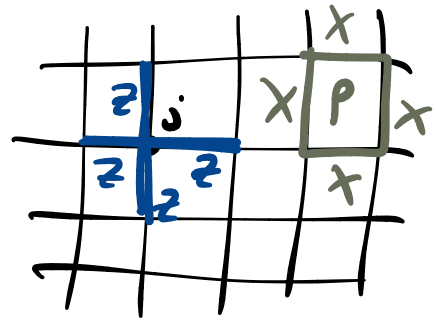
§1 Toric Code & Homology



$$\mathcal{H}_{TC} = \bigotimes_{\text{links}} \mathcal{H}_2$$

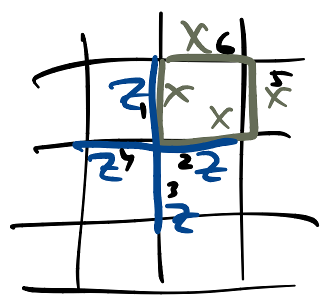
$$X_\ell \equiv (\sigma^x)_\ell \qquad Z_\ell = (\sigma^z)_\ell$$

$$\begin{cases} X\,|0\rangle = |1\rangle \\ X\,|1\rangle = |0\rangle \end{cases} \qquad \begin{cases} Z\,|0\rangle = |0\rangle \\ Z\,|1\rangle = -|1\rangle . \end{cases}$$

each site $j \longrightarrow A_j = \prod_{\ell \in v(j)} Z_\ell$

$v(j) \equiv \{\text{links } \ell \text{ w/ } \partial\ell \ni j\}$

each plaquette $p \rightarrow B_p = \prod_{\ell \in \partial p} X_\ell$



$$H_{TC} = -\sum_j A_j - \sum_p B_p$$

$$A^2 = B^2 = \mathbb{1} . \qquad [A_j, A_{j'}] = 0 \qquad [B_p, B_{p'}] = 0.$$

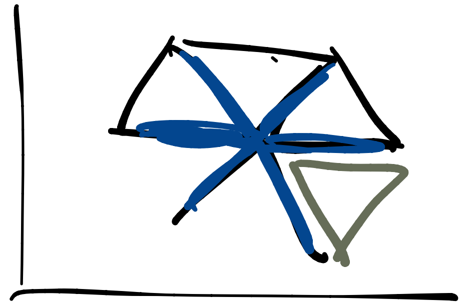
$$AB = Z_3 Z_4 Z_1 \underline{\underline{Z_2}} \; X_1 \underline{X_2} X_5 X_6 = BA . \Rightarrow [A, B] = 0 .$$

$$ZX = -XZ \qquad X_1 X_2 Z_1 Z_2 = Z_1 Z_2 X_1 X_j$$

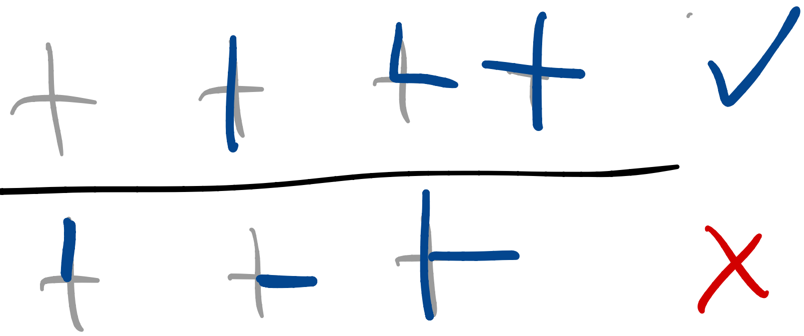
Z basis:

$|\underset{e}{\cdots}\rangle = |Z_e = 1\rangle$

$|\underset{e}{—}\rangle = |Z_e = -1\rangle$



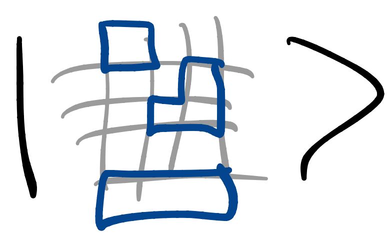
which states satisfy $A_j = 1$? closed strings.



✓ $A_j = +1$

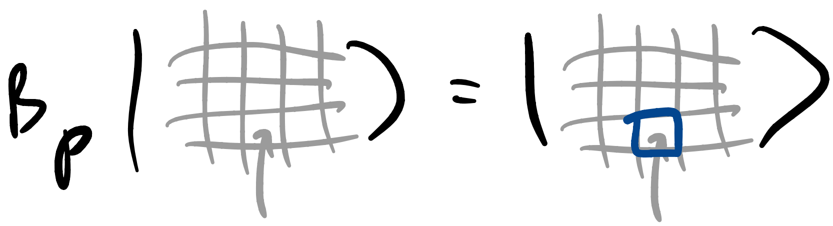
✗ $A_j = -1$

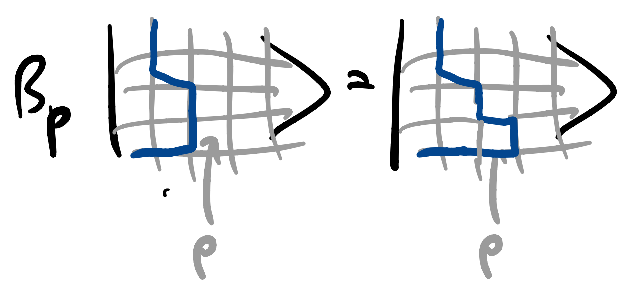
states satisfying $A_j = 1$ $= \sum_{\text{closed loops } C} \Psi(C)\,|C\rangle$ =

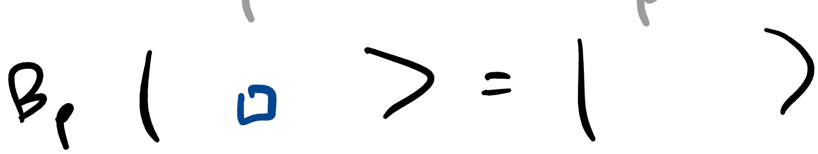


example of C

$[A_j, B_p] = 0 \Rightarrow B_p\,|\text{closed string}\rangle = |\text{closed string}'\rangle$







$$\boxed{B_p |c\rangle = |c + \partial p\rangle}$$
$\uparrow$ mod 2.

which $\Psi(c)$ in $\sum_c \Psi(c)|c\rangle = |\Psi\rangle$

satisfy $B_p|\Psi\rangle = |\Psi\rangle$. ?

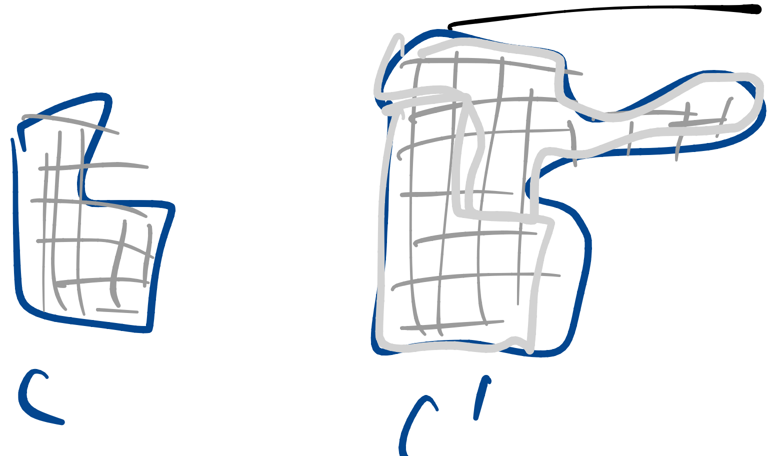
$$\Psi(c) = \Psi(c+\partial p) \quad \underline{\forall p}.$$

$$= \Psi(c')$$

where c & c' are related

by adding or removing

<u>contractable</u> curves. $\equiv$ the boundary of a collection of plaquettes.

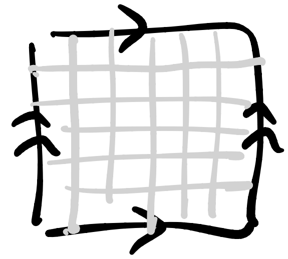
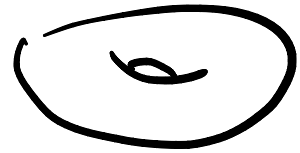


<u>If</u> the space were <u>simply connected</u> = all closed curves are contractable

$= |\#\rangle$

$$\Rightarrow \exists! \; |gs_0\rangle = \sum_c |c\rangle \propto \prod_p \tfrac{1}{2}(1+B_p) \bigotimes_x |0\rangle_x = P|\#\rangle$$

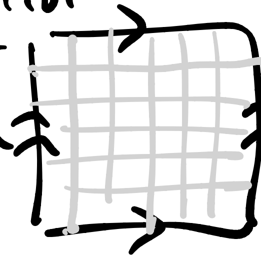
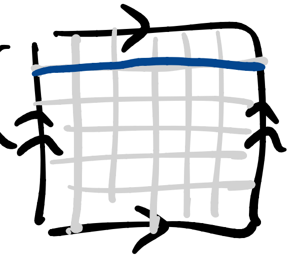
Topological order :

eg  T^2 = 

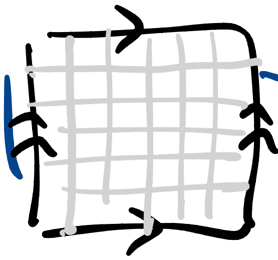
the condition

$B_p|\Psi\rangle = |\Psi\rangle$

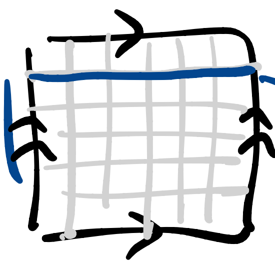
does not violate

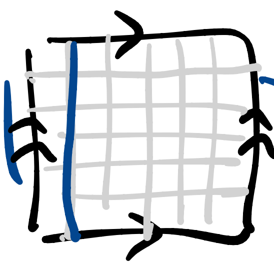
$\Psi($  $)$ to $\Psi($  $)$

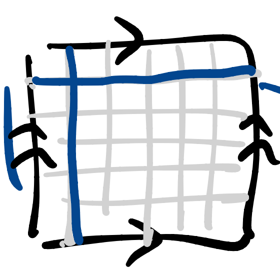
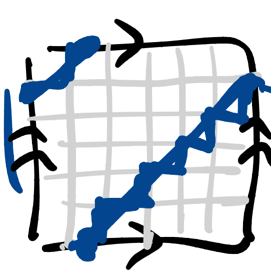
$\subset$ 

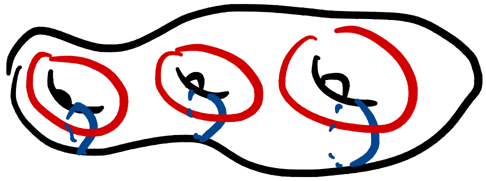
$|gs_{00}\rangle = P(|$  $\rangle)$

$$P = \prod_p \left(\frac{1+B_p}{2}\right)$$

$|gs_{10}\rangle = P(|$  $\rangle)$

$|gs_{01}\rangle = P(|$  $\rangle)$

$|gs_{11}\rangle = P(|$  $\rangle) = P(|$  $\rangle)$



genus g.

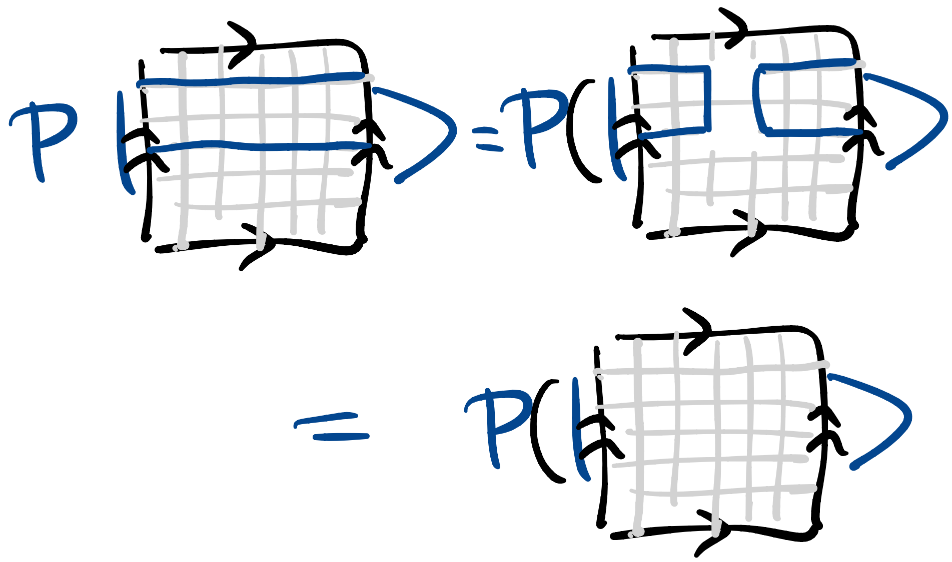
$\rightarrow 2^{2g}$ groundstate.

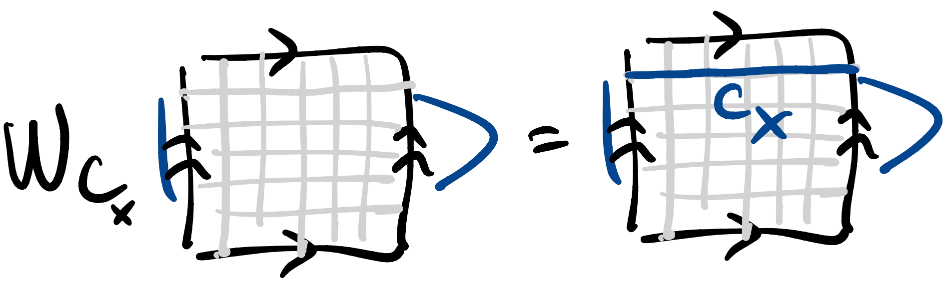
Stability

"Wilson loop"

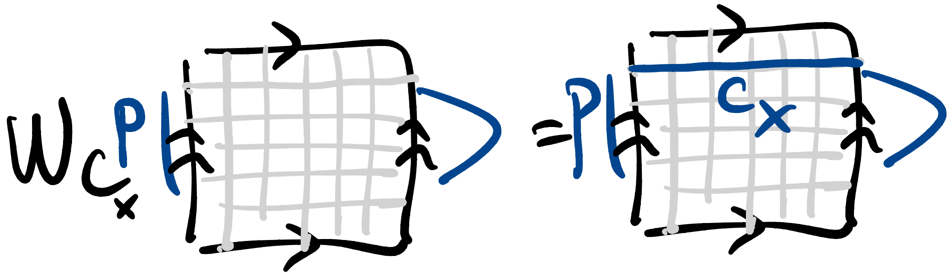
$$W_C = \prod_{l \in C} X_l$$

$$[W_C, B_p] = 0$$





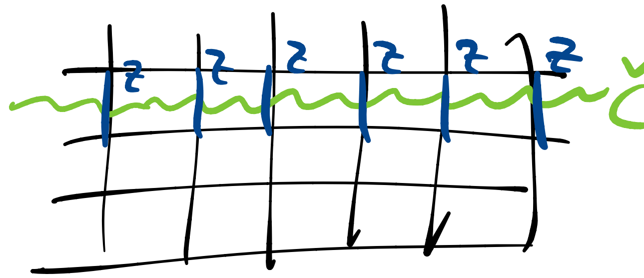
$|gs_{10}\rangle =$



$= W_{C_x} |gs_{00}\rangle$

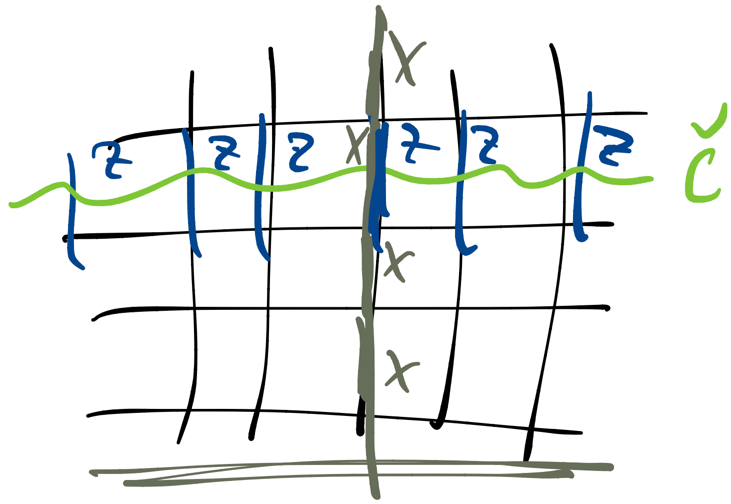
W_C is not a local op!

$$V_{\check{C}} = \prod_{\ell \perp \check{C}} Z_\ell$$

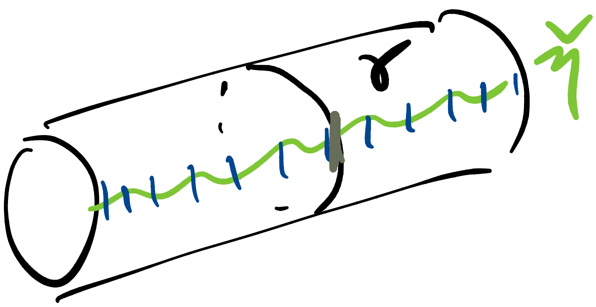


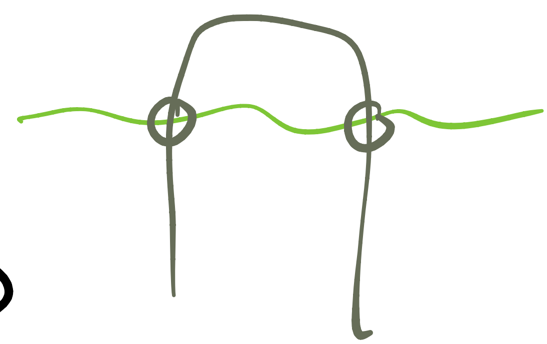
claim: $\begin{cases} [H_{TC}, V_{\check{C}}] = 0 \\ [H_{TC}, W_C] = 0. \end{cases}$

$$W_C V_{\check{C}} = (-1)^{\#(C \cap \check{C})} V_{\check{C}} W_C$$



C





$$|gs_0\rangle = \sum_{\text{contractible curves}} |c\rangle$$

$$W(\gamma)\,|gs_0\rangle = |gs_1\rangle$$

$$V(\check{\eta})\,|gs_0\rangle = |gs_0\rangle.$$

$$V(\check{\eta})\,|gs_1\rangle = \underbrace{V(\check{\eta})\,W(\gamma)}\,|gs_0\rangle = -W(\gamma)\overbrace{V(\check{\eta})\,|gs_0\rangle}^{|gs_0\rangle}$$

$\Rightarrow V(\check{\gamma})|gs_1\rangle = -|gs_1\rangle.$

V, W act like $\underline{\underline{Z \,\&\, X}}$ on a <u>protected</u> qubit. $(ZX = -XZ)$

topological quantum <u>memory</u>.

$H = H_{TC} + \underbrace{\Delta H}_{\text{local operators}}$ can't lift this degeneracy. (for small ΔH)

<u>eg</u>: $H = H_{TC} - g\sum_\ell X_\ell - h\sum_\ell Z_\ell$

$[H, W] \neq 0 \quad [H, V] \neq 0.$

$\overbrace{\langle gs_1| H |gs_0\rangle} = \Gamma \neq 0.$

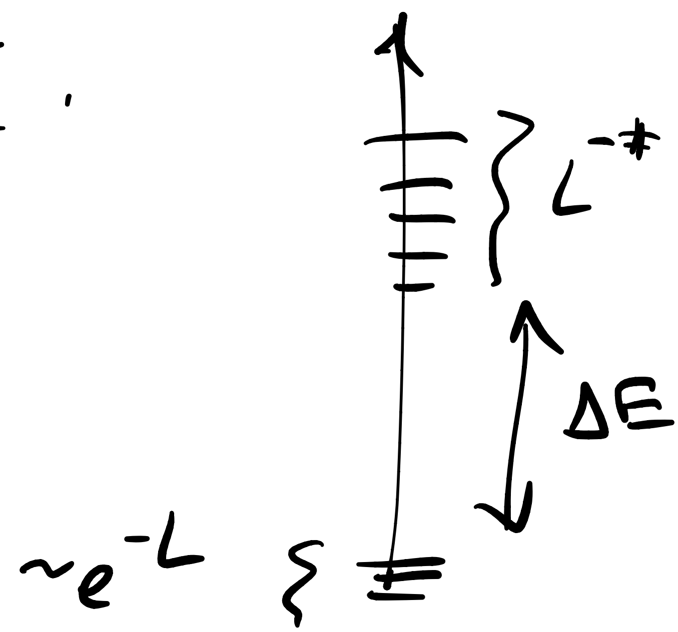
<u>Q</u>: how to get from $|gs_0\rangle$ to $|gs_1\rangle = W(\gamma)|gs_0\rangle$ by powers of ΔH?

$$\Gamma \sim \frac{\langle gs_0 | (-gX_L) \cdots (-gX_3)(-gX_2)(-gX_1) | gs_1 \rangle}{4 \cdots 4}$$

$$\sim \left(-\frac{g}{4}\right)^L = e^{-L \underbrace{|\log g/4|}_{>0 \text{ if } g<4.}}$$

$L = \text{length}(\gamma)$

decays exp'lly w/ L.



Spontaneous breaking of one-form symmetries:

ordinary (0-form) symmetry acts on $\mathcal{H}$ by $U = \prod_{\text{all } x} u_x$ $\quad [H, U] = 0$

(eg: Ising magnet $U = \prod_j X_j$ takes $Z \to -Z$.)

SSB when $U|gs)$ is not $\propto |gs\rangle$.

The only difference here is
V, W have support on curves
$\equiv$ 1-form sym in 2+1 dims.

T.O. $\equiv$ SSB of discrete p-form sym..

<u>string condensation</u>:

$\langle gs | W_C | gs \rangle = 1$ [ordinary sym $\langle Z \rangle \neq 0$ is an order param.

contractible

W_C creates a string on the curve

an object <u>condenses</u> $\equiv$ its creation op has an expectation value.

(away from H_{TC}, $\langle gs | W_C | gs \rangle \neq 0$.)

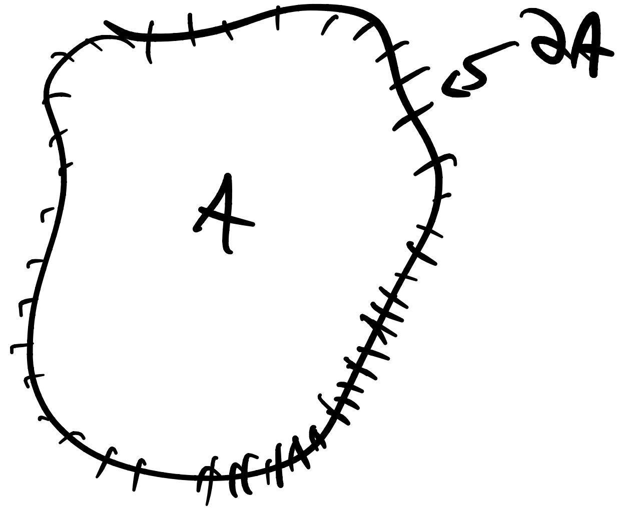
<u>ordinary SSB</u>: $|gs\rangle = \otimes |\uparrow\rangle$ is a product state.

<u>Here</u>: 1-form SSB $\Rightarrow$ Long-range entanglement.

In trivial phase:

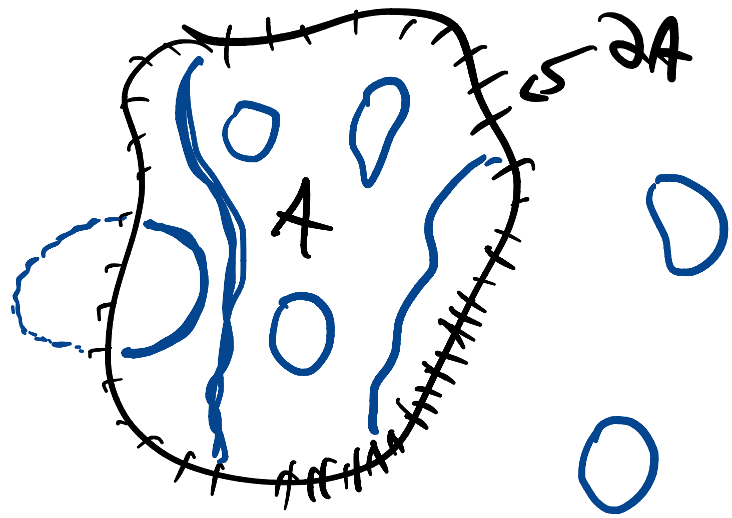
$$S(A) = -\mathrm{tr}\,\rho \ln \rho$$

$$= \frac{\ell(\partial A)}{\epsilon}$$



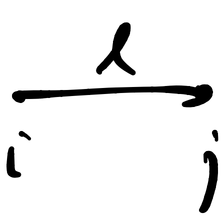
In the toric code phase:

$$S(A) = \frac{\ell(\partial A)}{\epsilon} - \underline{\log 2}$$



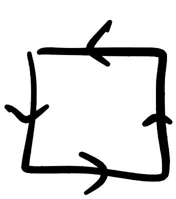
Gauge theory Notation: "$X_{ij} = e^{i\int_i^j \vec{a}\cdot d\vec{s}}$"

$\ell = \langle ij \rangle$



$$X_{ij} = e^{i\pi a_{ij}} \qquad a_{ij} = 0, 1.$$

$$B_{\square} = \prod_{\ell \in \partial\square} X_\ell = \text{"}\, e^{i\oint_{\partial\square} \vec{a}\cdot d\vec{\ell}} \,\text{"}$$



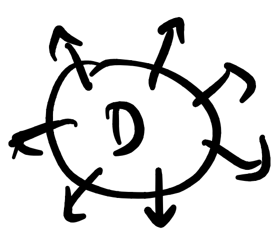
$$= e^{i\pi \sum a} \overset{\text{stokes}}{=} e^{i\pi b_{\square}}$$

Recall: E&M $\quad [\vec{A}_i(x), \vec{E}_j(y)] = i\delta_{ij}\,\delta(x-y)$

$$Z_\ell = e^{i\pi e_\ell} \qquad e_\ell = 0, 1$$

$$XZ = -ZX \Longleftarrow [a, e] = i\delta$$

$$A_+ = \prod_{\ell\in +} Z_\ell = e^{i\pi \underline{\sum_{\ell\in +} e_\ell}} = e^{i\pi(\Delta\cdot e)_+}$$



$$\int_D \vec{\nabla}\cdot\vec{e} = \oint_{\partial D} d\vec{\ell}\times\vec{e}$$

star condition:

$$1 = \prod_{\ell\in +} Z_\ell \longleftrightarrow (\Delta\cdot e)_+ = 0 \mod 2.$$

Gauss' law mod 2

In $\mathbb{Z}_2$ gauge theory. " is a constraint

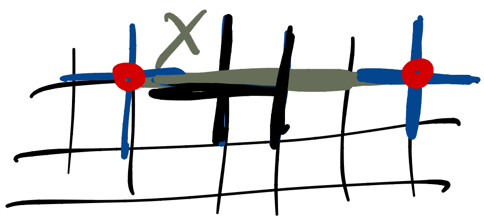
In TC it is imposed energetically.

A site where $(\Delta\cdot e)(i) = 1 \mod 2$ is a $\mathbb{Z}_2$ charge.

Excitations: 2 kinds:

violations of $A_j = 1$

e particle



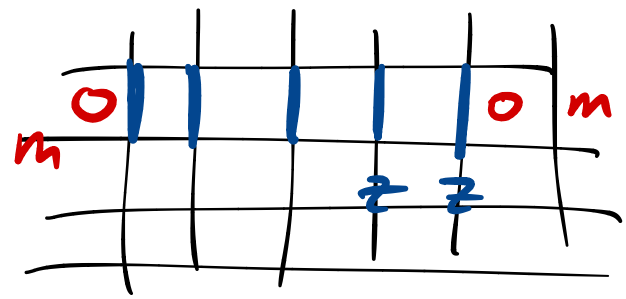
created in **pairs**

by W_c w/ $\partial c \neq 0$.

$$= \prod_{\ell \in c} X_\ell$$

violations of $B_p = 1$

m particle



created in pairs

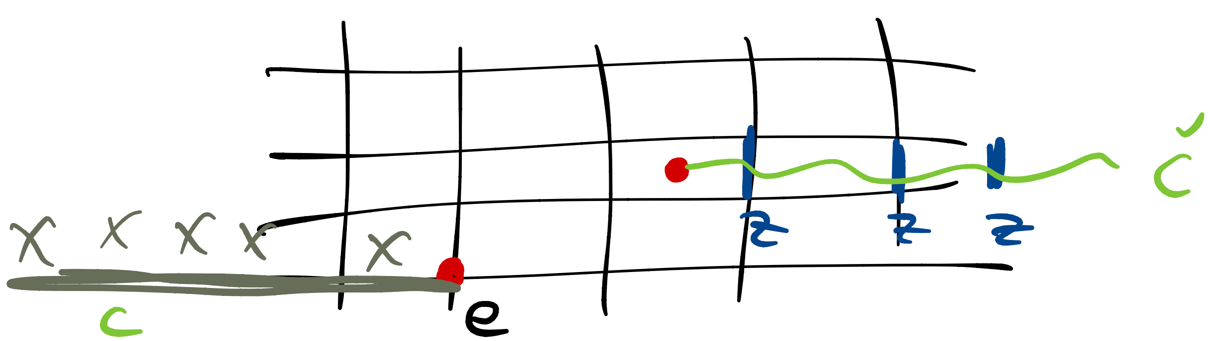
by $V_{\check{C}}$ w/ $\partial \check{C} \neq 0$

$$= \prod_{\ell \perp \check{C}} Z_\ell .$$

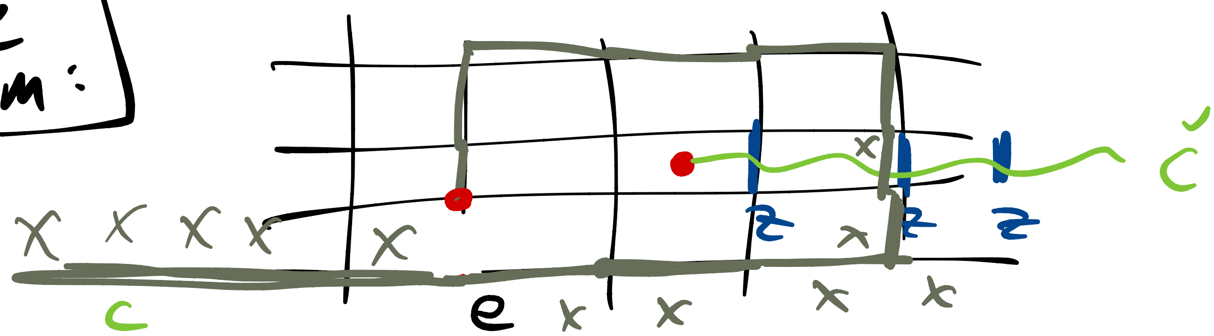
these are bosons.

(albeit their own antiparticles)

But they are **mutual semions** ie $\frac{1}{2}$ **anyons**



Move e around m:

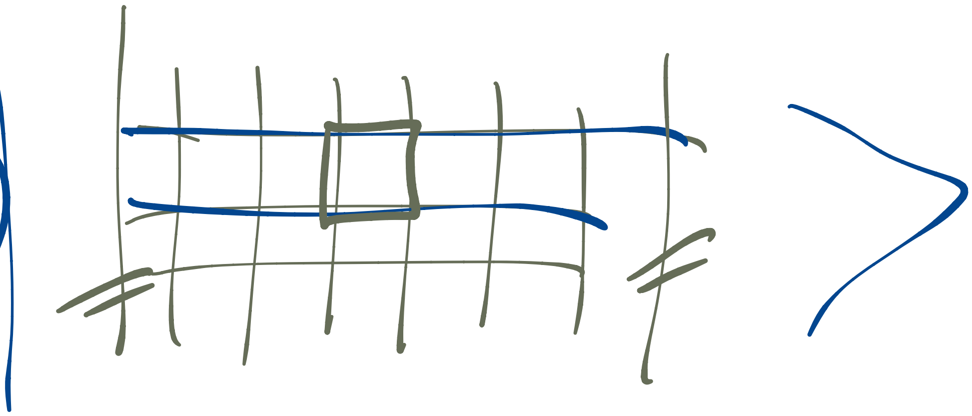


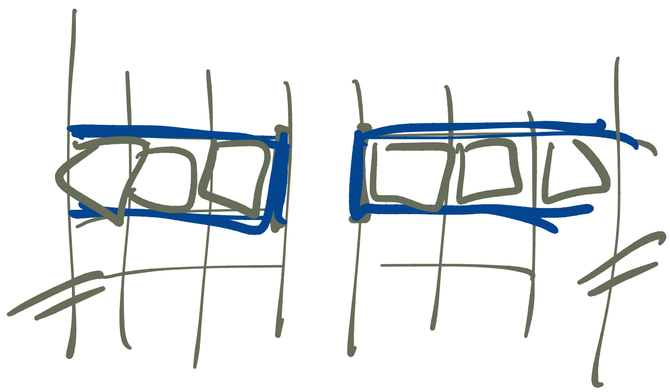
Comes back w/ a (-1).

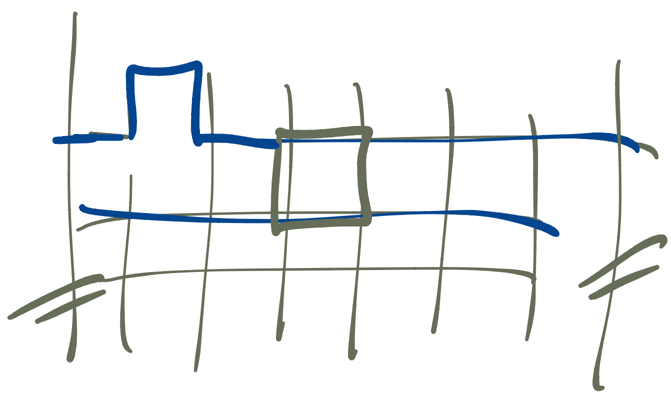
$\Longrightarrow$ a boundstate of e & m (F) is a fermion. i.e.

statistics of top. excitations $\Longleftrightarrow$ G.S. degeneracy.

$$U = e^{i\frac{1}{2}\theta\,\hat{n}\cdot\vec{\sigma}} \qquad \vec{\sigma} = (V, iVW, W)$$

$\prod_p (1+B_p)$ 





case 1 : $\hat{U}(c) = \hat{U}(c + \partial p)$ ← 1-form sym.

(on some subspace)

case 2 : $\hat{U}(c) \neq \hat{U}(c + \partial p)$ ← subsystem sym?

$$\mathcal{U}(c) = e^{i\oint_c A}$$

$$\mathcal{U}(c+\partial p) = e^{i\oint_{c+\partial p} A}$$

$$= e^{i\oint_c A + i\int_p F}$$

on groundstates $\underline{B_p = e^{i\int_p F} = 1}$