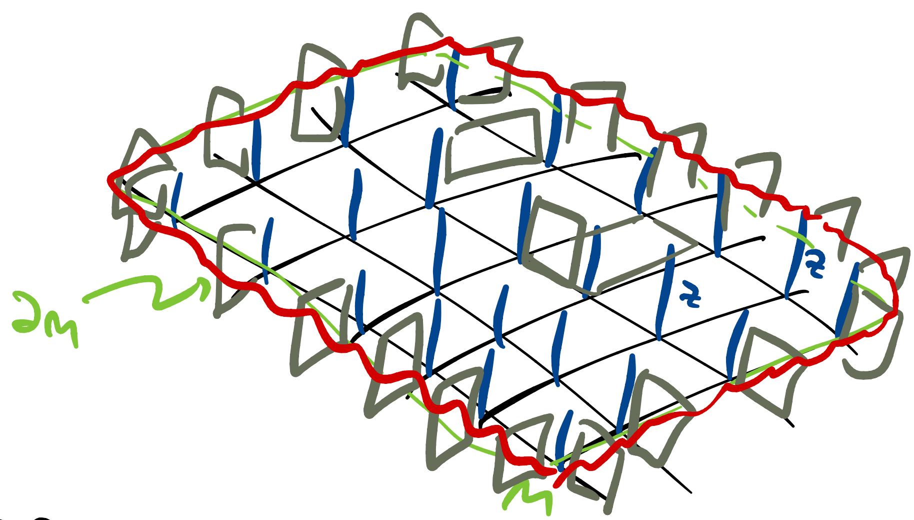
excitations of 3d toric code:



- violations of A_j are e-particles created by

$$W_c = \prod_{\ell \in c} X_\ell$$

at $\underline{\partial c}$.



- violations of B_p.

$$V_M = \prod_{\ell \perp M} Z_\ell$$

if M is closed $\partial M = 0$

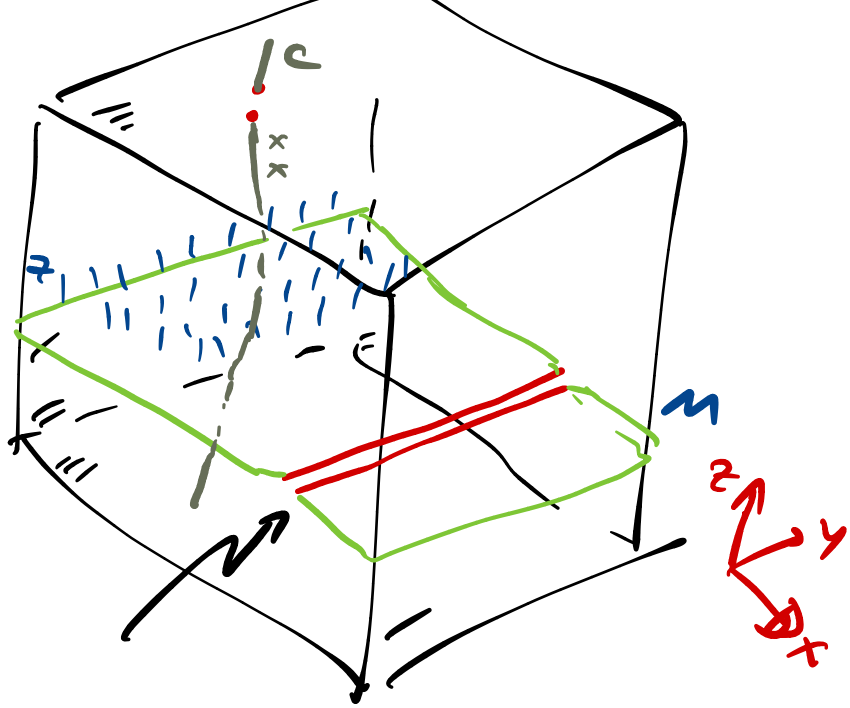
then $[V_M, H] = 0$.

V_M creates a <u>string excitation</u> on ∂M.

<u>The boundary</u> of <u>a boundary is empty</u>

$\underline{\partial^2 M = 0}$. $\Rightarrow$ string is <u>closed</u>.

$T^3 = S^1 \times S^1 \times S^1$



$V_M W_C = - W_C V_M$

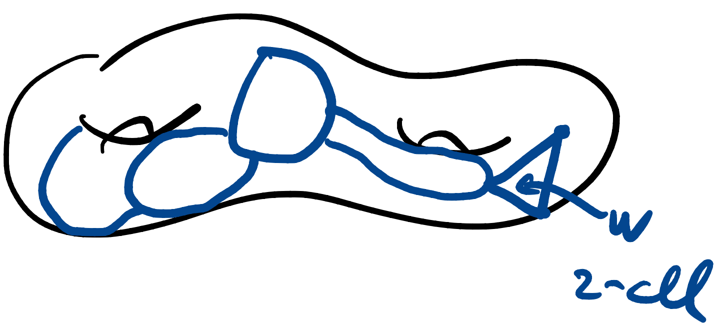
e & m string particle braid nontrivially.

$\Longleftrightarrow$ nontrivial GSD on T^3.

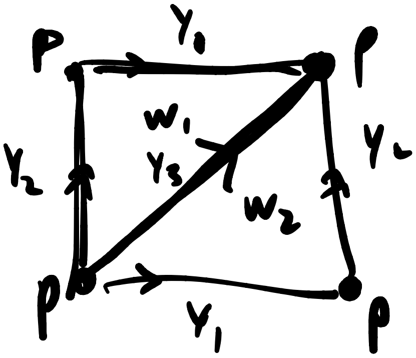
1.1 Cell complexes & homology

Take a d-manifold X & drop it into **cells**.

each is $\cong$ a ball.



choose an abelian group A.
eg $\mathbb{Z}_2 = \langle 0, 1 \rangle$. $1+1=0$.



2 cells w_1, w_2

1 cells $Y_1\ Y_2\ Y_3$

0 cells p

$$\begin{cases} \partial W_1 = Y_3 - Y_1 - Y_2 \\ \partial W_2 = Y_1 + Y_2 - Y_3 \end{cases}$$

$$\begin{cases} \partial Y_1 = p - p = 0 = \partial Y_2 = \partial Y_3 \\ \partial p = 0. \end{cases}$$

$$\Omega_k \equiv \Omega_k(\underset{\substack{\uparrow \\ \text{triangulation} \\ \text{of } X}}{\Delta}, A) \equiv \mathrm{span}_A \{ \sigma, \ \sigma \in \Delta_k \}$$

$\Delta_k \equiv \{k\text{-cells in } \Delta\}$

$C \in \Omega_k \equiv$ formal linear combination of k-cells

$\equiv$ k-chain.

$C + C' \in \Omega_k$

<u>ORientation</u>: $\underline{\sigma_{\bar{\ell}} = -\sigma_\ell}$. $\quad \bar{\ell} \equiv$ reverse orientation

<u>Bdy map</u>: $\partial_k : \Omega_k \to \Omega_{k-1}$

<u>linear</u> $\quad \sigma_k \mapsto (\partial \sigma_k)$

This data is called a <u>cell complex</u>.

<u>The bdy of a bdy is empty</u> $\quad \partial_{k-1} \circ \partial_k \equiv \partial^2 = 0.$

$\Rightarrow$ A cell complex is a <u>chain complex</u>

A chain $(\partial c = 0)$ is called a cycle
closed

i.e. $c \in \ker \partial$.

$$\partial^2 = 0 \implies \ker \partial_k \supset \operatorname{Im} \partial_{k+1}$$

$$\cdots \longrightarrow \Omega_{k+1} \xrightarrow{\partial_{k+1}} \Omega_k \xrightarrow{\partial_k} \Omega_{k-1} \longrightarrow \cdots$$

$$c_{k+1} \xmapsto{\partial_{k+1}} \partial_{k+1}(c_{k+1}) \xmapsto{\partial_{k+1}} 0$$

Homology of this chain complex

$$\equiv H_k(\Delta, A) = \frac{\ker(\partial_k : \Omega_k \to \Omega_{k-1})}{\operatorname{Im}(\partial_{k+1} : \Omega_{k+1} \to \Omega_k)}$$

Thm: $\dim H_k$ are top. inv'ts of X.

Compare to toric code: $\Omega_1(\Delta, \mathbb{Z}_2)$ label a basis of $\mathcal{H}_{TC}$ on Δ

1-chain $\sigma \longrightarrow \left| \begin{array}{l}\text{draw a blue} \\ \text{line on 1-} \\ \text{cells in } \sigma\end{array} \right\rangle \in \mathcal{H}_{TC}$.

$\ker \partial_1 \subset \Omega_1$ $\longleftrightarrow$ closed string states satisfying $A_j = 1$

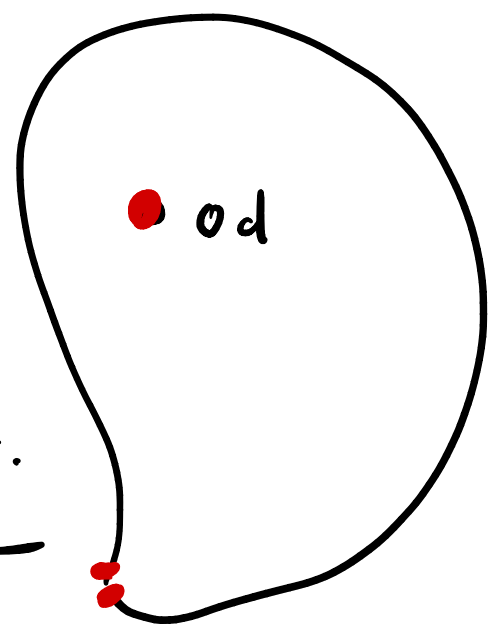
$\mathrm{Im}\, \partial_2 : \Omega_2 \to \Omega_1$ $\longleftrightarrow$ states makeable by the action of $\{B_p\}$.

H_1 $\longleftrightarrow$ groundstates.

so far $A = \mathbb{Z}_2$ and $p = 1$

<u>next:</u> general abelian A, general p.

<u>In 2d:</u>



1d worldline links w/ a d-2 dim'l object.

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p-dim'l worldvolume links w/ a d-1-p dim'l object

<u>In 3d:</u>

eg: in $d=4$ string braids w/ string.