Physics 239: Topology from Physics
Winter 2021

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0.1 Introductory remarks

Goals. The primary, overt goal of this class is to use simple physical systems to introduce some important mathematical concepts, mostly in algebraic topology. For starters, this will include homology, cohomology and homotopy groups. When I say physical systems what I really mean is toy models of physical systems.

The secondary, hidden goal is to smuggle in as much physics as possible. This will certainly include physics of topological phases of matter, about which there is a lot to say and a lot which is not yet understood. We will also talk about supersymmetry, a beautiful idea still looking for its rightful place in observable physics, but which has many deep connections to geometry and topology.

So although the primary goal is mathematical, this is not a math course in many ways. One is that I will try to restrict myself to subjects where I think physical insight is helpful (or where I can at least find another good excuse).

A brief overview of topology in many-body physics. There are many different manifestations of topology in physics, even just within condensed matter physics. Probably the manifestation of which the largest number of people are aware these days is band topology, or topological insulators. This is an example where the physics is extremely simple – it involves free fermions, so everything can be solved completely – but the mathematics is fancy (twisted equivariant K-theory). Maybe we'll get there.

I am going to start instead with some situations where the physics is fancy or exotic – in the sense that it requires interactions or hasn't been found in earth-rocks yet – but the mathematics is stuff everyone should know: homology, cohomology, homotopy. I'm pretty excited that all of these things (which are squarely the subject of *e.g.* Hatcher's *Algebraic Topology* book) can be explained quite adequately using only familiar ideas from physics. In particular, all the (forbidding, homological) algebra of algebraic topology will take place in the comfort of a friendly Hilbert space.

Generally covariant theories. There are many ways in which a physical system can be topological. One definition of *topological* is *independent of a choice of metric* (and therefore insensitive to distances between points).

By this definition, historically the first topological theory then is actually general relativity. With general relativity (GR), we have a system defined on some smooth manifold without a choice of metric, because the metric is a fluctuating degree of freedom. In the language of path integrals (perhaps not entirely well-defined for GR), the metric is just an integration variable. (In low-enough dimensions this statement is known to be correct.)

Towards the end of last century a large new collection of topological theories came

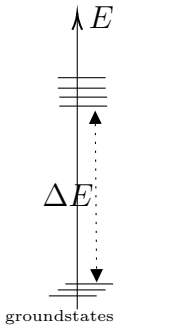
from several different directions. One is the study of solutions of equations which simply don't require a choice of metric. An example is Chern-Simons theory, with action $S[A] = \frac{k}{4\pi}\int_M \mathrm{tr}\left(A\wedge F + \frac{2}{3}A\wedge A\wedge A\right)$. My point in writing the action here is to show that the metric does not appear. I hope I will say more later. Another, harder, example is the self-dual Yang-Mills equations – Donaldson theory of 4-manifolds.

A more sophisticated origin of topological systems is supersymmetric field theories. Witten defined a procedure called *twisting* by which one can construct a set of observables which do not depend on the metric. The fact that Donaldson theory also arises this way allows one to use the Seiberg-Witten solution of 4d $\mathcal{N}=2$ supersymmetric gauge theory to compute Donaldson invariants. I'll talk about first steps in this direction in §2. These topological field theories generally have the shortcoming that they are not unitary. What I mean by this is that they cannot arise as a low-energy description of a condensed matter system.

Gapped phases of matter. This leads to a third origin of topological physics: gapped phases of quantum matter.

First let's define the notion of a gapped quantum phase. A nice context is: consider a Hilbert space $\mathcal{H} = \otimes_x \mathcal{H}_x$ made from finite-dimensional Hilbert spaces distributed over space, and a local Hamiltonian $H = \sum_x H_x$. Local means H_x acts nontrivially only on degrees of freedom near x.

Roughly, a groundstate has a gap if the energy difference ΔE to the first excited state stays finite in the thermodynamic limit ($L\to\infty$, where L is the linear size of the system).



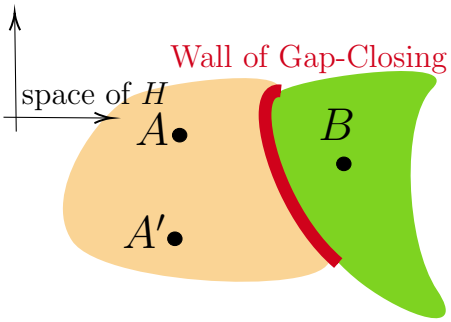
In contrast, a massless field in a box of linear size L has a level spacing of order $1/L^z$, which vanishes in the thermodynamic limit.

More precisely, we will allow some number of states below the gap, with a level spacing that decays faster than any power of L. (It is simplest when the number of such states is finite. But in fact in gapped *fracton* phases the number of these states diverges exponentially with L.) Most interesting will be the situation when any state obtained by acting on a groundstate by (superpositions of) local operators has a finite energy above the groundstates in the thermodynamic limit. If the putative groundstates were related by acting with a local operator, $\langle\psi_1|\,\mathcal{O}_x\,|\psi_2\rangle \neq 0$, we could add that operator to the Hamiltonian $\Delta H = \sum_x c\mathcal{O}_x$ and split the degeneracy by a finite amount, so it would not be a stable situation.

You may think this notion of having a gap is a property of the Hamiltonian and not just of the groundstate, but in fact a groundstate knows whether it is gapped or not. One signature is exponential decay of equal-time correlators of local operators. (I

don't know how to prove this; it is a piece of folklore.)

Different gapped states are in different *phases* if we can't deform the Hamiltonian to get from one to the other without closing the gap. The idea is that their groundstates are related by adiabatic evolution. So it is tempting to say that a gapped phase is an equivalence class of Hamiltonians. In the figure at right, $[A] = [A'] \neq [B]$.



Before you get too excited about the Wall of Gap-Closing: note that the closing of the gap does not by itself mean a quantum critical point: at a first order transition, just the lowest levels cross each other at some random point in parameter space. The two states which become degenerate are related by some horrible global rearrangement and not by acting with local operators, but the situation is unstable. So it's not necessarily true that any gapped state with a finite number of levels below the gap represents a phase of matter. A gapped state which does represent a phase of matter has an energy gap above a *stable groundstate subspace*. By *stable* I mean that there is an open set in the space of Hamiltonians in which the dimension of this subspace doesn't change.

Actually, there is an important extra equivalence relation that we must include: We don't care if on top of some nontrivial phase of matter someone sprinkles a dust of decoupled qubits which are totally inert and do nothing at all. This represents the same phase of matter. Then, further, we are allowed to adiabatically deform the Hamiltonian[1] including these decoupled bits, so that they can interact with the original degrees of freedom. So: in addition to allowing adiabatic variation of couplings, we also allow the addition of decoupled local degrees of freedom[2].

This definition is non-empty. An example of a gapped phase of quantum matter is obtained by putting a qubit on every site of some lattice, and taking $\mathbf{H} = \mathbf{H}_0 = -\sum_s X_s$ where $X_s \equiv \sigma_s^x$ acts only on the qubit at site s as the Pauli x operator. Its groundstate is the product state $\otimes_s |{\rightarrow}\rangle_s$, no matter what lattice we choose. The equivalence class of states obtained by deforming this Hamiltonian, or equivalently, by acting on this completely unentangled state with finite-depth local unitaries, is called the *trivial phase*.

[1] Adiabatically here means without closing the gap, so we can make the change slowly enough not to create excitations in an amount of time independent of the system size. This means that the unitary operator taking one groundstate to the other ($U(T) = \mathcal{T} e^{\mathbf{i} \int_0^T dt H(t)}, H(0) = H_A, H(T) = H_{A'}$) has *finite depth* $\sim T$. Note that different sectors of the groundstate subspace evolve independently under such finite-depth unitaries, and are not mixed by them.

[2] This enlarging of the space of equivalent states can be called *stable equivalence* after the very analogous equivalence relation on vector bundles in K-theory. More on this next quarter.

You might be bothered by the following: it is hard to imagine checking that there is no way around the wall of gaplessness between two potentially-distinct states. Don't we have to consider every possible change to the Hamiltonian, after adding arbitrarily many ancillary bits? The way to avoid spending our lives doing this is to find sharp characterizations of such states, like integer labels, which cannot change smoothly (*e.g.* under adiabatic variation). This is the very definition of topology. An important goal in condensed matter physics is to figure out labels that can be put on states, computable just from the states themselves, that can distinguish them as distinct phases of matter.

There are roughly three classes of such labels that have been found: (1) symmetry-breaking, (2) topological order, (3) edge modes. (1) The first is ancient: the idea is that whether or not a symmetry is broken is a yes or no question with nothing in between. An Ising magnet has two groundstates in the broken phase and only one in the unbroken phase. Actually phases labelled by what symmetries they break can already teach us about topology: their defects are classified by homotopy groups[3]. (3) The third is the subject of topological insulators and symmetry-protected topological (SPT) phases, which I'll put off to last. (2) We'll learn to think of topological order as a generalization of symmetry-breaking, and this is where we'll focus our efforts for a while (§1, 3.9). We can already see an example of a topological label on phases of matter in the definition of gapped state above: the dimension of the stable groundstate subspace is an integer which cannot change smoothly.

Now, to highlight the value of such labels, consider the following. Think of a lattice model of the form described here as arising by discretizing some continuous system, as a short-distance regulator – by chopping a continuous space X up into small disk-shaped regions. How does the metric on space, or more generally the shape of the space, enter into a lattice model? In two ways: the coupling constants multiplying terms in the Hamiltonian, and the arrangement of the degrees of freedom. Now what is the effect of making small, smooth changes in the metric on one of our topological labels? Essentially by definition, such changes fall under the class of adiabatic variations (adding in decoupled bits and varying couplings without closing the gap) that cannot change within the phase. This means that our topological labels (such as the dimension of the stable groundstate subspace) are not only topological labels on phases of matter but also topological invariants of the space X on which we defined our system. If the same phase of matter on two spaces X and Y have distinct labels, then X and Y are topologically distinct.

Since the topologists, attempting to decide which spaces can be reached from each other by continuous deformations, have exactly the same problem as we do (where the

[3] I had planned to cover this, but decided to postpone it until next quarter's class

analog of the Wall of Gap-Closing is the Wall of Tearing), such labels on spaces are valuable. We'll see that many of the labels on spaces we get this way, at least from simple examples of topological phases, have already been discovered by the mathematicians.

A final word about mathematical underpinnings. This being a physics class it is not as essential that we say precisely what we are talking about as it would be in a math class. In §3.1 I plan to have some mathematical self-defense training. Part of the goal is to be a little more precise, but a more important goal is to arm ourselves to be able to get more out of the math literature. Here is a quote from Mermin articulating the principle we will follow:

> *I rely heavily on the reader's firm intuitive grasp of the notion of continuity, and invite readers possessing the appropriate blend of ingenuity and perversity to add whatever assumptions of regularity are needed to exclude whatever pathological counterexamples they may come up with. This is, admittedly, a dangerous game to play, but it has had a long and honorable history of successful practice. In my opinion the substantial gain in clarity it achieves more than compensates for the reduction in certainty. Bridges would not be safer if only people who knew the proper definition of a real number were allowed to design them.*

[End of Lecture 1]

0.2 Conventions

For some of us, eyesight is a valuable commodity. In order not to waste it, I will often denote the Pauli spin operators by

$$\mathbf{X} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathbf{Y} \equiv \begin{pmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix} \quad \mathbf{Z} \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(rather than $\sigma^{x,y,z}$) in the Z basis. I'll write $|0\rangle$, $|1\rangle$ for the Z eigenstates, $Z|0\rangle = |0\rangle$ and $Z|1\rangle = -|1\rangle$ and $|\pm\rangle$ for the states with $X|\pm\rangle = \pm|\pm\rangle$.

$\equiv$ means 'equals by definition'. $A \stackrel{!}{=} B$ means we are demanding that $A = B$. $A \stackrel{?}{=} B$ means A probably doesn't equal B.

The convention that repeated indices are summed is always in effect unless otherwise indicated.

A useful generalization of the shorthand $\hbar \equiv \frac{h}{2\pi}$ is

$$đk \equiv \frac{\mathrm{d}k}{2\pi}.$$

I will also write $\not{\delta}(q) \equiv (2\pi)^d \delta^d(q)$.

I try to be consistent about writing Fourier transforms as

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{ikx} \tilde{f}(k) \equiv \int đ^d k \; e^{ikx} \tilde{f}(k) \equiv f(x).$$

WLOG $\equiv$ without loss of generality.
IFF $\equiv$ if and only if.
RHS $\equiv$ right-hand side. LHS $\equiv$ left-hand side. BHS $\equiv$ both-hand side.
IBP $\equiv$ integration by parts.
$+\mathcal{O}(x^n) \equiv$ plus terms which go like x^n (and higher powers) when x is small.
iid $\equiv$ independent and identically distributed.

We work in units where $\hbar$ and k_B are equal to one unless otherwise noted.

Please tell me if you find typos or errors or violations of the rules above.

0.3 Sources

This list will grow with the notes.

Bott and Tu, *Differential Forms in Algebraic Topology*.

A. Hatcher, *Algebraic Topology*.

G. Bredon, *Topology and Geometry*.

M. Nakahara, *Geometry, Topology and Physics*. I was not a big fan of this book when I was a student because I thought it was superficial. Looking at it again now, I see its virtues more clearly. It has lots of useful things in it and it is mostly written for physicists.

Nash and Sen, *Geometry and Topology for Physicists*.

G. Moore, *Some Remarks on Topological Field Theory*.

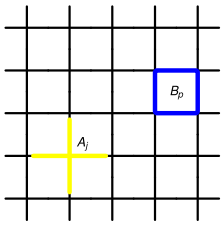
1 The toric code and homology

We begin with an example of a nontrivial gapped phase of quantum matter in the sense defined above. It is an example of a *spin liquid* phase.

An example of a spin system which emerges gauge theory. [Kitaev, quant-ph/9707021, §1,2] The example we'll begin with is a realization of $\mathbb{Z}_2$ lattice gauge theory. Gauge theory has a long history which I won't discuss right now. This model produces the same physics at low energies but avoids many of the confusions by having a Hilbert space which is just a product of local Hilbert spaces, just like an ordinary spin system. It is called the toric code, for no good reason.

To define the Hilbert space, put a qubit on every link of some graph. Let us begin by thinking about the square lattice, but, crucially for our purposes, the model is defined much more generally.

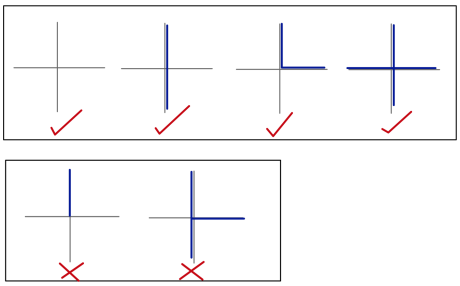
A term in the Hamiltonian is associated with
each site $j \to A_j \equiv \prod_{l \text{ ends on } j} Z_l$
each plaquette $p \to B_p \equiv \prod_{l \in \partial p} X_l$.



$$\mathbf{H} = -\sum_j A_j - \sum_p B_p.$$

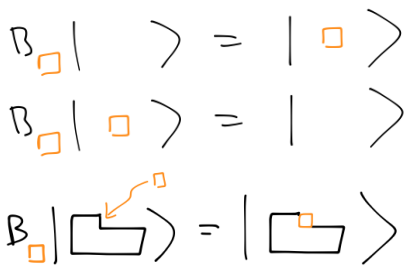
These terms all commute with each other because they all share an even number of Z_ls and X_ls (which anticommmute). That means we can diagonalize the Hamiltonian by minimizing one term at a time.

Which states satisfy the 'star condition' $A_j = 1$? In the Z basis there is an extremely useful visualization: we say a link ℓ of the lattice is covered with a segment of string (an electric flux line) if $Z_\ell = -1$ and is not covered if $Z_\ell = +1$: $\left| \ell \right\rangle \equiv |Z_\ell = -1\rangle$. In the figure at right, we enumerate the possibilities for a 4-valent vertex. We conclude that $A_j = -1$ if a flux line *ends* at the site j.



So the subspace of $\mathcal{H}$ satisfying the star condition at every site is spanned by closed-string states, of the form $\sum_{\{C\}} \Psi(C) |C\rangle$, where C is a collection of closed loops on the lattice, indicating which qubits are in the state $|1\rangle$, while the rest are in the state $|0\rangle$.

Because $[B_p, A_j] = 0$ B_p acts within the subspace of closed-string states. Now we look at the action of B_p on this subspace of states:



Recall that $X|0\rangle = |1\rangle, X|1\rangle = |0\rangle$. If C does not intersect p, it creates a little loop of string on the links in the boundary of p, which we'll denote ∂p. If C includes ∂p it erases it. If C includes part of ∂p, it erases that part and covers the rest. These rules can be summarized as

$$B_p|C\rangle = |C + \partial p\rangle$$

where the addition is understood mod two.

The condition that $B_p|\text{gs}\rangle = |\text{gs}\rangle$ is the equivalence relation defining homology. We will devote a lot of attention to this point, but first we'll talk about the physics. In words, the eigenvalue equation $\mathbf{B}_\square = 1$ says $\Psi(C) = \Psi(C')$ whenever $C' = C + \partial p$ for some plaquette p in the lattice. By repeatedly applying this rule, this means the wavefunction is the same whenever C' and C can be continuously deformed into each other by attaching or removing any collection of plaquettes.

If the lattice were simply connected[4] – if all curves are the boundary of some region contained in the lattice – then this would mean that there is a unique groundstate $\Psi(C) = 1$

$$|\text{gs}_0\rangle = \sum_C |C\rangle = \prod_p \frac{1}{2}(1 + B_p) \otimes_x |0\rangle_x ,$$

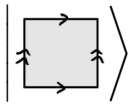
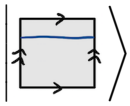
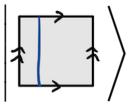
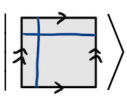
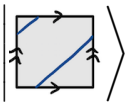
the uniform superposition of all contractable loops. In the second expression we act on a trivial product state (satisfying the closed-string condition) with a projector onto $B_p = 1$ for all p. You can see that its binomial expansion creates the superposition.

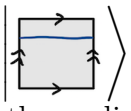
Topological order. In contrast, if the space has non-contractible loops, as for example if we impose periodic boundary conditions on our square lattice, then the local eigenvalue equation $B_p = 1$ does not determine the relative coefficients of loops of different topology! On a space with $2g$ independent non-contractible loops, there are 2^{2g} independent groundstates.

Let me be more explicit about the torus groundstates. The torus is just a square with periodic boundary conditions in both directions, which I denote by  . Let $P \equiv \prod_p \frac{1}{2}(1 + B_p)$ be the projector onto all plaquette operators equal one. Then the

[4] A distracting persnicket that you should ignore: You might complain that the only simply-connected space without boundary in two dimensions is the 2-sphere. But it is not possible to have a perfect square lattice on a sphere. This is forbidden by Euler's theorem which says that if we chop up a surface into simply-connected pieces, the number of vertices V, edges E and faces F satisfy $V - E + F = \chi = 2 - 2g$. But on a perfect square lattice, each face is associated with one vertex and two edges, so $\chi = 0$. One way out is to allow a boundary; that works but requires saying something about boundary conditions, which I postpone until §1.6. Another way out is to allow some of the faces not to be squares, which we will see momentarily is quite innocuous.

groundstate subspace is spanned by

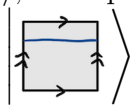
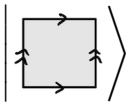
P  $= |\mathrm{gs}_0\rangle$, P  , P  , P  $= P$  .

For example the groundstate P  is a superposition of all closed loops, an odd number of which wind around the x direction (and an even number of which wind around the y direction).

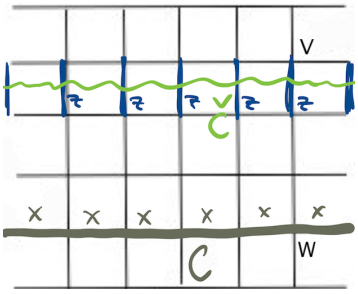
The number of these groundstates is an integer which distinguishes this phase from the trivial phase, which has a unique groundstate (the representative is $\otimes_x |{\rightarrow}\rangle_x$) on any space.

To see that this degeneracy is stable to local perturbations of the Hamiltonian, observe that no local operator mixes these groundstates. Instead, they are connected only by the action of W_C – Wilson loops:

$$W_C \equiv \prod_{\ell \in C} X_\ell,$$

where C is a non-contractable curve – a closed curve which is not the boundary of a collection of plaquettes in the lattice. Acting on the state which is the uniform superposition of contractable loops $|\mathrm{gs}_0\rangle$, this operator creates a non-contractable loop. Note that W_C commutes with P, so P  $= PW_{C_x}$  $= W_{C_x} = P|\mathrm{gs}_0\rangle$.

To see that the resulting state must be independent of $|\mathrm{gs}_0\rangle$ and to understand the multiplicity better, define the conjugate (magnetic) loop operator



$$V_{\check{C}} \equiv \prod_{\ell \perp \check{C}} Z_\ell,$$

where $\check{C}$ is a path which passes in between the lattice points, and $\ell \perp \check{C}$ means ℓ is a link crossed by this path. (Soon we will learn to view $\check{C}$ as a path in the *dual lattice*.)

The operators V and W commute with $\mathbf{H}_{\mathrm{TC}}$ and don't commute with each other – specifically W_C anticommutes with $V_{\check{C}}$ if C and $\check{C}$ intersect an odd number of times

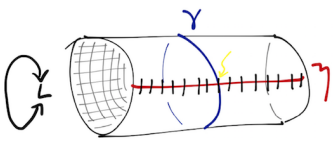
$$V_{\check{C}} W_C = (-1)^{\#C \cap \check{C}} W_C V_{\check{C}}. \tag{1.1}$$

This algebra (sometimes called a Heisenberg algebra) must therefore be represented on the groundstates (actually on every energy level), and it has no one-dimensional

representations. I'll say more about the representations in a simple example in a moment.

The resulting degenerate space of groundstates is protected: the operators which take one of these states to another are not local operators. Rather, the *logical operators* which take one groundstate to another are the loop operators V, W. A hostile environment coupling to the quantum system probably couples to it just by local operators, and will act trivially on the degenerate subspace. This is the idea of topologically protected quantum memory. Each pair of V and W that anticommute act as Pauli X and Z on a protected qubit.

Consider the infinite cylinder. There is a nontrivial class of loops, call a representative γ. Let η be a line running along the long direction of the cylinder, as in the figure. A new groundstate is generated from $|\mathrm{gs}_0\rangle$ by the action of the 'Wilson loop operator'



$$W(\gamma) \equiv \prod_{\ell \in \gamma} X_\ell.$$

in the sense that

$$|\mathrm{gs}_1\rangle = W(\gamma)\,|\mathrm{gs}_0\rangle \ . \tag{1.2}$$

This is also a groundstate (of $\mathbf{H}_{\mathrm{TC}}$) since there is no plaquette that violates $\mathbf{B}_p$ or star that violates $\mathbf{A}_j$ (more simply: $[\mathbf{H}_{\mathrm{TC}}, W(\eta)] = 0$). These two states can be *distinguished* by the action of

$$V(\eta) \equiv \prod_{\ell \text{ crossed by } \eta} Z_\ell$$

in the sense that

$$V(\eta)\,|\mathrm{gs}_\alpha\rangle = (-1)^\alpha\,|\mathrm{gs}_\alpha\rangle\,, \quad \alpha = 0, 1.$$

How did I know this? Since $V^2 = W^2 = 1$, the eigenvalues of V and W are ± 1. And $\{V(\eta), W(\gamma)\} = 0$ because they share a single link (the one pointed to by the yellow arrow in the figure). If we diagonalize V, the two eigenstates are exchanged by the action of W:

$$V\left(W\,|V=-1\rangle\right) \overset{(1.1)}{=} -WV\,|V=-1\rangle = +W\,|V=-1\rangle \quad \text{implies } W\,|V=-1\rangle = |V=+1\rangle\,.$$

Now consider a perturbation of the toric code hamiltonian:

$$\mathbf{H} = \mathbf{H}_{\mathrm{TC}} - g\sum_\ell X_\ell - h\sum_\ell Z_\ell.$$

At finite g, h (and in finite volume), there is tunneling between the topologically degenerate groundstates, since in that case

$$[\mathbf{H}, \prod_{\ell \in \gamma} X_\ell], [\mathbf{H}, \prod_{\ell \in \eta} Z_\ell] \neq 0.$$

This means

$$\langle \mathrm{gs}_0 | \, \mathbf{H} \, | \mathrm{gs}_1 \rangle \equiv \Gamma \neq 0,$$

which will lead to a splitting of the topological degeneracy. However, arriving at a nonzero amplitude Γ in perturbation theory requires the creation of a particle excitation on some site (*i.e.* a site j with $A_j = -1$, which costs energy 2 times the coefficient of the star term which I set to 1; actually it must also have a partner, so it costs energy 4), which then must hop (using the $-gX$ term in $\mathbf{H}$) all the way along the path γ, of length L (and annihilate with its partner), to cancel the action of $W(\gamma)$. The amplitude for this process goes like

$$\Gamma \sim \frac{\langle \mathrm{gs}_0 | \, (-gX_1)\,(-gX_2)\cdots(-gX_L) \, | \mathrm{gs}_1 \rangle}{4 \cdot 4 \cdot \ldots 4} \sim \left(\frac{g}{4}\right)^L = e^{-L|\log g/4|}$$

which is *extremely tiny* in the thermodynamic limit (if $g < 4$). The way to think about this is that the Hamiltonian is itself a local operator, and cannot distinguish the groundstates from each other. It takes a non-perturbative process, exponentially suppressed in system size, to create the splitting.

Spontaneous breaking of one-form symmetries. An interpretation of this phenomenon which may make it seem more familiar is the following. Recall that spontaneous symmetry breaking happens when the groundstate does not respect a symmetry of the Hamiltonian. A symmetry G of the hamiltonian is implemented on the Hilbert space by a collection of unitary operators $\mathbf{U}$ which form a representation of G. The symmetry is broken if $\mathbf{U} \, |\mathrm{gs}\rangle \neq \lambda \, |\mathrm{gs}\rangle$ for some phase λ, meaning that $\mathbf{U} \, |\mathrm{gs}\rangle$ is a linearly independent groundstate.

This looks just like what happened in (1.2). The only difference is the following. Ordinary symmetries, associated with a transformation that acts everywhere in space, are represented by operators of the form $\mathbf{U} = \prod_x u_x$ where u_x has support near the point x, and the product runs over all space. For example, in the Ising ferromagnet, there is a $\mathbb{Z}_2$ symmetry acting by $Z_i \to -Z_i$ for all i, generated by $\mathbf{U} = \prod_i X_i$. In contrast, the unitary symmetry operator $W(\gamma)$ is supported only on the curve γ. Moreover, the action of W on the space of groundstates is not changed if we deform

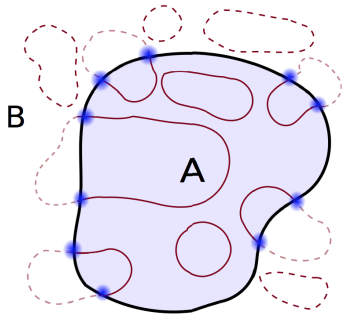
the path η by adding collection of plaquettes. This is called a one-form symmetry[5][6] Topological order is spontaneous breaking of discrete higher-form symmetries.

String condensation. Notice that the phase with topological order involves the condensation of the electric flux strings, in the sense that the operators $\mathbf{B}_\square$ (or more generally W_C for contractable loops) which create these strings have a nonzero ground-state expectation value:

$$\langle \mathrm{gs} | \, \mathbf{B}_\square \, | \mathrm{gs} \rangle \overset{g=\infty}{=} 1.$$

This is the order parameter for the 1-form symmetry-breaking. As with an ordinary condensate of bosons, away from the zero-correlation-length limit ($g = 0$), the condensate will not be exactly 1, since nonzero g suppresses configurations with electric flux. But within the topologically-ordered phase it will be nonzero.

Unlike condensation of particles and breaking of 0-form symmetries, which can be realized by product states, string condensation means *long-range entanglement*: The picture at right shows why – in a state described by fluctuating *closed* strings – there is a contribution to the entanglement entropy of region A which is independent of the size of A: if a string enters A and it is known to be closed, then it must leave again somewhere else; this is one missing bit of freedom, so $S \sim L/\epsilon - \log 2$.



[fig: Tarun Grover]

It is nice to try to incorporate topological order into the Landau paradigm for ordered phases, but here is a small warning. Higher form symmetries do not always behave in the same way as ordinary (0-form) symmetries. In particular, notice that away from the toric code fixed point, the one-form symmetry is not exact, since $[H, W] \neq 0, [H, V] \neq 0$. However, this microscopic violation of the 1-form symmetry heals itself at low energies – it is an emergent symmetry.

Gauge theory notation. Why do I call the operators W Wilson loops? To make it look more like gauge theory familiar from high energy physics, regard the variable

[5] Notice that the operators W_C and $W_{C+\partial p}$ act in the same way on the groundstates. Such string operators are *deformable*. This distinguishes one-form symmetry from what is called *subsystem symmetry*: this is when the unitary operator implementing a symmetry is supported on a subsystem, but the detailed geometry of *which* subsystem makes a difference. This results in a much larger symmetry group and is associated with fracton physics.

[6] In the previous footnote, I said the word 'group'. It's true that the string operators form a group. But note that the composition of two string operators wrapping *e.g.* conjugate cycles of the torus gives an operator which is supported on a region which is not a manifold, since the two strings cross. More generally, one must allow symmetry operators supported on graphs.

X appearing in the plaquette operator as

$$\text{“} \qquad X_{ij} = e^{\mathbf{i}\int_i^j \vec{a}\cdot d\vec{s}} \qquad \text{”}$$

the holonomy of some fictitious continuum gauge field integrated along the link. More precisely, let

$$X_{ij} \equiv e^{\mathbf{i}\pi \mathbf{a}_{ij}}, \quad \mathbf{a}_{ij} = 0, 1.$$

Then the plaquette operator is

$$\mathbf{B}_\square = \prod_{\ell \in \square} X_\ell \text{ “ } = e^{\mathbf{i}\oint_{\partial\square} \vec{a}\cdot d\vec{l}} \text{ ” } = e^{\mathbf{i}\pi \sum\!\!\!\!\!\!\int_{\partial\square} \mathbf{a}} \overset{\text{Stokes}}{=} e^{\mathbf{i}\pi \mathbf{b}_\square},$$

where $\mathbf{b}_\square$ is the (discrete) magnetic flux through the plaquette $\square$. In the penultimate expression, the symbol $\sum\!\!\!\!\!\!\int$ is intended as an analog of $\oint$ to emphasize that we are summing the $\mathbf{a}$s around a closed loop.

In the Hamiltonian description of gauge theory, the field momentum for $\vec{\mathbf{a}}$ is the electric field $\vec{\mathbf{e}}$. So, we call

$$Z_\ell \equiv e^{\mathbf{i}\pi \mathbf{e}_\ell}, \quad \mathbf{e}_\ell = 0, 1.$$

The idea is that $X_\ell Z_\ell = -Z_\ell X_\ell$ then follows from the canonical commutation relation $[\mathbf{a}_\ell, \mathbf{e}_{\ell'}] \propto \mathbf{i}\delta_{\ell\ell'}$ upon demanding that both $\mathbf{a}$ and $\mathbf{b}$ are periodic and discrete variables. The star operator is

$$\mathbf{A}_+ = \prod_{\ell \in +} Z_\ell = e^{\mathbf{i}\pi \sum_{\ell \in +} \mathbf{e}_\ell} \equiv e^{\mathbf{i}\pi (\Delta \cdot \mathbf{e})_+}$$

where Δ is a lattice divergence operator. (Think about applying the divergence theorem $\int_D \vec{\nabla}\cdot\vec{e} = \oint_{\partial D} d\vec{\ell}\times\vec{e}$ on a small disk D around the point.) In this notation, the constraint is

$$1 = \prod_{\ell \in +} Z_\ell \quad \leftrightarrow \quad \Delta \cdot \mathbf{e} = 0 \bmod 2.$$

This is *binary electrodynamics*, electrodynamics mod two. Electric charges are violations of the Gauss' Law constraint: if

$$(\Delta \cdot \mathbf{e})\,(i) = 1 \bmod 2$$

at some site i, we say there is a $\mathbb{Z}_2$ charge at site i.

Excitations. There are two kinds of particle excitations in the 2d toric code: violations of $A_s = 1$ and violations of $B_p = 1$. Notice that the former kinds of defects would be strictly forbidden in 'pure gauge theory' since $A_s = 1$ is the Gauss' law constraint. So pure $\mathbb{Z}_2$ gauge theory is the limit where the coefficient of A_s goes to infinity.

But there is something funny about these excitations: it is not possible to create a single excitation from the groundstate. Instead, the excitations are created by the endpoints of open Wilson lines. Again there are two kinds:

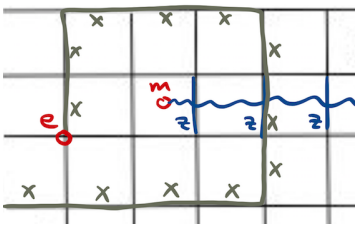
$$W_C = \prod_{\ell \in C} X_\ell, \quad V_{\check{C}} = \prod_{\ell \perp \check{C}} Z_\ell.$$

Here C is a curve in the lattice, and $\check{C}$ is a curve in the dual lattice, but now we allow endpoints. Endpoints of W_C violate A_s and are called e-particles. The $\Delta H = -g \sum_\ell X_\ell$ term is a kinetic term for the e-particles – when $g = 0$ they are localized in the sense that they have a flat dispersion; a small g splits this degeneracy into a band.

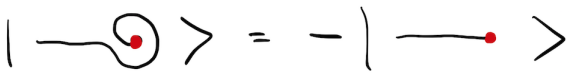
Endpoints of $V_{\check{C}}$ violate B_p and are called m-particles. At a plaquette around which $-1 = B_p = e^{\mathbf{i} \oint_{\partial p} a} = e^{\mathbf{i} \int_p \vec{\nabla} \times a}$ is some localized magnetic flux adding up to π – a lump of π-flux. In the presence of the $\Delta H = -h \sum_\ell Z_\ell$ term, this is a dynamical particle which can hop around.

The e-particles and m-particles are both bosons in the sense that we can make a collection of many e particles and the wavefunction is symmetrized. (Don't be confused by the fact that they are their own antiparticles.)

But the e-particles and m-particles are *mutual semions*. What I mean by this is that if we put an m particle somewhere in the lattice and move an e particle around it, its wavefunction acquires a minus sign. You can see this because the W_C and $V_{\check{C}}$ which create these particles and move one around the other must share a single link and hence will anticommute. From the gauge theory point of view, this is just the Bohm-Aharonov effect of moving an electric charge around a clump of π-flux.



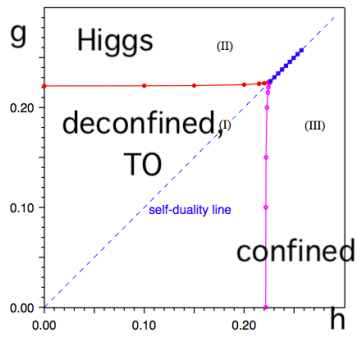
This also means that a boundstate of e and m (which is called ϵ) is a fermion. A fermion is a particle where



[End of Lecture 2]

The mutual statistics of e and m implies the topological groundstate degeneracy. This is because we can regard the V and W operators (whose algebra generates the groundstate subspace) as arising by creating a pair of anyons, moving one member of the pair around a non-contractable loop, and then re-annhilating them.

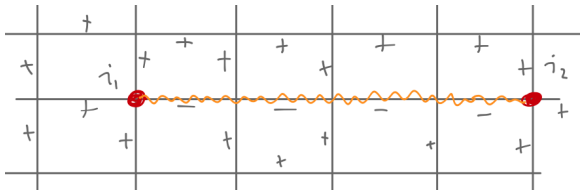
The phase diagram. Perturbations $\Delta H = -\sum_l (gX_l + hZ_l)$ produce a nonzero correlation length. These couplings h and g are respectively an electric string tension, and a hopping amplitude and fugacity for the e particles. Make these too big and the model is confined or higgsed, respectively. These are actually adiabatically connected [Fradkin-Shenker 1979]: Both are connected to the trivial state where e.g. $H = \sum_l X_l$ whose groundstate is a product $\otimes_l |\rightarrow_l\rangle$.



from Tupitsyn-Kitaev-Prokof'ev-Stamp, 2008

Confinement versus Higgsing. Consider for a moment the limit where h is large and the coefficient of the gauss law (star) term is large, both compared to the plaquette term. In that case, we can make the big terms happy just by setting $Z = 1$: no electric flux. Inserting a pair of charges is accomplished by violating the star term at two sites – this forces an odd number of the nearby links to have $Z = -1$. What's the lowest energy state with this property, as a function of the separation between the two charges?

Its energy isthe potential between static charges $V(x)$. To find it, we need to minimize



$$\begin{aligned} \mathbf{H}(h \to \infty) &= -h \sum_l Z_l \\ &= E_0 + 2h\mathsf{L}(\text{string}) \ . \end{aligned}$$

Here $E_0 = -h2N$ is the energy of the state with no electric flux and no external charges, where $2N$ is the number of links. L(string) is the length of the electric flux string: the string can be said to have a nonzero tension (energy per unit length), $2h$. Clearly this minimization is accomplished by a straight line, and the potential between the charges is

$$V(x) = +2hx$$

which is linearly rising with the separation x between the charges, and implies a constant attractive force

$$F = -\partial_x V = 2h.$$

This is confinement.

Taking large h is adding lots of m-particles and letting them hop around. If instead we take large g, this is adding lots of e-particles – charges of the gauge theory – and letting them hop around. Very roughly the transition happens when their dispersion

curve touches the groundstate energy. Then they condense. Condensing the charges leads to the Higgs phase of the gauge theory. In this model the confined phase at $h \to \infty$ and the Higgs phase at $g \to \infty$ are adiabatically connected – they are both the trivial phase, and their representatives $\otimes_\ell |0\rangle_\ell$ and $\otimes_\ell |+\rangle_\ell$ are related by a local unitary rotation.

Stability of topologically ordered phases of matter. I've argued that the special solvable toric code groundstate represents a phase of matter. My argument was that the topological groundstate degeneracy can't be lifted by adding local operators to the hamiltonian because the degenerate groundstates are related only by the action of string operators. This argument can be strengthened by showing what happens to the string operators when perturbing the hamiltonian away from the solvable limit. They are no longer supported on just a single curve of links, but rather develop some thickness of order the correlation length. They can be constructed by a procedure called *quasiadiabatic filtering* [Hastings-Wen 2004].

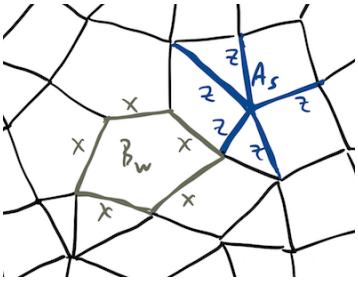
Renormalization group comment. The toric code is a special representative of this phase, where the correlation length is zero (because all the terms in the Hamiltonian commute, nothing moves). In general, any gapped phase should have such a special representative, which is easier to understand: it arises as the limit of any renormalization group (RG) flow starting from a point within the phase. Recall that at a fixed point of the RG, the correlation length must be infinity or zero; for a gapped fixed point, it is zero. The consolation prize is that such a fixed point is attractive from all directions. The fixed point Hamiltonian for the trivial phase is $-\sum_j X_j$, which we reach from the toric code in the limit $h \to \infty$.

A little more about gauge theory. I said earlier that the difference between pure gauge theory and the toric code is just that in gauge theory states the violate the gauss law are strictly forbidden, while in the toric code, they are only discouraged energetically. It is also true that the (perturbed) toric code is identically the same as $\mathbb{Z}_2$ gauge theory with charged massive bosonic matter fields, ϕ_i, living on the sites and creating or annihilating the e particles. In the presence of charged particles, gauss' law is modified to $\nabla \cdot \vec{E} = 4\pi\rho$; the binary, lattice version of this is $A_j(-1)^{n_j^\phi} = 1$ where $n^\phi = 0, 1$ is the number operator for e particles. This model with ϕ_i has a gauge redundancy (generated by $A_j(-1)^{n_j^\phi}$) acting by $\phi_i \to s_i\phi_i, X_{ij} \to s_i X_{ij} s_j$, $s_i = \pm 1$. The perturbation $\Delta H = \sum_\ell g X_\ell$ is just a hopping term for these particles $\phi_i X_{ij} \phi_j$, written in unitary gauge, where we choose s_i to set $\phi_i = 1$.

General graphs. So far we've focussed on the toric code on the 2-dimensional square lattice. Here is a simple and vast generalization. First, from the point of view of the toric code, there is nothing special about the square lattice. The model is perfectly

well-defined on any *cell complex* (and we'll see soon that the space of groundstates and the excitation types are independent of the details of the triangulation). Rather than defining right now what I mean by cell complex (wait for §1.1), let me just say that we can put the toric code on an arbitrary graph with some extra data. As before, put qubits on the links.

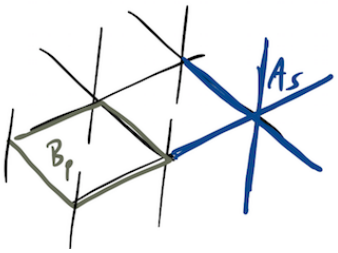
The star operator associated with a site ("0-cell") s is $A_s \equiv \prod_{\ell \in v(s)} Z_\ell$ where $v(s) \equiv \{\ell | s \in \partial \ell\}$, the set of links which end on the site s ('v' is for 'vicinity').



The extra data, required to define the plaquette operators, is some notion of *faces* of the graph: which 2-dimensional regions w with boundary a collection of links in the graph do we include. The plaquette operator associated with a face w is $B_w \equiv \prod_{\ell \in \partial w} X_\ell$. These star and plaquette operators all still commute because they share either zero or two links. Again the Hamiltonian is minus the sum of all star and plaquette operators.

So we can indeed put the system on a contractable space without boundary, such as the sphere. And we would find a unique groundstate.

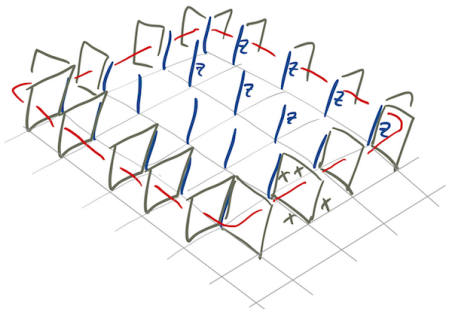
Notice that this definition does not require that the graph is 2-dimensional (though for a planar graph it is obvious what we mean by faces). For example, we can put the toric code on the cubic lattice in 3 dimensions. We choose the faces to be just the square faces of the cubes. In that case, the star and plaquette operators are as in the figure at right.



Let's talk about the excitations of the 3d toric code. A violation of the star operator is still created by an open-string operator $W_C = \prod_{\ell \in C} X_\ell$. The ends of the string still create e-particles.

But now the conjugate object is quite different. It is an operator supported on a *membrane* M, a two-dimensional subset of links:

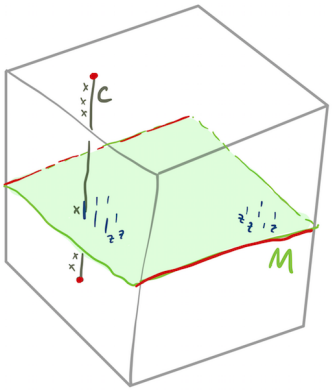
$$V_M \equiv \prod_{\ell \perp M} Z_\ell.$$



At right I've drawn the operator associated with a piece of flat membrane M in the cubic lattice.

As long as M has no boundary, $\partial M = 0$, this operator commutes with $\mathbf{H}_{\rm TC}$. On the boundary, ∂M, however, it fails to commute with the plaquette operators (the ones

indicated in the figure) and creates a 1-dimensional locus of excitations – a magnetic flux loop. Note that it must be a loop since $\partial^2 M = 0$, the boundary of a boundary is empty.



In 3d the analog of braiding of e and m particles is now braiding of the e particle around the m flux loop; again they get a minus sign because the string operator C and membrane operator M which accomplish this operation share a single link, as in the figure at right on the three-torus $T^3 = S^1 \times S^1 \times S^1$ (*i.e.* periodic boundary conditions in all three directions).

1.1 Cell complexes and homology

Let's extract the purely mathematical idea here. Take a d-dimensional manifold X whose topology is of interest and chop it up into simply-connected *cells*. By "simply-connected" here I just mean that each cell can be deformed into a ball. For $d = 2$ *e.g.* this means a triangulation (or squarulation or $\cdots$) into a set of 2-cells which are triangles (or squares...), 1-cells which are intervals, and 0-cells which are points. It is what physicists might call a lattice, though no translation symmetry is actually required or assumed here. But it has more structure – it knows how it is glued together. This gluing data is encoded in a *boundary map* ∂, which we define next. Let Δ_k be the set of k-cells in the triangulation of X, and choose an an abelian group A (*e.g.* $\mathbb{Z}_2$). Define a vector space

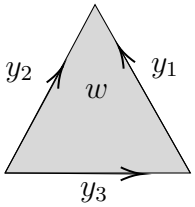
$$\Omega_k \equiv \Omega_k(\Delta, A) \equiv \mathrm{span}_A\{\sigma \in \Delta_k\}$$

to be spanned by vectors associated with k-cells σ, with coefficients in A. (We are writing the group law of A additively, so *e.g.* for $\mathbb{Z}_2$ it is $1+1=0$.) It does no harm to introduce an inner product where these vectors σ are orthonormal. An element $C \in \Omega_k$ is then a formal linear combination of k-cells, and is called a *k-chain* – it's important that we can *add* (and subtract) k-chains, $C + C' \in \Omega_k$. A k-chain with a negative coefficient can be regarded as having the opposite orientation.

The boundary map takes the vector space Ω_k to the corresponding vector space for the (k-1)-cells, Ω_{k-1}:

$$\partial_k : \Omega_k \to \Omega_{k-1}$$

This map ∂ is linear and takes a basis vector associated with a k-cell to the linear combination (with signs for orientation and multiplicity) of cells the union of which lie

in its boundary. (And it takes a basis vector associated to a *collection* of k-cells to the sum of vectors.) For example, in this figure, 

we have $\partial w = y_1 - y_2 + y_3$ (where I denote the vectors associated with the simplices by the names of the simplices, why not?). This construction is called a cell complex[7]. A chain C satisfying $\partial C = 0$ is called a *cycle*, and is said to be *closed*.

The fact that the boundary of a boundary is empty makes this series of vector spaces connected by linear maps into a chain complex, meaning that $\partial^2 = 0$. So the image of $\partial_{p+1} : \Omega_{p+1} \to \Omega_p$ is a subspace of $\ker(\partial_p : \Omega_p \to \Omega_{p-1})$. This allows us to define the *homology* of this chain complex – equivalence classes of p-cycles, modulo boundaries of $p+1$ chains:

$$H_p(\Delta, A) \equiv \frac{\ker(\partial : \Omega_p \to \Delta_{p-1}) \subset \Omega_p}{\mathrm{im}(\partial : \Omega_{p+1} \to \Omega_p)}.$$

These objects depend only on the topology of X and not on how we chopped it up. Below we'll discuss several points of view on this independence of homology on the triangulation.

$H_p(\Delta, A)$ is a vector space over A. In the case when A is a field (such as $\mathbb{Z}_p$ for p prime) the dimensions of these vector spaces over A are called the Betti numbers of X. When A is not a field there can be more information called *torsion*, which we'll discuss.

Note that $H_p(X, A)$ is also itself a group. The group law is just addition of representatives: if C and C' are cycles, then the sum of their equivalence classes modulo boundaries is $[C]+[C'] = [C+C']$. This is independent of the choice of representatives.

So notice that states in $\Omega_1(X, \mathbb{Z}_2)$ label a basis of the Hilbert space of the $\mathbb{Z}_2$ toric code with qubits on the 1-cells of X: an element of Ω_1 is specified by a 0 or 1 for each link of the cell complex. $\ker \partial_1$ is the subset of *closed* loops, and $\partial_2 : \Omega_2 \to \Omega_1$ determines the action of the plaquette operator, so $H_1(X, \mathbb{Z}_2)$ labels a basis of groundstates. The kernel condition comes from the star operator, the image condition comes from the plaquette operator. In the next subsection, we'll define the p-form toric code, with degrees of freedom on the p-cells of a cell complex, whose hamiltonian is determined exactly by the data of ∂_p and ∂_{p+1}, and its groundstate subspace is $H_p(X)$. So the whole homology of X is equivalent to a collection of p-form toric codes for every $p = 0...d$.

[7] There are many very closely related constructions (such as simplicial complex or Δ-complex or semi-simplicial complex or CW-complex) but I will not distinguish between them. One distinction is that we don't care that all the cells are triangles or their higher-dimensional generalizations.

[End of Lecture 3]

1.2 p-form $\mathbb{Z}_N$ toric code

Consider putting a spin variable on the p-cells of $\mathbf{\Delta}$. More generally, let's put an N-dimensional hilbert space $\mathcal{H}_N \equiv \text{span}\{|n\rangle, n = 0..N-1\}$ on each p-cell (the argument of the ket is understood mod N), on which act the operators

$$\mathbf{Z} \equiv \sum_{n=0}^{N-1} |n\rangle\langle n|\omega^n = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & \omega & 0 & \dots \\ 0 & 0 & \omega^2 & \dots \\ 0 & 0 & 0 & \ddots \end{pmatrix}, \qquad \mathbf{X} \equiv \sum_{n=0}^{N-1} |n\rangle\langle n+1| = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots \\ 1 & 0 & 0 & \dots \end{pmatrix}$$

where $\omega \equiv e^{2\pi \mathbf{i}/N}$ is an Nth root of unity, $\omega^N = 1$. These satisfy the Heisenberg or clock-and-shift algebra: $\mathbf{XZ} = \omega \mathbf{ZX}$. For $N = 2$ $\omega = -1$ and these are Pauli matrices. The new ingredient for $N > 2$ is that this requires us to choose an orientation of each p-cell. We define $\mathbf{Z}_{-\sigma} = \mathbf{Z}_\sigma^{-1}$. The choice of orientation for each p-cell is completely arbitrary, but making a different choice would change the form of the boundary map and hence of the Hamiltonian[8].

Because of the orientation, we need to define more carefully the 'vicinity' map v which goes in the opposite direction from ∂ (but is not the inverse):

$$\begin{aligned} v : \Omega_p &\to \Omega_{p+1}, \\ \sigma &\mapsto v(\sigma) \equiv \{\mu \in \Delta_{p+1} | \partial\mu = +\sigma + \text{anything}\} \end{aligned} \tag{1.3}$$

– it picks out the $p+1$-cells in whose boundary the p-simplex appears with a $+1$[9]. To be more precise, the property we really need for v is that it is the *adjoint* of ∂ with respect to the innocent little inner product we introduced on the complex Ω:

$$\langle \sigma, v\mu \rangle \equiv \langle \partial\sigma, \mu \rangle . \tag{1.4}$$

[8] Note that we can do this even if the manifold we are triangulating is not orientable, like the Klein bottle or $\mathbb{RP}^n$ for n even. Non-orientability is a global problem about parallel transporting a choice of orientation around cycles of the manifold. Here we are just choosing an orientation locally, just a reference orientation of each p-cell.

[9] For this to work out, it will be useful to assume that the coefficients in the boundary map are only $\pm 1, 0$. I think the general definition is

$$v(\sigma_p) = \sum_{\mu \in \Delta_{p+1}, \partial\mu = k\sigma_p + \cdots} k\mu.$$

Consider the Hamiltonian

$$\mathbf{H} = -J_{p-1} \sum_{s \in \Delta_{p-1}} A_s - J_{p+1} \sum_{\mu \in \Delta_{p+1}} B_\mu - \Gamma_p \sum_{\sigma \in \Delta_p} \mathbf{Z}_\sigma + h.c.$$

with

$$A_s \equiv \prod_{\sigma \in v(s) \subset \Delta_p} \mathbf{Z}_\sigma, \qquad B_\mu \equiv \prod_{\sigma \in \partial \mu} \mathbf{X}_\sigma \ .$$

(The "+h.c." means "plus hermitian conjugate" and makes the hamiltonian hermitian. Unlike the case $N = 2$, the star and plaquette operators are not hermitian for $N > 2$.) I claim that

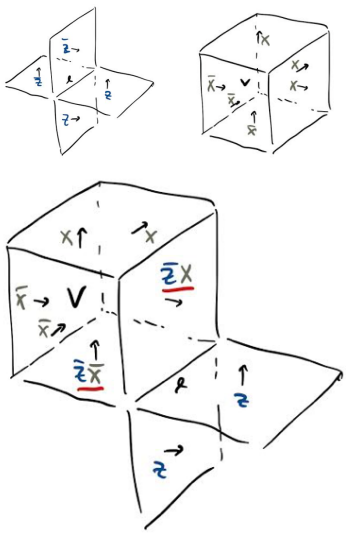
$$0 = [A_s, A_{s'}] = [B_\mu, B_{\mu'}] = [A_s, B_\mu], \quad \forall s, s' \in \Delta_{p-1}, \mu, \mu' \in \Delta_{p+1}$$

so that for $\Gamma_p = 0$ this is exactly solvable. Here's why: the nontrivial one is

$$B_\mu A_s = \prod_{\sigma' \in \partial \mu} X_\mu \prod_{\sigma \in v(s)} Z_\sigma = A_s B_\mu \prod_{\sigma \in v(s)} \prod_{\sigma' \in \partial \mu} \omega^{\langle \sigma, \sigma' \rangle} = A_s B_\mu \omega^{\langle v(s), \partial \sigma' \rangle} \overset{(1.4)}{=} A_s B_\mu \omega^{\langle s, \partial^2 \sigma' \rangle} \overset{\partial^2 = 0}{=} A_s B_\mu.$$

So the terms commute because of the fact that $\partial^2 = 0$, the boundary of a boundary is empty. This is the same reason that the homology of the complex Ω is well-defined. I find this very satisfying.

For example, in the figures at right we show the case of $d = 3$ and $p = 2$ – the 2-form toric code on the cubic lattice. The degrees of freedom live on the plaquettes. We can orient each plaquette so the normal points in the $+\hat{x}, +\hat{y}$ or $+\hat{z}$ direction. A star operator is associated with each link and involves the four adjacent faces, two are Z_σ and two are $Z_\sigma^\dagger$. The analog of the 'plaquette operator' is now a volume operator associate to each cube. It involves the six faces of the cube, three X_σs (for the ones with normals pointing out of the cube) and three $X_\sigma^\dagger$s (for the ones pointing in). Any pair of star and volume operators shares an even number of faces. If they share two faces, one pair has XZ and the other pair has $X^\dagger Z$, so the product commutes.



Here's the solution: Suppose for motivation that $J_{p-1} \gg J_{p+1}, \Gamma_p$ so that we should satisfy $A_s = 1$ first. This equation is like a gauss law, but instead of flux *lines* in the $p = 1$ case, we have flux *sheets* for $p = 2$ or ... whatever they are called for larger p. The condition $A_s = 1$ means that these sheets satisfy a conservation law that the total flux going *into* the $p - 1$ simplex vanishes. So a basis for the subspace of states satisfying this condition is labelled by configurations of closed sheets. For $N = 2$ there

is no orientation, and each p-simplex is either covered ($\mathbf{Z}_\sigma = -1$) or not ($\mathbf{Z}_\sigma = 1$) and the previous statement is literally true. For $N > 2$ we have instead sheet-nets (generalizing string nets), with N kinds of sheets labelled by $k = 0...N-1$ (including the trivial one with $k = 0$) which can split and join as long as they satisfy

$$\sum_{\sigma \in v(s)} k_\sigma = 0 \text{ mod } N, \forall s \in \Delta_{p-1}. \tag{1.5}$$

This is the Gauss law of p-form $\mathbb{Z}_N$ gauge theory.

The analog of the plaquette operator B_μ acts like a kinetic term for these sheets. In particular, consider its action on a basis state for the $A_s = 1$ subspace $|C\rangle$, where C is some collection of (N-colored) closed p-sheets – by an N-colored p-sheet, I just mean that to each p-simplex we associate an integer k_σ (mod N), and this collection of integers satisfies the equation (3.3). The action of the plaquette operator in this basis is

$$B_\mu |C\rangle = |C + \partial\mu\rangle$$

Here $C + \partial\mu$ is another collection of p-sheets differing from C by the addition (mod N) of a sheet on each p-simplex appearing in the boundary of μ. The eigenvalue condition $B_\mu = 1$ then demands that the groundstate wavefunctions $\Psi(C) \equiv \langle C|\text{groundstate}\rangle$ have equal values for chains C and $C' = C + \partial\mu$. But this is just the equivalence relation defining the pth homology of $\mathbf{\Delta}$. Distinct, linearly-independent groundstates are labelled by pth-homology classes of $\mathbf{\Delta}$. More precisely, they are labelled by homology with coefficients in $\mathbb{Z}_N$, $H_p(\mathbf{\Delta}, \mathbb{Z}_N)$.

We can reinterpret the toric code above as a p-form $\mathbb{Z}_N$ gauge theory with 'electric' charged matter by associating an $\mathcal{H}_N$ to each $\ell \in \Delta_{p-1}$; I'll call its $\mathbf{Z}$ operator Φ_ℓ. Notice that $\Phi_{-\ell} = \Phi_\ell^\dagger$. We can introduce $\mathbb{Z}_N$ gauge transformations

$$\Phi_\ell \mapsto \omega_\ell \Phi_\ell, \quad \mathbf{Z}_\sigma \mapsto \prod_{\ell \in \partial\sigma} \omega^\ell \mathbf{Z}_\sigma$$

(notice that the latter generalizes the transformation of a link variable, in which case the boundary of the link is the difference of the two sites at its ends), in which case the coupling

$$\mathbf{H}_\text{e} = \sum_{\sigma \in \Delta_p} \prod_{\ell \in \partial\sigma} \Phi_\ell \mathbf{Z}_\sigma + h.c.$$

is gauge invariant. Now notice that we may choose *unitary gauge* where we completely fix the gauge redundancy by setting $\Phi_\ell = 1$. This produces the p-form toric code above. (For the case $p = 1, N = 2$, this is explained in Fradkin's book, 2d edition.)

It is interesting to consider other possibilities the collection of simplices on which the matter resides. For example, put a spin on every simplex. With the appropri-

ate hamiltonian, this should compute the whole homology complex $H_\bullet(\mathbf{\Delta}, \mathbb{Z}_N) = \oplus_{p=0}^{d} H_p(\mathbf{\Delta}, \mathbb{Z}_N)$.

Ferromagnets and H_0. An important special case is when the degrees of freedom live on the 0-cells. The resulting model is a model of a ferromagnet: since there are no (-1)-cells, there is no star term. The analog of the plaquette terms are just

$$-\sum_{\ell \in \Delta_1} \prod_{s \in \partial \ell} Z_s = -\sum_{\langle ij \rangle} Z_i Z_j^\dagger$$

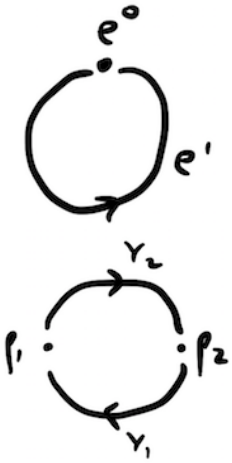
for any pair of neighboring sites $\langle ij \rangle$. This term is minimized when Z_i and Z_j point in the same direction.

What mysterious topological invariant is computed by the groundstates of a ferromagnet? If the space X is *connected* (meaning that there is a path of edges connecting any site to any other), then there are $|A|$ groundstates which tell us nothing about the topology of X. But if X has more than one connected component, then each component may independently choose a direction in A, and there will be $|A|^x$ groundstates where $x = H_0(X)$ is the number of connected components of X. This is the point in life of $H_0(X)$.

1.3 Some examples

The simplest possible example is complex with only a single 0-cell, a point. This has $H_0(\text{pt}, A) = A$, and all other $H_{n>0}$ vanish. If our cell complex were k 0-cells, we would find $H_0(k \text{ pts}, A) = A^k$ in agreement with the discussion above about ferromagnets.

Circle. Consider the cell complex at top right. This is a cellulation of a circle with one 1-cell and one 0-cell. The boundary map is $\partial e_1 = e_0 - e_0 = 0$. The kernel is everyone and the image is no one. So the homology (with integer coefficients) is $H_0(S^1, A) = A = H_1(S^1, A)$. Another cell decomposition of the circle is the bottom figure at right. Now there are two 1-cells and two 0-cells with boundary map $\partial y_1 = p_1 - p_2 = -\partial y_2$. Now the complex looks like



$$0 \to A^2 \xrightarrow{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}} A^2 \to 0.$$

The kernel of ∂ is generated by $y_1 + y_2$. The complement of the image is generated by $p_1 \simeq p_2 \mod \partial$. So we find the same answer for the homology as above, $b_0(S^1) = b_1(S^1) = 1$.

Ball. Consider what happens if we add to the first example a 2-cell e^2 whose boundary

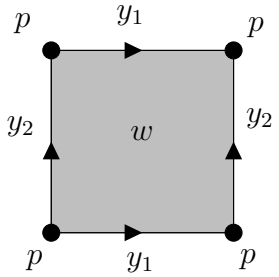
is e^1 – *i.e.* fill in the interior of the circle in the picture. This makes a cellulation of a 2-ball. The complex is

$$0 \to A \xrightarrow{1} A \xrightarrow{0} A \to 0.$$

In that case, $e^1 \in \mathrm{Im}\partial_2$, so it kills the first homology – Ω_1 and Ω_2 eat each other. This complex has the same homology as a point. We'll see later that this is because they are related by homotopy – a family of continuous maps which starts at one and ends at the other.

An important point: a demand we make of our cellulations is that each k-cell is topologically a k-ball.

Torus. Consider the cell complex at right: It has one 2-cell w, two 1-cells y_1, y_2 and two 0-cells p_1, p_2. Opposite sides are identified. This is a minimal cell complex for the 2-torus, $T^2 = S^1 \times S^1$.
The boundary map on 2-cells is $\partial w = y_2 + y_1 - y_2 - y_1 = 0$. One 1-cells it is $\partial y_1 = p - p = 0, \partial y_2 = p - p = 0$.



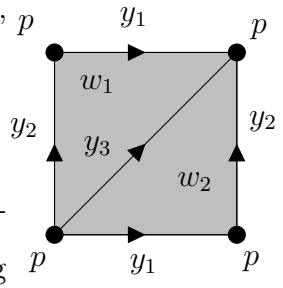
All the boundary maps are zero! The chain complex is

$$0 \to A \xrightarrow{0} A^2 \xrightarrow{0} A \to 0$$

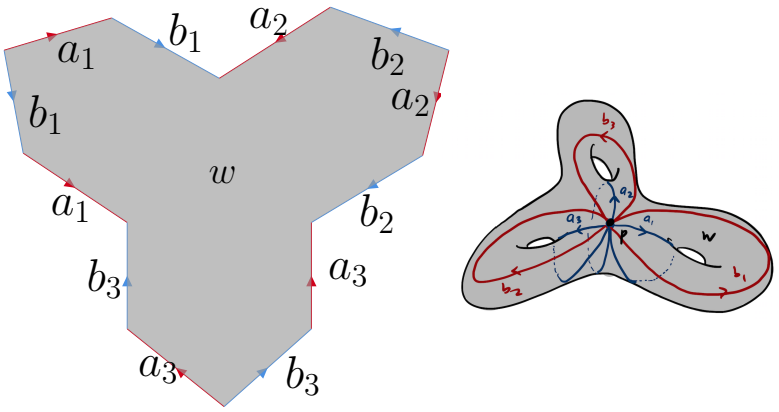
This means that every generator of the cell complex is a generator of homology, and we have $H_0(T^2, A) = A, H_1(T^2, A) = A^2, H_2(T^2, A) = A$ (the betti numbers are $b_0(T^2) = 1, b_1(T^2) = 2, b_2(T^2) = 1$.
We can also choose a less-minimal cellulation, as at right. The boundary maps are $\partial w_1 = y_3 - y_1 - y_2, \partial w_2 = y_1 + y_2 - y_3$, $\partial y_i = 0$. Now the complex is

$$0 \to A^2 \xrightarrow{\partial_2} A^3 \xrightarrow{0} A \to 0$$

with $\partial_2 = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$. Clearly ∂_2 has rank 1, so the extra 2-chain and the extra 1-chain just eat each other leaving behind the same homology as before.



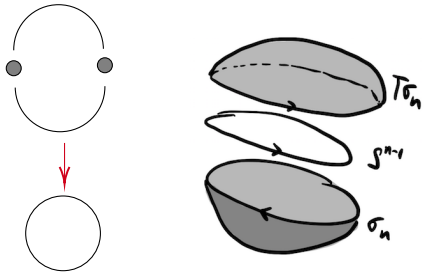
More generally, we can make a cellulation of a genus g Riemann surface Σ_g using a single plaquette, $2g$ 1-cells, and a single 0-cell. (The torus is the case $g = 1$.) At right is a cellulation of a genus 3 Riemann surface. Again the boundary maps are all trivial, and we see that $b_0(\Sigma_g) = 1 = b_2(\Sigma_g), b_1(\Sigma_g) = 2g$. You can see that we're losing some information here by choosing an abelian group.



Spheres. Generalizing in another direction, we can make a sphere $S^n, n \geq 1$ starting with an n-dimensional ball B_n– a single n-cell – and identifying all the points on its boundary[10] to make a single 0-cell: $S^n = B_n/\partial B_n$. The boundary map for this complex is again trivial. So $b_0(S^n) = b_n(S^n) = 1$ and all others are zero.

[End of Lecture 4]

Alternatively, we can make a sphere iteratively. Start with an S^0 (two points ($S^0 = \{x|x^2 = 1\}$) which I'll call σ_0 and $T\sigma_0$, where T stands for anTipodal map), and glue in two 1-cells (intervals, B_1, which I'll call σ_1 and $T\sigma_1$) as in the figure at right, so that $\partial\sigma_1 = \sigma_0 - T\sigma_0$ and $\partial(T\sigma_1) = T\sigma_0 - \sigma_0$. This makes an S^1 as before. Now glue on two 2-cells (disks, B_2, which I'll call σ_2 and $T\sigma_2$) so that $\partial\sigma_2 = \sigma_1 + T\sigma_1 = -\partial(T\sigma_2)$. You see that this can go on forever with an alternation in the sign $\partial\sigma_k = \sigma_{k-1} + (-1)^k T\sigma_{k-1}$ so that $\partial^2 = 0$.



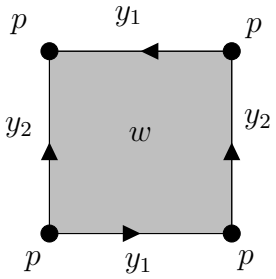
For example, for the 4-sphere we find the complex

$$0 \to A^2 \xrightarrow{\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}} A^2 \xrightarrow{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}} A^2 \xrightarrow{\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}} A^2 \xrightarrow{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}} A^2 \to 0.$$

Now the boundary map in each dimension but the first and last has a 1d kernel and a 1d image, so no homology. So we get the same homology as above.

[10] Note that a single n-cell is not by itself an acceptable complex, since that n-cell has a boundary and the boundary map needs somewhere to go.

An example with torsion. Consider the cell complex at right: It has one 2-cell w, two 1-cells y_1, y_2 and two 0-cells p_1, p_2. Opposite sides are identified, but top and bottom are identified with a twist. This is a minimal cell decomposition for the Klein bottle, an example of an unoriented closed surface.



The boundary map on 2-cells is $\partial_2 w = y_2 + y_1 - y_2 + y_1 = 2y_1$. On 1-cells it is $\partial_1 y_1 = p - p = 0 = \partial_1 y_2$.

Here is the first place where we have to say something about the choice of A. If our coefficient group were $\mathbb{Z}_2$, the map ∂_2 would just be zero, and we would find the same answers for $H_{0,1,2}(\Delta, \mathbb{Z}_2)$ as for the torus. With *e.g.* $\mathbb{Z}_3$ coefficients, however, $2y_1 = y_1$ mod 3, so we find no generator of $H_2(\Delta, \mathbb{Z}_3)$, and only one generator of $H_1(\Delta, \mathbb{Z}_3)$. With integer coefficients, we find

$$H_2(\Delta, \mathbb{Z}) = 0, H_1(\Delta, \mathbb{Z}) = \langle y_1, y_2 | 2y_1 = 0 \rangle = \mathbb{Z}_2 \oplus \mathbb{Z}, H_0(\Delta, \mathbb{Z}) = \mathbb{Z} = \langle p \rangle \, .$$

(Here I am using an additive notation for these abelian groups, since we add the coefficients.) The finite-group summands are called torsion homology.

With $A = \mathbb{Z}_6$ we find

$$H_2(\Delta, \mathbb{Z}_6) = \langle 3w | 6w = 0 \rangle = \mathbb{Z}_2, H_1(\Delta, \mathbb{Z}_6) = \langle y_1, y_2 | 2y_1 = 0 \rangle = \mathbb{Z}_2 \oplus \mathbb{Z}_6, H_0(\Delta, \mathbb{Z}_6) = \mathbb{Z}_6 = \langle p | 6p = 0 \rangle \, .$$

The reason $A = \mathbb{Z}_6$ can detect the torsion is because $\mathbb{Z}_6$ contains zero-divisors, a nontrivial *torsion subgroup* $TG = \{g \in G | ng = 0, n \geq 1\}$. In contrast, if we choose the abelian group to be a field (such as $\mathbb{Z}_p$ with p prime or the rationals $\mathbb{Q}$), which by definition has no zero-divisors, the information about torsion is lost, as you can see in the examples above.

You can see that the homology with coefficients in $\mathbb{Z}_n$ is not just the integer homology mod n. Below I'll say a little more about how they are related.

It is sometimes useful to think about the data specifying the boundary map as an *attaching map* describing how the cell complex is assembled starting from the 0-cells and working up in dimension, as the following examples illustrate. These examples also show that torsion homology can occur for oriented manifolds.

$\mathbb{RP}^n$. Real projective space $\mathbb{RP}^n$ is the space of lines through the origin in $\mathbb{R}^{n+1}$. Such a line is specified by a vector up to rescaling by a nonzero real number: $\mathbb{RP}^n = \{\vec{v} \in \mathbb{R}^{n+1}\} / (\vec{v} \sim \lambda \vec{v}) \, , \lambda \in \mathbb{R} \backslash \{0\}$. By rescaling, we can pick a gauge where $|\vec{v}| = 1$; this leaves just the sign of λ unfixed, so $\mathbb{RP}^n = S^n / (\hat{v} \sim -\hat{v})$ – the sphere with antipodal points identified. The upper hemisphere (a B_n) is a fundamental domain for this $\mathbb{Z}_2$ action, but the $\mathbb{Z}_2$ still acts on the equator, which is a S^{n-1}:

$$\mathbb{RP}^n = B_n / \left(\hat{v} \sim -\hat{v} \text{ on } \partial B_n = S^{n-1} \right) .$$

So we see that the boundary of the ball is itself $\mathbb{RP}^{n-1}$.

So we obtain a cell complex for $\mathbb{RP}^n$ from one for $\mathbb{RP}^{n-1}$ by attaching a single n-cell. What is the attaching map? Well, we're going to again divide up S^{n-1} into two hemispheres, each of which will be associated with a single $(n-1)$-cell, σ_{n-1}. This one $(n-1)$-cell is obtained from the cell complex we made above for S^n by identifying its two $(n-1)$-cells, σ_{n-1} and $T\sigma_{n-1}$. There is one tricky point about the orientation here. Let's do the first couple: $\partial\sigma_1 = \sigma_0 - T\sigma_0 = \sigma_0 - \sigma_0 = 0$. But as you can see from the figure at right $\partial\sigma_2 = \sigma_1 + T\sigma_1 = 2\sigma_1$.

In fact $\partial\sigma_3 = \sigma_2 - T\sigma_2 = \sigma_2 - \sigma_2 = 0$ – it couldn't be a plus sign because then we'd get $\partial^2\sigma_3 = 4\sigma_1 \neq 0$, not a chain complex. The point is that the antipodal map in dimension n reverses the orientation if n is even. So $\Delta(\mathbb{RP}^n) = \sigma_0 \cup \sigma_1 \cdots \cup \sigma_n$ where $\partial\sigma_i = (1 + (-1)^i)\sigma_{i-1}$. Torsion up the wazoo. So the complex is

$$\cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0.$$

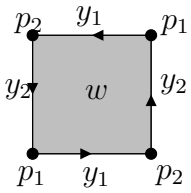
This gives

$$H_i(\mathbb{RP}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 0 \\ \mathbb{Z}_2, & i \text{ odd}, < n, \\ \mathbb{Z}, & i = n, n \text{ odd}, \\ 0, & \text{else} \end{cases}.$$

You can check this answer for $n = 2$ with the cell complex at right, which gives the complex

$$0 \to \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z}^2 \xrightarrow{\partial_1} \mathbb{Z}^2 \to 0$$

with $\partial_2 = (2, 2), \partial_1 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.



This is overkill on this example, but one way to compute the homology of a complex is using the software Macaulay2. Here are the necessary commands for this example, with integer coefficients:

```
d1=matrix{{1,-1}, {-1,1}}
d2=matrix{{2},{2}}
C = new ChainComplex; C.ring = ZZ;
C #0 = target d1; C #1 = source d1; C #2 = source d2;
C.dd #1 = d1; C.dd #2 = d2;
answer = HH C;
prune answer
```

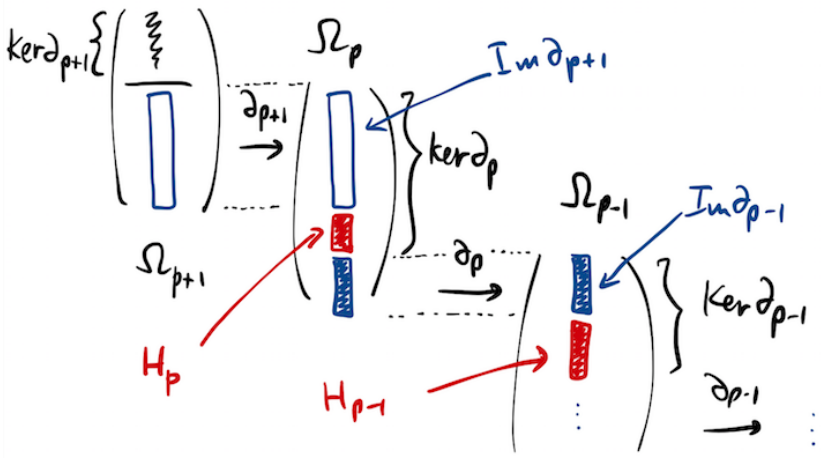
Incidentally, the group manifold of the rotation group $\mathsf{SO}(3)$ is $\mathbb{RP}^3$.

$\mathbb{CP}^n$. Complex projective space $\mathbb{CP}^n$ is the space of complex lines (copies of $\mathbb{C}$) through the origin in $\mathbb{C}^{n+1}$, $\mathbb{CP}^n = \{\vec{z}\}/\left(\vec{z} \sim \lambda\vec{z}\right), \lambda \in \mathbb{C}\setminus\{0\}$. We can choose a gauge where $|\vec{z}| = 1$, leaving just a phase ambiguity: $\mathbb{CP}^n = S^{2n+1}/\left(\vec{z} \sim \lambda\vec{z}\right), |\lambda| = 1$. To fix the phase, consider the region where $z^{N+1} \neq 0$. Then we can use λ to set $z^{N+1} > 0$, so that a general point is of the form $\vec{z} = (\vec{w}, \sqrt{1-|\vec{w}|^2}), |\vec{w}|^2 \leq 1$. But the set of points $\{\vec{w} \in \mathbb{C}^n, |\vec{w}|^2 \leq 1\}$ is a B_{2n}. Its boundary occurs when $z^{N+1} = 0$, which means $|\vec{w}| = 1$, which is an S^{2n-1}. On this locus, the phase redundancy still acts. So:

$$\mathbb{CP}^n = B_{2n}/\left(\hat{w} \sim \lambda\hat{w} \text{ on } \partial B_{2n} = S^{2n-1}\right).$$

Therefore the boundary is a copy of $\mathbb{CP}^{n-1}$. So a cell complex for $\mathbb{CP}^n$ is $\Delta(\mathbb{CP}^n) = \sigma_0 \cup \sigma_2 \cup \cdots \cup \sigma_{2n}$, and the boundary map is just zero. $b_i(\mathbb{CP}^n) = 1$ for i even and $b_i(\mathbb{CP}^n) = 0$ for i odd.

At right is a visualization of homology which I find useful.



Euler-Poincaré theorem:

$$\chi(X) \equiv \sum_{p=0}^{d}(-1)^p I_p = \sum_{p=0}^{d}(-1)^p b_p.$$

Here I_p is simply the *number* of p-simplices in the triangulation. We've seen that this is sometimes saturated by the minimal cellulation, *i.e.* no cancellation is required at all.

Proof: $I_p = \dim\Omega_p = \dim\ker\partial_p + \dim\mathrm{Im}\,\partial_p$. This is made clear by the visualization above. Now when we add these up with alternating coefficients, we get the alternating sum of the betti numbers $b_p = \dim\ker\partial_p - \dim\mathrm{Im}\,\partial_{p+1}$, using the fact that $0 = \dim\mathrm{Im}\,\partial_{d+1}$. This gives a proof that the euler character is a topological invariant, independent of the triangulation. ■

1.4 Higgsing, change of coefficients, exact sequences

[Bredon §IV.5] Suppose we've been talking about the toric code with gauge group A_1 so far. And suppose $A_2 \subset A_1$ is a nontrivial subgroup (*i.e.* not A_1 itself or the trivial group with one element). (Since A_1 is abelian, it is a normal subgroup.) How are the toric codes with gauge group A_1 and A_2 related? One way they are related is by Higgsing: For example, if $A_1 = \mathbb{Z}_{pq}$, $A_2 = \mathbb{Z}_p$ for primes p, q, and X_ℓ is the generator of A_1, then we can add $\Delta H = -h \sum_\ell X_\ell^q$. If we add such a term with large enough h this term will cause a phase transition to a phase where the charges in A_2 are proliferated, a Higgs phase. Strings labelled by elements of A_2 (multiples of q in the example) can break, so they don't produce nontrivial topological sectors. We end up with a toric code with gauge group A_1/A_2 ($= \mathbb{Z}_q$ in the example).

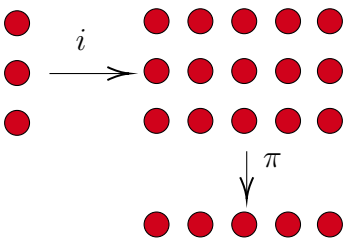
Consider the *exact sequence*

$$0 \to A_2 \xrightarrow{i} A_1 \xrightarrow{\pi} A_1/A_2 \to 0 \tag{1.6}$$

where the map i is just inclusion, and $\pi(g)$ is the projection onto the equivalence class $[g]$ modulo A_2. Exact means that the image of i is the kernel of π.

Some brief comments: An exact sequence of groups like (1.6) is called a *group extension*, more specifically, an extension of $G = A_1/A_2$ by A_2.

In general, a short exact sequence (meaning it has only three nodes) can be visualized as a fiber bundle. The first node is the fiber, the middle node is the total space the last node is the base. A nice visualization I learned from Greg Moore is at right.



This short exact sequence produces a corresponding short exact sequence on the chain complexes

$$0 \to \Omega_\bullet(A_2) \xrightarrow{i} \Omega_\bullet(A_1) \xrightarrow{\pi} \Omega_\bullet\left(A_1/A_2\right) \to 0 \tag{1.7}$$

where by putting $\bullet$ what I mean is that it's true for any p. (Sometimes these maps are denoted $i_\sharp, \pi_\sharp$ but I don't feel a strong urge to distinguish them from the maps on groups.) As long as the maps i, π commute with the boundary operator ∂ (such things are called *chain maps*), any such a short exact sequence on chain complexes produces the following *long exact sequence* on their homology:

$$\begin{array}{ccccc} H_p(A_2) & \longrightarrow & H_p(A_1) & \longrightarrow & H_p(A_1/A_2) \\ & & & \searrow^{\partial_\star} & \\ H_{p-1}(A_2) & \longrightarrow & H_{p-1}(A_1) & \longrightarrow & H_{p-1}(A_1/A_2) \end{array}$$

I've only drawn one step of it, but it is *long* in the sense that it continues above and below what I've drawn. The horizontal maps are just the maps i and π acting on representatives of homology. The vertical maps are the boundary maps ∂. The mystery is the connecting homomorphism $\partial_\star$, sometimes (in this case) called the *Bockstein*. Here's the idea: Consider the following commuting diagram, which is just (1.7) with the boundary maps ∂ written too:

$$\begin{array}{ccccccccc}
 & & & & b & & c & & \\
0 & \longrightarrow & \Omega_p(A_2) & \xrightarrow{i} & \Omega_p(A_1) & \xrightarrow{\pi} & \Omega_p(A_1/A_2) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \\
0 & \longrightarrow & \Omega_{p-1}(A_2) & \xrightarrow{i} & \Omega_{p-1}(A_1) & \xrightarrow{\pi} & \Omega_{p-1}(A_1/A_2) & \longrightarrow & 0 \\
 & & a=\partial_\star c & & \partial b & & 0 & & \\
 & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \\
0 & \longrightarrow & \Omega_{p-2}(A_2) & \xrightarrow{i} & \Omega_{p-2}(A_1) & \xrightarrow{\pi} & \Omega_{p-2}(A_1/A_2) & \longrightarrow & 0 \\
 & & 0 & & 0 & & & &
\end{array}$$

Start with an element $c \in \ker \partial \subset \Omega_p(A_1/A_2)$. Since the sequence is exact, this means $c = \pi(b)$ for some $b \in \Omega_p(A_1)$. Now act with ∂ on this guy: $\partial(b)$ has the property that it vanishes modulo A_2, that is $\pi(\partial(b)) = 0 \in \Omega_{p-1}(A_1/A_2)$, since π commutes with ∂ in the upper right box. But by exactness of the sequence, this means $\partial(b) \in \text{Im}\,(i : \Omega_{p-1}(A_2) \to \Omega_{p-1}(A_1))$, that is, there exists $a \in \Omega_{p-1}(A_2)$ with $i(a) = \partial b$. Using $[i, \partial] = 0$ in the lower left box, we have $\partial a = 0$, so a determines an element of $H_{p-1}(A_2)$ – this is who we were looking for: $\partial_\star c = a$. This kind of argument of homological algebra is called a *diagram chase*. A less explicit way to say what is the connecting homomorphism $\partial_\star$ is $\partial_\star[c] = [i^{-1}\partial\pi^{-1}(c)]$.

It remains to show that the resulting sequence is exact, so we have a little more diagram-chasing to do. This step, of taking a short (2-step) exact sequence on the complex and making from it a long-exact sequence on homology (or homotopy!) groups is powerful and ubiquitous. Notice that it didn't use any information about the nature of the maps, except that they were chain maps.

Let me change notation for generality and hopefully clarity here. Suppose our short exact sequence of chain maps is

$$0 \to A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{\pi} C_\bullet \to 0.$$

We want to show that

$$\ldots \xrightarrow{\partial_\star} H_p(A) \xrightarrow{i_\star} H_p(B) \xrightarrow{\pi_\star} H_p(C) \xrightarrow{\partial_\star} H_{p-1}(A) \xrightarrow{i_\star} \cdots$$

is exact. This involves three statements: exactness at each of the three kinds of nodes. These arguments are harder to read than they are to figure out. At each step there's basically only one thing you can do.

To see exactness at $H_\bullet(A)$, suppose $i_\star[a] = 0$ for some $a \in A_p$. This means $i(a) = \partial b$ for some $b \in B_{p+1}$. What can we do with such a b? We can take π of it: $\pi(b) = c$. Now what is $\partial c = \partial\pi b = \pi\partial b = \pi i a = 0$ by exactness of the short exact sequence. So c represents a homology class, and by construction $\partial_\star[c] = [a]$.

To see exactness at $H_\bullet(C)$, suppose $\partial_\star[c] = 0$ for some $c \in C_p$. This means there's $b \in B_p$ with $\pi b = c$, and $\partial b = ia$, with $[a] = \partial_\star[c]$. Now this means $a = \partial a'$ for some $a' \in A_p$. But $\partial b = ia = i\partial a' = \partial i a'$ means $\partial(b - ia') = 0$ and $\pi(b - ia') = c$, so we find $[c] = \pi_\star[b - ia']$.

Finally, to see exactness at $H_\bullet(B)$, suppose $\pi_\star[b] = 0$ for some $b \in B_p$. This means $\pi b = \partial c$ for some $c \in C_{p+1}$. By the fact that π is onto, this is $c = \pi(b')$ for some $b' \in B_{p+1}$. But now what is $\partial b'$? It's not b. Consider

$$\pi(b - \partial b') = \pi b - \partial\pi b' = \partial c - \partial c = 0.$$

But now $\ker\pi = \mathrm{Im} i$ implies that $(b - \partial b') = ia$ for some $a \in A_p$ which is closed because $i(\partial a) = 0$ and i is injective. Therefore $i_\star[a] = [b]$. ■

You can check that our answers for the homology of the Klein bottle above with coefficients in $\mathbb{Z}_{2,3,6}$ are consistent with the long exact sequence induced by the short exact sequence $0 \to \mathbb{Z}_2 \xhookrightarrow{i} \mathbb{Z}_6 \to (\mathbb{Z}_6/\mathbb{Z}_2 = \mathbb{Z}_3) \to 0$.

Universal Coefficients Theorem. I've explained the toric code for $\mathbb{Z}_N$ gauge group. Any discrete abelian group is of the form

$$A = \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z} \times \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \cdots . \tag{1.8}$$

It's a theorem (the *Universal Coefficients Theorem*) that taking $A = \mathbb{Z}$ contains all the information we would get by choosing other abelian groups. So although it's a bit pathological in that the local hilbert spaces are infinite-dimensional, the toric code with gauge group $\mathbb{Z}$ is universal in this sense. I'm not going to explain this theorem in lecture because it's a bit complicated and I'm not sure I have any physics insight about it to share[11]. [End of Lecture 5]

[11] Here's how to find $H_p(X, A)$ given $H_p(X, \mathbb{Z})$. There's an exact sequence (meaning a complex of maps (the image of each one is in the kernel of the next) with no homology)

$$0 \to H_p(X, \mathbb{Z}) \otimes A \to H_p(X, A) \to \mathrm{Tor}(H_{p-1}(X, \mathbb{Z}), A) \to 0.$$

If we know $H_\bullet(X, \mathbb{Z})$, we know the first and third steps, so we know the middle step. Now what is this $\mathrm{Tor}(B, A)$? It is a gadget that asks whether the abelian groups B and A have zero-divisors in

$\mathbb{Z}_N$ toric code with $N \to \infty$? The toric code with the gauge group (1.8) is just a collection of toric codes for each of the factors, sitting on top of each other, but some of gauge group $\mathbb{Z}$. You can ask it makes sense to study the toric code with gauge group $\mathbb{Z}$. The clock and shift operators are a bit singular in this limit. In particular, the phases of the eigenvalues of Z get closer and closer together as $N \to \infty$ – the space of eigenvalues approaches $\mathsf{U}(1)$. This is the familiar fact that a discrete variable is conjugate to a periodic variable (at finite N both conjugate variables are discrete *and* periodic): think of the phase of $Z = e^{\mathbf{i}2\pi\frac{n}{N}}$ as the position of a particle on a circle of unit radius, and then $X = e^{\mathbf{i}\hat{p}}$ is the translation operator.

In the limit $N \to \infty$ (to the extent that it can be said to exist) the state space runs over all integers $\mathcal{H}_\infty = \text{span}\{|n\rangle, n \in \mathbb{Z}\}$. This is the Hilbert space of a U(1) rotor. Another useful basis is the theta-vacua, aka Bloch waves:

$$|\theta\rangle = \sum_n e^{\mathbf{i}n\theta} |n\rangle , \qquad \theta \equiv \theta + 2\pi \ .$$

We can think of θ as the direction in which the rotor is pointing.

In this case, we need no longer write the 'mod N' in the star condition

$$0 = \sum_{\ell \in v(s)} n_\ell \ .$$

---

common.

To be more explicit about the definition of Tor: Any discrete abelian group is of the form $\mathbb{Z}^{m_2}$ modulo some relations which say that some elements are zero divisors. A *free resolution* of B is an exact sequence of the form

$$0 \to \mathbb{Z}^{m_1} \to \mathbb{Z}^{m_2} \to B \to 0$$

which encodes these relations in second map. If we tensor this sequence with A, we find

$$A^{m_1} \to A^{m_2} \to A \otimes B \to 0$$

that there can be a kernel in the first map. That kernel is defined to be $\text{Tor}(B, A)$:

$$0 \to \text{Tor}(B, A) \to A^{m_1} \to A^{m_2} \to A \otimes B \to 0$$

is exact. $\text{Tor}(B, A)$ is independent of the choice of free resolution and is symmetric in its two arguments. For example, a free resolution of $B = \mathbb{Z}_2$ is

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}_2 \to 0.$$

Tensoring with $A = \mathbb{Z}_n$ gives

$$0 \to \text{Tor}(\mathbb{Z}_2, \mathbb{Z}_n) \to \mathbb{Z}_n \xrightarrow{2} \mathbb{Z}_n \to \cdots$$

so we learn that $\text{Tor}(\mathbb{Z}_2, \mathbb{Z}_n) = \mathbb{Z}_{\text{gcd}(n,2)}$. If you want to learn more about this I highly recommend this concise and clear video. (Amazingly, Prof. Borcherds seems to be producing one such video every day.)

This is more obviously a lattice version of the gauss law condition $0 = \vec{\nabla} \cdot \vec{E}$ for E&M. A term which imposes this condition energetically just as well as $A_s = 1$ is the first term in

$$\mathbf{H} = -J \sum_{s \in \Delta_{p-1}} \left(\sum_{\sigma \in v(s)} n_\sigma \right)^2 - \sum_{\mu \in \Delta_{p+1}} \prod_{\sigma \in \partial \mu} e^{\mathbf{i}\theta_\sigma} \quad + h.c..$$

where

$$[n_\sigma, e^{\pm \mathbf{i}\theta_{\sigma'}}] = \pm e^{\mathbf{i}\theta_\sigma} \delta_{\sigma,\sigma'}$$

– *i.e.* $e^{\pm \mathbf{i}\theta}$ are raising and lowering operators. This second term more obviously approaches $\cos \vec{\nabla} \times \vec{A}$ in the continuum limit. Near the minima of the cosine, where $(\nabla \times \vec{A})^2$ is small, we can Taylor expand and the Hamiltonian looks like $\mathbf{H} = \int_{d^2x} \left(J \left(\nabla \cdot E \right)^2 + B^2 \right)$, not quite the Maxwell energy. If we take $J \to \infty$ (to impose Gauss' law exactly) and perturb by

$$\Delta H = -g \sum_\ell \left(Z_\ell + h.c. \right) = -g \sum_\ell \cos n_\ell \simeq \text{const} \ + \frac{g}{2} \sum_\ell n_\ell^2 + \cdots \simeq \int \frac{g}{2} E^2 + \cdots$$

we get the Maxwell energy. As Polyakov showed, in low dimensions, ignoring the nonlinear terms in the cosines is not always a good idea. If it is a good idea, the theory has a gapless photon. If it's not a good idea then the theory confines, which means (in the absence of symmetry) that it is in the trivial phase. For more on this see §7.2 of these lecture notes.

Notice that for any finite N there are two conjugate $\mathbb{Z}_N$ operations we might consider, one generated by $\mathbf{X}$, which acts by $\mathcal{O} \to \mathbf{X}\mathcal{O}\mathbf{X}^\dagger$, so in particular

$$\mathbb{Z}_N^X : \mathbf{Z} \to \omega \mathbf{Z}, \mathbf{X} \to \mathbf{X},$$

and one generated by $\mathbf{Z}$, which acts by $\mathcal{O} \to \mathbf{Z}\mathcal{O}\mathbf{Z}^\dagger$,

$$\mathbb{Z}_N^Z : \mathbf{Z} \to \mathbf{Z}, \mathbf{X} \to \omega \mathbf{X}.$$

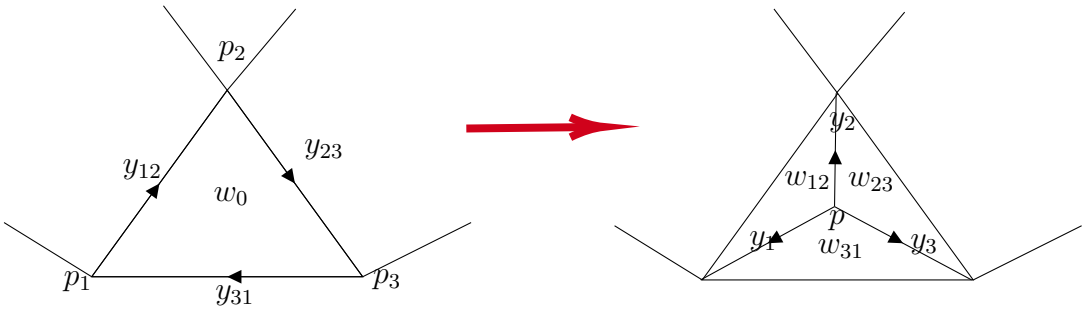
In the limit $N \to \infty$, one of these acts by the U(1) transformation $\theta \to \theta + \epsilon$.

1.5 Independence of cellulation

We've seen in a few examples that subdividing a cell into two cells does not change the homology of the complex.

Here is a mathy version of the general argument. Begin with a cellulation of a manifold, which produces a cell complex $\Omega_\bullet$. (The $\bullet$ is meant to indicate the collection of all possible indices.) Consider subdividing a single cell. In the figure we do this for

a two-dimensional space, by replacing a 2-cell w_0 by three 2-cells w_{12}, w_{23}, w_{31}, three 1-cells $y_i, i = 1..3$ and one new 0-cell p. This produces a new cell complex $\hat{\Omega}_\bullet$.



These complexes participate in a short exact sequence:

$$0 \to \Omega_\bullet \xrightarrow{i} \hat{\Omega}_\bullet \to \Omega'_\bullet \to 0$$

where each $\Omega'_q \equiv \hat{\Omega}_q/\Omega_q$ is defined by the demand that the sequence is exact. The map $i : \Omega_q \hookrightarrow \hat{\Omega}_q$ is an inclusion defined by $|w_0\rangle \mapsto \sum_i |w_i\rangle$ (and any other cell just maps to the cell with the same label). Since this map is injective (and any equivalence class $[\sigma] \in \hat{\Omega}_q/\Omega_q$ has a representative $\sigma \in \hat{\Omega}_q$), the sequence is exact.

As in the previous subsection, this short exact sequence on chain complexes produces a long exact sequence on homology:

$$\begin{array}{ccccc} H_p(\Omega) & \longrightarrow & H_p(\hat{\Omega}) & \longrightarrow & H_p(\Omega') \\ \downarrow & & \downarrow & & \downarrow \\ H_{p-1}(\Omega) & \longrightarrow & H_{p-1}(\hat{\Omega}) & \longrightarrow & H_{p-1}(\Omega') \end{array} \tag{1.9}$$

But now I claim that $H_\bullet(\Omega') = 0$. The exactness of the sequence (1.9) then implies that $H_\bullet(\Omega) = H_\bullet(\hat{\Omega})$. To be completely explicit: if we have an exact sequence of the form $0 \to A \xrightarrow{\phi} B \to 0$, exactness at A says $\ker \phi = 0$ and exactness at B says $\mathrm{Im}\,\phi = \ker(0) = B$, so ϕ is an isomorphism. We conclude that the homology is unchanged by subdivision.

To see that $H_\bullet(\Omega') = 0$ notice that because $\Omega' \equiv \hat{\Omega}/\Omega$, it only contains those cells which are added by the subdivision. In the example above,

$$\Omega'_0 = \langle p \rangle \, \Omega'_1 = \langle y_i \rangle \, , \Omega'_2 = \left\langle w_{ij} | \sum_{ij} w_{ij} = 0 \right\rangle .$$

By $\langle x \rangle$ I mean 'the vector space over A generated by x' The relation among the 2-cells comes from the fact that $\sum_{ij} w_{ij} = w_0$ is the original cell, which is in Ω, and hence

equivalent to zero in $\Omega' = \hat{\Omega}/\Omega$. The boundary operator is:

$$\partial w_{ij} = y_i - y_j + y_{ij} = y_j - y_i \bmod \Omega, \quad \partial y_i = p - p_i = p \bmod \Omega$$

where in both cases we set to zero components of the boundary which are part of Ω. So the relevant part of the complex Ω' (with an extra generator at Ω'_2) is

$$\mathbb{Z}^3 \xrightarrow{M} \mathbb{Z}^3 \xrightarrow{N} \mathbb{Z} \to 0,$$

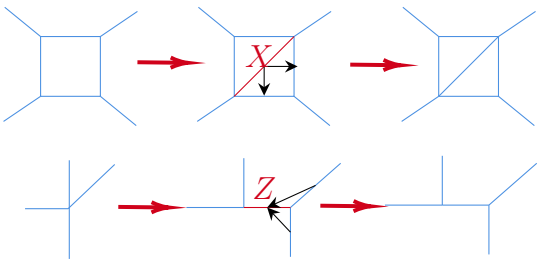
with (in the basis $(12, 23, 31)$ for the faces)

$$M = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}.$$

As you can see, $NM = 0$. Since $p \in \mathrm{Im}\partial_1 = \mathrm{Im}N$, $H_0(\Omega') = 0$. On Ω'_1, $\partial_1 = N$ has a 2-dimensional kernel, but $\partial_2 = M$ has a 2-dimensional image from Ω'_2. (M has rank 2 since, *e.g.*, its singular values are $(\sqrt{3}, \sqrt{3}, 0)$). The only kernel of $\partial_2 = M$ is $\sum_{ij} w_{ij} = 0$ in Ω'_2.

More poetically, the complex Ω' describes a triangulation of a disc, but with the nontrivial homology (in $p = 0$) removed.

But for our purposes, there is a better way to demonstrate the subdivision invariance.



The figure is describing a process of adding an ancilla qubit and incorporating it into the lattice model. In each case, the ancilla begins its life in a trivial decoupled state. In the top row, its hamiltonian is $H_0 = -cX$, in the bottom row, it is $H_0 = -cZ$, where c is some large energy. Recall that adding decoupled bits is allowed and doesn't change the phase.

We act on the hamiltonian with a (brief!) series of 2-qubit gates. You should think of this as varying the coupling constants of the hamiltonian within the phase. The specific gates are as follows: the black arrows indicate CNOT gates, which act on 2-qubits as

$$\mathsf{CX} \equiv |0\rangle\langle 0|_C \otimes \mathbb{1}_T + |1\rangle\langle 1|_C \otimes X_T$$

where C is for 'control' and T is for 'target'. The arrow points from the control bit to the target bit, $C \to T$. When conjugating operators the CNOT gate acts as

($\mathcal{O} \leftrightarrow \mathsf{CX}\mathcal{O}\mathsf{CX}$)

$$
\begin{aligned}
1_C Z_T &\leftrightarrow Z_C Z_T \\
1_C X_T &\leftrightarrow 1_C X_T \\
Z_C 1_T &\leftrightarrow Z_C 1_T \\
X_C 1_T &\leftrightarrow X_C X_T
\end{aligned}
$$

It is a fun exercise to convince yourself that this maps the TC Hamiltonian on the initial graph to a Hamiltonian with the 'stabilizer algebra' of the final graph. (That little outpouring of jargon was necessary because the terms in the resulting H are not exactly the same; rather we get terms like $B_{p_1} B_{p_2} + B_{p_1}$ where p_1 and p_2 are the new plaquettes. But the set of groundstates is the same.) That is

$$
H_{\text{coarse}} = H_{\text{TC}} \otimes \mathbb{1} + \mathbb{1} \otimes H_0 \;\; \mapsto \;\; H_{\text{fine}} \equiv U H_0 U^\dagger
$$

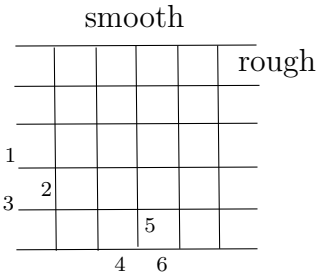
which has a groundstate subspace isomorphic to H_{TC} on the finer lattice.

You can convince yourself that any two cell decompositions of a given manifold are related by a sequence of these two operations.

These operations were written down in this paper by Vidal and Aguado. They were thinking of it as *entanglement renormalization*: a sequence of steps starting from the groundstate of a given hamiltonian on a fine lattice to produce a groundstate of the hamiltonian on a coarser lattice (times a bunch of decoupled bits). Since the operations are unitary, they are completely reversible and can be regarded either as subdivision or coarse-graining.

1.6 Gapped boundaries and relative homology

So far I haven't said much about spaces with boundary. Many boundary conditions are possible on the toric code. Two are special in that they are *gapped*.



At a *rough* boundary, plaquettes are missing a link; we still include the broken plaquette operator $B_{123} = X_1 X_2 X_3$ in the hamiltonian. At a *smooth* boundary, stars are missing a link; we still include the broken star operator $A_{456} = Z_4 Z_5 Z_6$.

All these operators still commute, and the groundstates still have the same form, except: for rough boundaries, the strings (over which we sum in the groundstates) are allowed to end at the boundary – this doesn't violate any star operators.

Relative homology. Consider a space X and a subspace Y which includes its boundary. Cellulate X and Y into associated cell complexes C^X and C^Y. The inclusion $Y \subset X$ means that there is a short exact sequence on the complexes:

$$0 \to C_\bullet^Y \stackrel{i}{\to} C_\bullet^X \stackrel{\pi}{\to} C_\bullet^{X/Y} \to 0. \tag{1.10}$$

If you like, the complex at right, the quotient, is defined by making this sequence exact. The idea is that in the complex $C^{X/Y}$, a chain in Y is regarded as zero[12]. We already studied an example of this in our discussion of subdivision. The homology of this complex is called the *relative homology* $H_\bullet(X/Y)$ (of X relative to Y).

But this is just what we get by taking the toric code on a cellulation of X and cutting out (erasing) the subspace Y so that X ends with with all-rough boundary conditions. (In fact the answer with all-smooth boundary conditions is not really different; they are related by a duality. But the description is different and not quite the same as relative homology.)

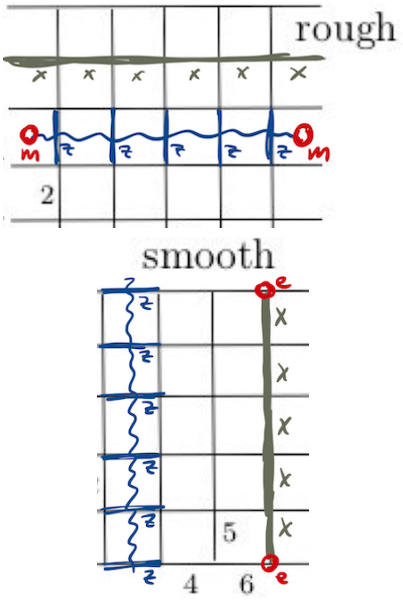
As you might have expected, the short exact sequence (1.10) induces a long exact sequence relating the homology of X, of Y and the relative homology:

$$\cdots \to H_{p+1}(X/Y) \stackrel{\partial_*}{\to} H_p(Y) \stackrel{i_*}{\to} H_p(X) \stackrel{\pi_*}{\to} H_p(X/Y) \stackrel{\partial_*}{\to} H_{p-1}(Y) \to \cdots$$

So if you know the homology of all of X and of Y (which by the way, has smooth boundary conditions), then you can use this exact sequence to figure out the relative homology. Or, more often, if you know the homology of Y and the relative homology, you can find the homology of all of X.

[12] Warning about notation: I called this complex $C^{X/Y}$ because it enjoys an equivalence relation where cells in Y are regarded as zero. A distinct thing we could mean by X/Y is the space where we say that each point in Y is equivalent to every other, *i.e.* identify Y to a point. Perhaps a better notation would have been $C^{X \setminus Y}$. Thanks to Ahmed Akhtar for pointing out this ambiguity.

Physical picture of boundary conditions. Consider how the smooth and rough boundary conditions interact with the excitations of the toric code. At a rough boundary, an e-particle (the end of an electric string) simply disappears into the boundary. But an m-particle gets stuck – there is still a (broken) plaquette operator to be violated at the boundary. At a smooth boundary, the situation is reversed: an m-particle is absorbed into the boundary without violating any terms in the hamiltonian, while the e-particle gets stuck (still violates the broken star operator).



A smooth boundary is like removing a row of faces, saying we don't care if those terms are violated, while a rough boundary is like removing a row of sites.

We can say that at a rough boundary the e-particle is *condensed*. Recall that saying that an excitation p is *condensed* in state $|\psi\rangle$ means that its creation operator $\mathcal{O}_p$ has an expectation value: $\langle\psi|\mathcal{O}_p|\psi\rangle \neq 0$. A creation operator for an e-particle at site i is just X_{ij} for any link ij ending at i. We can make this operator condense by adding a term to the hamiltonian $\Delta H = -\Gamma_{\text{big}} X_{ij}$. Then this will persuade the qubit on this link to be in the state $|+\rangle$ with $X_{ij}|+\rangle = |+\rangle$, meaning that acting on this state we can regard X_{ij} as 1. Then in any plaquette operator in which it participates, $B_{ijkl} = X_{ij}X_{jk}X_{kl}X_{li}$, we can replace it with 1, leaving behind $X_{jk}X_{kl}X_{li}$, the broken plaquette defined above.

So a good way to think about the relative cohomology of X relative to Y is to imagine adding $\Delta H = -\Gamma_{\text{big}} \sum_{\ell \in Y} X_\ell$ to condense e particles everywhere in Y and put the system into the higgs phase in that region. Then we don't care if we violate the star operators in that region because they're already violated.

[End of Lecture 6]

Rough and smooth boundary conditions are *dual* to each other in a way I will explain in §1.7.

Oracular comment on larger picture. [Kitaev-Kong 1104.5047] So we've seen that there is an interesting interplay between the data of the bulk excitations and their behavior at a gapped boundary. Which subsets of the anyons can be condensed? To

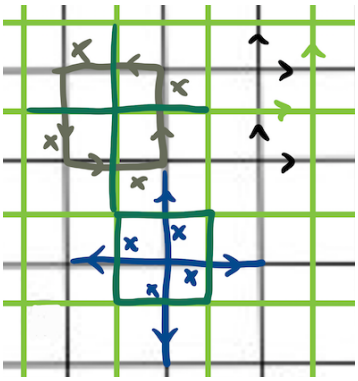
condense two anyon types in the same state requires the anyons to have trivial mutual statistics – to be *mutually local*. It turns out that the possible boundary data actually determines the bulk topological order up to a certain equivalence relation ('Morita equivalence') which turns out to be just duality.

1.7 Duality

You may notice an apparent asymmetry between our descriptions of rough and smooth boundary conditions on the toric code: the electric strings (over which we sum in the groundstate wavefunction) are allowed to end on a rough boundary and not on a rough boundary. The asymmetry arises because we wrote the wavefunction in the Z-basis, where the star operator is diagonal.

Let's write the wavefunction in the X-basis and see what happens. A configuration which satisfies the plaquette condition $B_p = 1$ then has $\prod_{\ell \in \partial p} x_\ell = 1$. We can interpret this again as a closed-string condition in the following way: introduce the *dual lattice*. Let's do it first in two dimensions. For each 0-cell in the original complex Δ, introduce a 2-cell in the dual complex $\Delta^\vee$. For each 2-cell in the original complex, introduce a 0-cell in $\Delta^\vee$. For each 1-cell in Δ, we have a 1-cell in $\Delta^\vee$. It remains to define the boundary map. Let's postpone the general description for a moment and show how it works for the square lattice.

In the figure at right, the black edges are the original square lattice; the green edges are the dual lattice. You can see that the plaquette operator on the original lattice is a star operator on the dual lattice, up to the exchange of Z_ℓ for X_ℓ. But this is just a local basis transformation, accomplished by an on-site unitary operator, $\mathsf{H}Z\mathsf{H} = X, \mathsf{H}X\mathsf{H} = Z$, $\mathsf{H} = |0\rangle\langle+| + |1\rangle\langle-| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \mathsf{H}^\dagger = \mathsf{H}, \mathsf{H}^\dagger\mathsf{H} = \mathbb{1}, \mathsf{H}^2 = \mathbb{1}$ (the Hadamard gate). The plaquette operator on Δ also becomes the star operator on $\check{\Delta}$, with the same interchange.



It's not hard to check that the orientations work out in the $\mathbb{Z}_N$ case. For $A = \mathbb{Z}_N$, the matrix which generalizes H is the character table of G (normalized to make it unitary)[13]. For example, for $N = 3$, it is

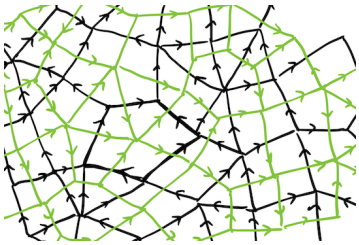
$$\mathsf{H} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}.$$

[13] In case you've forgotten, the character table is $\chi_a^\alpha \equiv \mathrm{tr}_{R_a} U(g_\alpha) = \mathrm{tr}_{R_a} U(h g_\alpha h^{-1})$, where α labels a conjugacy class of A, while a labels an irreducible representation (irrep) of A.

So we see that the Z basis is labelled by conjugacy classes while the X basis is labelled by irreps. For an abelian group there is not much difference between either of these and elements of the group. But this is a hint about the generalization to nonabelian groups (§3.9), where these are very different things.

Let's go back to writing the groundstate wavefunctions in the X basis: this realization restores the symmetry between electric and magnetic excitations. If we draw links of the original lattice when they are in a state $X = -1$, the plaquette condition is hard to interpret. But suppose instead we draw the links of the dual lattice. Then the plaquette condition (of the original lattice) is just the condition that the strings on the dual lattice are closed. And the star operator A_s on the original lattice hops these strings across the plaquette of the dual lattice corresponding to the site s. That is: the wavefunction in the X-basis is just uniform superpositions of closed (magnetic) strings on the dual lattice. Just as the electric string on a curve C is created from $\prod_\ell |0\rangle_\ell$ by the operator W_C, the magnetic string on a curve $\check{C}$ is created from $\prod_\ell |+\rangle_\ell$ by the operator $V_{\check{C}}$.

This lattice duality operation extends to more general cell complexes, not just a regular lattice. A useful orientation convention is that the orientation of the new green link perpendicular to a link of the original lattice makes a right turn relative to the orientation of the original link.



More generally, for a d-dimensional cell complex (where d is the dimension of the largest-dimension cells), we replace p-cells in Δ with $(d-p)$-cells in $\Delta^\vee$. A good way to think about the duality map is that to each p-cell σ_p of $\mathbf{\Delta}$ we associate a $(d-p)$-cell $\sigma^\vee_{d-p}$ roughly in such a way that they combine to form a local volume element. More precisely, I believe the boundary map on $\mathbf{\Delta}^\vee$ can be defined by

$$\boxed{\partial\left((\sigma_p)^\vee\right) \equiv \left(v(\sigma_p)\right)^\vee \in \Delta^\vee_{d-p-1}}\,.$$

If the space being triangulated is not oriented, I don't know how to make this work for $\mathbb{Z}_N, N > 2$. We need a choice of orientation in saying which way was a 'right turn' in defining the boundary operator.

This operation we've defined here isn't really a duality from the physics point of view – it's just a relabelling of the local variables. Mathematically, it says something interesting, however: it says that when this operation of constructing the dual cell complex makes sense (it certainly makes sense for $A = \mathbb{Z}_2$ (or any other field) and for cellulations of oriented manifolds X[14]), there is a dual description of the same space

[14]Full disclosure: I'm still not entirely sure what goes wrong if X is not oriented, but we've already

of groundstates (hence the (torsion-free) homology of X) which interchanges the roles of p-cells and $(d-p)$-cells. This implies (let's just say with $\mathbb{Z}_2$ coefficients to be safe) that $H_p(X, \mathbb{Z}_2) \simeq H_{d-p}(X, \mathbb{Z}_2)$. This is a version of Poincaré duality.

Corollary: $\chi(M_n) = 0$ if n is odd.

In 3d, for example, this operation exchanges the 1-form toric code (with degrees of freedom on the links) and the 2-form toric code (with degrees of freedom on the plaquettes). Here is a way to make this less surprising: Write the groundstate wavefunctions for the original 1-form toric code in the X-basis. How to describe the plaquette condition? As in 2d, the best way is in terms of the dual lattice (note that here I won't switch X and Z on the dual lattice since the model is not self-dual anyway): If a link is in the state $X = -1$, cover the plaquette of the dual lattice that it pierces. The plaquette operator demands that the set of covered plaquettes is closed. The star operator for a site s $A_s = \prod_{\ell \in v(s)} Z_\ell$ hops these plaquettes across the volume cell dual to s. Thus, the wavefunctions of the 1-form toric code in the X-basis are uniform superpositions of closed membranes.

In any dimension duality exchanges the 0-form toric code, *i.e.* the ferromagnet, with the d-form toric code, which is (at least on an oriented space) also just a ferromagnet.

Finite temperature quantum memory. In 4d something interesting happens: the 2-form toric code is self-dual in the same way the 1-form toric code in 2d is self-dual. This means there are no particle excitations (created by string operators). This model is extremely interesting for the following reasons[15]. All of the topological phases of matter we've discussed so far are zero-temperature phases – properties of the groundstate. What happens if we put them at finite temperature (*i.e.* consider the mixed state $\rho = e^{-\beta H}/Z$ for $\beta < \infty$)? Do they still represent a phase distinct from the trivial product state that one reaches at $\beta \to 0$, infinite temperature?

The answer, in all the examples so far until this one, is no: At any finite temperature such topological order is destroyed by the proliferation of defects of the appropriate nature. For 1-form gauge theory in any dimension, the appropriate defects are particles; the topologically-protected degenerate groundstates differ by the action of the holonomy of these particles around topologically non-trivial curves (the Wilson loop operators). Any finite temperature introduces a system-size-independent density of these particles, $n(T) \propto e^{-\Delta/T}$, where Δ is the energy gap. Away from the zero-correlation-

seen some examples where $H_p(X, A) \neq H_{d-p}(X, A)$, such as $X = K$, Klein bottle, $A = \mathbb{Z}_6$, where $H_0 \neq H_2$. Another example is $X = K \times S^1$ and $A = \mathbb{Z}_6$, where $H_1 \neq H_2$. Thanks to Bowen Shi for discussions of this point.

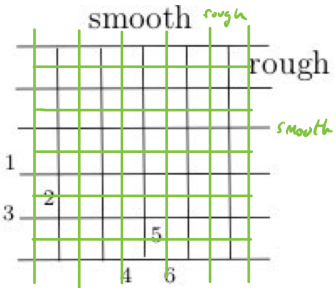
[15] I think this was first pointed out in Dennis-Kitaev-Landahl-Preskill, 2001; the following discussion comes from here. Perhaps later I'll have a chance to explain what we were doing in that paper. Another paper which discusses these issues is this one

length fixed point, these defects can move around and they can move around the nontrivial cycles of space, and their holonomy produces the logical operators which mix the degenerate groundstates.

The situation for $(p \geq 2)$-form gauge theory in $d > 3$ is different. ($d > 3$ is required because as we just saw, 2-form gauge theory in $d = 3$ dimensions can be dualized to 1-form gauge theory.) The defects which destroy the topological order are large, closed strings (more generally, $p-1$-dimensional objects). On a generic space, they must be large (scaling with system size) because they must wrap topologically non-trivial one-cycles. The Boltzmann factor therefore provides a system-size-dependent suppression of the density of such defects.

The statistical mechanics of strings whose energy is dominated by a tension term is governed by a Hagedorn equation of state: the entropy at fixed energy is linear in the energy, $S(E) = aE$, with a coefficient determined by the string tension. The idea is that the energy goes like the length $E \sim \sigma L$, while the number of states of the string e^S grows exponentially in the length (for example, a random walk independently chooses a direction at each step). Therefore, the free energy $F = E - TS = (1 - aT)E$. This Peierls-type argument implies a transition at some 'Hagedorn' temperature above which the canonical ensemble in terms of strings breaks down. Above this temperature, strings are condensed, and in the 2-form gauge theory context, the topological order is destroyed. Conversely, below the Hagedorn temperature, the entropic contribution is overwhelmed by the tension, and the ensemble is dominated by small strings. Hence, there is a temperature below which the 2-form topological order persists.

Notice that this Poincaré duality exchanges rough and smooth boundary conditions. This is consistent with the fact that in the 2d toric code it exchanges e and m particles. Notice also that it exchanges the couplings g and h and therefore acts as a diagonal reflection on the phase diagram, exchanging Higgs and confined phases. In this example, these phases are actually the same phase. The line where $g = h$ is self-dual and some interesting things happen there, as first found here. Some very recent progress is here.



Kramers-Wannier-Wegner duality. Discrete gauge theory participates in another, it seems to me quite distinct, duality, also involving the dual lattice.

Consider the *quantum clock model* on one of the cell complexes described above. This model has $\mathcal{H} = \otimes_{s \in \Delta_0} \mathcal{H}_N$, N states on each site. The Hamiltonian is

$$\mathbf{H}_{\text{clock}} = -\Gamma \sum_{\ell \in \Delta_1, \partial \ell = i - j} \mathbf{X}_i \mathbf{X}_j^\dagger - g \sum_{i \in \Delta_0} \mathbf{Z}_i + h.c. \ . \tag{1.11}$$

For $N = 2$, this is the quantum transverse-field Ising model[16]. Notice that if we diagonalize $\mathbf{X}_i = e^{\mathbf{i}2\pi k_i/N}, k_i = 1..N$, then

$$-\mathbf{X}_i\mathbf{X}_j^\dagger + h.c. = -2\cos\frac{2\pi}{N}(k_i - k_j)$$

– it is minimized when $k_i = k_j$.

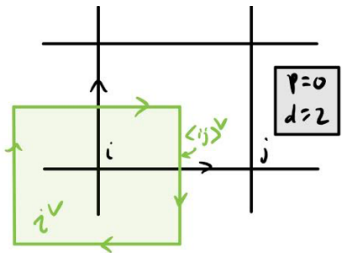
In any d, the following duality (due to Wegner, for a review see Savit) maps this model to a gauge theory on the dual cell complex $\mathbf{\Delta}^\vee$. I will describe first the case where the variables live on the sites of $\mathbf{\Delta}$ and then we can figure out the general construction. The idea is to put a variable on each links of $\mathbf{\Delta}$ (or equivalently each $(d-1)$-cell of $\mathbf{\Delta}^\vee$) that keeps track of the *change* of $\mathbf{X}$ across the link; for $N = 2$, this is a sign which is -1 only if $\mathbf{X}_i$ and $\mathbf{X}_j$ disagree:

$$\mathbb{Z}_2: \quad \boldsymbol{\sigma}^z_{ij} \equiv \mathbf{X}_i\mathbf{X}_j.$$

For $N > 2$, $\mathbf{X}_i$ and $\mathbf{X}_j$ can differ by a phase, so the domain wall takes values in N-th roots of unity:

$$\mathbb{Z}_N: \quad \boldsymbol{\sigma}^z_{ij} \equiv \mathbf{X}_i\mathbf{X}_j^\dagger.$$

The edge $\langle ij\rangle \in \Delta_1 = \Delta^\vee_{d-1}$ corresponds to a $(d-1)$-cell of the dual lattice – a *wall*. At right is the picture for $d = 2$.



Not all configurations of the $\boldsymbol{\sigma}^z_\ell$s are attainable by this map. If we label the $(d-1)$-cells of the dual lattice by a color according to k_ℓ in $\boldsymbol{\sigma}^z_\ell = \omega^{k_\ell}$, then these walls form closed membranes in the same sense as above (*i.e.* literally closed, unoriented loops for $d = 2, N = 2$, and more generally $\mathbb{Z}_N$-string-nets or brane-nets in $d > 2$). This is just the familiar fact that level-sets of a function are collections of closed curves. The allowed configurations of the $\boldsymbol{\sigma}_\ell \equiv e^{\frac{2\pi\mathbf{i}m_\ell}{N}}$ satisfy

$$1 = \prod_{\ell\in\partial w}\boldsymbol{\sigma}^z_\ell, \quad \forall w \in \Delta_2 = \Delta^\vee_{d-2}, \quad \text{or} \quad \sum_{\ell\in v^\vee(s)} m_\ell = 0 \bmod N, \quad \forall s \in \Delta^\vee_{d-2}.$$

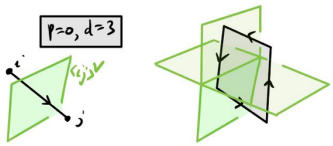
This is just the star condition, *i.e.* the gauss law, for a $(d-1)$-form gauge theory on the dual lattice. In 1+1 dimensions, this condition is empty and the model is self-dual (up to global issues), as observed by Kramers and Wannier for $N = 2$.

[16]The *quantum Potts model* is defined on the same Hilbert space, but with the Hamiltonian

$$\mathbf{H}_{\text{Potts}} = -\beta\sum_{\ell\in\Delta_1,\partial\ell=i-j}\sum_{p=1}^{N}\mathbf{X}_i^p\left(\mathbf{X}_j^p\right)^\dagger - \Gamma\sum_{i\in\Delta_0}\sum_{p=1}^{N}\mathbf{Z}_i^p + h.c.\ .$$

They both reduce to the quantum transverse-field Ising model for $N = 2$.

Here is the picture for $d = 3$: A link of Δ is dual to a 2-cell of $\Delta^\vee$. A product of $\mathbf{X}_i\mathbf{X}_j^\dagger$s around a loop equals one, which means that the product of σ_w^zs in the right figure is one. This is the star condition for 2-form gauge theory on the dual lattice.



What about the $\mathbf{Z}_i$ term in (1.11)? σ_{ij}^z counts the number of domain walls on the link of the dual lattice separating sites i and j of the original lattice. But $\mathbf{Z}_i$ *changes* the value of $\mathbf{X}_i$ fixing $\mathbf{X}_j^\dagger$ and hence *creates* domain walls on each of the links of the dual lattice surrounding the site i:

$$\mathbf{Z}_i = \prod_{ij\in v(i)} \sigma_{ij}^x = \prod_{\ell\in\partial^\vee(i^\vee)} \sigma_\ell^x$$

and $i^\vee$ is the plaquette of the dual lattice corresponding to the site i. This is the plaquette term of the dual gauge theory. [End of Lecture 7]

What I mean by global issues: the $\boldsymbol{\sigma}^z$s do not completely specify the configurations of the $\mathbf{X}_i$: if we act by $\prod_{i\in\Delta_0}\mathbf{Z}_i$, to rotate *every* site by ω, nothing happens to the $\boldsymbol{\sigma}_{ij}^z$. This is just one global integration constant in $\mathbb{Z}_N$. Conversely, the loops making up the groundstate of the dual gauge theory arise as boundaries of domains of values of the k_j variables. Hence they are always contractible – we only get one of the groundstates of the dual gauge theory.

More generally, to dualize a gauge theory with dofs on the p-cells, with

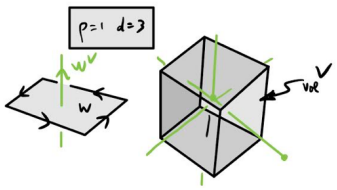
$$H = -\Gamma \sum_{\sigma\in\Delta_{p+1}} \prod_{\ell\in\partial\sigma} \mathbf{X}_\ell - g\sum_{\ell\in\Delta_p}\mathbf{Z}_\ell$$

(with the Gauss law constraint $1 = \prod_{\ell\in v(w)}\mathbf{X}_\ell, \forall w \in \Delta_{p-1}$ imposed as a constraint) we will want a duality equation like:

$$\forall\sigma\in\Delta_{p+1} = \Delta_{d-p-1}^\vee: \quad \boldsymbol{\sigma}_\sigma^z \equiv \prod_{\ell\in\partial\sigma}\mathbf{Z}_\ell \quad \text{which satisfy } 1 = \prod_{\sigma\in\partial w}\boldsymbol{\sigma}_\sigma^z, \ \forall w\in\Delta_{p+2} = \Delta_{d-p-2}^\vee$$

by virtue of $\partial^2 = 0$.

Here is the picture for $(p = 1)$-form gauge theory in $d = 3$: A plaquette of Δ is dual to a 1-cell of $\Delta^\vee$. A product of plaquette terms around the faces of a 3-cell equals one, which means that the product of σ_ℓ^zs in the right figure is one. A 3-cell of Δ is dual to a 0-cell of $\Delta^\vee$, so is the star condition for 1-form gauge theory on the dual lattice. So under this duality, 1-form gauge theory is self-dual in 3d.



This transformation then takes the term

$$\mathbf{Z}_\sigma = \prod_{w \in \partial(\sigma^\vee)} \sigma_w^x$$

to the analog of the plaquette term.

This exchanges degrees of freedom on the p-cells with degrees of freedom on the $(d-p-1)$-cells. It is like electric-magnetic or Hodge duality in spacetime; for example in $\mathsf{U}(1)$ p-form gauge theory in the continuum, the duality relation is

$$dA_p = \star d\check{A}_{d-p-1}.$$

$\star$ here is the Hodge duality operation of contracting with the $\epsilon_{i_1 \cdots i_D}$ tensor[17]. Here $D = d+1$ is the number of spacetime dimensions. So Hodge duality exchanges a p-form potential for a $(d-p-1)$-form potential. Under this transformation, Maxwell theory in $D = 3+1$ is self-dual. In $D = 2+1$ this duality relates a Maxwell field to a scalar (Goldstone) field, *i.e.* an XY model.

Physically, this change of variables is much more nontrivial and interesting than one that gives Poincaré duality. For example, when $g \gg \Gamma$ we are in the confined phase in the Δ variables (the single-link term is more important than the plaquette term). But in the $\Delta^\vee$ variables, the roles of those two couplings are interchanged, so this $g \gg \Gamma$ regime is in the *deconfined* phase of the $\Delta^\vee$ variables! But mathematically it is less interesting. In fact those 'global issues' I mentioned earlier exactly account for the fact that naively one side of the duality has topological order and the other side does not. If we are careful about the global issues we will see that this does not happen, just as in the $p = 0$ case, the duality forgot about the N degenerate groundstates related by the global symmetry.

[17] I'll say more about in the next section, but a little more precisely, the Hodge dual of a q-form ω is

$$(\star\omega)_{i_{q+1} \cdots i_D} = \frac{\sqrt{\gamma}}{q!} \epsilon_{i_1 \cdots i_D} \omega^{i_1 \cdots i_q}$$

where γ_{ij} is the metric on spacetime.

2 Supersymmetric quantum mechanics and cohomology, index theory, Morse theory

2.1 Supersymmetric quantum mechanics

Most of the contents of this chapter are from these papers: [Witten NPB 202 (1982) 253, NPB 188 (1981) 513, Journal of Differential Geometry, Volume 17, Number 4 (1982), 661-692].

Supersymmetry. [A good reference is Argyres's notes. You may also find useful chapter 10 of this book] A supercharge Q is an anticommuting symmetry generator. An interesting algebra that such an operator can generate is:

$$[Q, H] = 0, \quad \{Q^\dagger, Q\} = 2H, \quad \{Q, Q\} = 0. \tag{2.1}$$

A consequence of this supersymmetry algebra is that $H \geq 0$, the resulting hamiltonian has a spectrum bounded below by zero, since $\{Q^\dagger, Q\}$ is a positive operator[18].

In a many body system with Lorentz symmetry, the Hamiltonian H is the time component of a D-vector P_μ. The generalization of the supersymmetry algebra is

$$\{Q^\dagger_\alpha, Q_\beta\} = 2\gamma^\mu_{\alpha\beta} P_\mu + C^I_{\alpha\beta} Z_I \tag{2.2}$$

where α, β are spinor indices. Here C^I are invariant tensors for the Lorentz group and Z_I are called central charges. Let's set them to zero for now.

So Q is like a square root of translations. It is sometimes useful to regard it as a 'translation in a fermionic direction' whose coordinate is a grassmann variable. A space with such coordinates is called a superspace. This idea is actually useful for constructing supersymmetric actions.

Returning to the quantum mechanics ($D = 1$ spacetime dimensions) case (2.1), consider the spectrum of H: $H|n\rangle = E|n\rangle$. Acting on such an energy eigenstate, $\{Q^\dagger, Q\}|n\rangle = 2E_n|n\rangle$. If $E_n > 0$, let $a^\dagger \equiv \frac{Q^\dagger}{\sqrt{2E_n}}, a \equiv \frac{Q}{\sqrt{2E_n}}$ then these operators satisfy $\{a^\dagger, a\} = 1, a^2 = 0$, an ordinary fermionic creation-annihilation algebra of a single fermion mode. This algebra is represented on a two state system:

$$a|-\rangle = 0, a^\dagger|-\rangle = |+\rangle, a|+\rangle = |-\rangle, a^\dagger|+\rangle = 0 \ .$$

In this basis, the matrix elements are $a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, a^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Another important

[18] For any state $|\psi\rangle$, $\langle\psi|\{Q^\dagger, Q\}|\psi\rangle = \| Q|\psi\rangle \|^2 + \| Q^\dagger|\psi\rangle \|^2 \geq 0$. This inequality is only saturated if $Q|\psi\rangle = 0$ and $Q^\dagger|\psi\rangle = 0$, assuming unitarity, which implies no nonzero zero-norm states.

operator on this 2-state system is the fermion parity

$$(-1)^F \equiv 2a^\dagger a - 1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It satisfies $\{(-1)^F, a\} = 0 = \{(-1)^F, a^\dagger\}$ – the fermionic operators $a, a^\dagger$ change the fermion parity. The key point is that states with $E_n > 0$ come in pairs of opposite fermion parity.

In contrast, on states with $E = 0$, the supersymmetry algebra becomes $\{Q^\dagger, Q\} |E = 0\rangle = 0$ which has 1d irreps: $Q |E = 0\rangle = 0 = Q^\dagger |E = 0\rangle$ on which $(-1)^F$ (defined to anti-commute with Q and $Q^\dagger$) can have either sign.

We conclude that states with $E > 0$ come in bose-fermi pairs, while there can be any number of groundstates with $E = 0$.

Spontaneous supersymmetry breaking is said to occur if there exists no state with $E = 0$, in which case the groundstate has $E > 0$. This is not a bad name, since it means that the groundstate is acted upon nontrivially by the (super)symmetry generator Q. It is a strange phenomenon compared to spontaneous breaking of ordinary symmetries, however, because it can happen in finite volume[19]. In fact, there are examples where in the thermodynamic limit, the groundstate energy approaches zero, so that the spontaneous breaking *only* happens in finite volume! See the appendix of Witten's paper for an example.

A reason to care about supersymmetry breaking is that supersymmetry may be realized in Nature. The argument above extends to the relativistic case (2.2). But if $Q |\text{vacuum}\rangle = 0$, then Q is linearly realized on 1-particle states – for each bosonic particle, there is a fermionic particle of the same mass. But this is observed not to be the case, so if supersymmetry is a symmetry of the world, it must be spontaneously broken by our vacuum.

Here is a first example of a quantum system that realizes the algebra (2.1). Take

[19] Why do we expect symmetries to be restored in finite volume (systems with a finite number of degrees of freedom)? Because there can be tunneling events which mix the degenerate states.

There are other examples where this degeneracy is not lifted. One class of examples is represented by an $\mathsf{SU}(2)$ ferromagnet, where the order parameter is a conserved quantity. Another example more similar to what happens for supersymmetry was suggested by Zhengdi Sun: take a particle on a ring with π-flux through the ring. This model has a finite number of degrees of freedom (one) so is certainly not in the thermodynamic limit, but it has two degenerate groundstates which are related by time-reversal symmetry. Here the idea is that the contributions of tunneling events in the path integral destructively interfere because of the θ term in the action: an instanton with winding number Q contributes with a phase $e^{\mathbf{i}\pi \oint \dot{\phi}} = (-1)^Q$.

In the supersymmetric case, too, there is a cancellation of instanton contributions, which can be understood in terms of fermionic collective coordinates of the instantons.

the Hilbert space to be that of a single particle on the line times a complex canonical fermion ψ (meaning that $\{\psi^\dagger, \psi\} = 1, \psi^2 = 0$),

$$\mathcal{H} = \text{span}\{|x\rangle\} \otimes \text{span}\{|-\rangle, \psi^\dagger |-\rangle \equiv |+\rangle\}.$$

This is the Hilbert space of a particle in 1d with spin one-half. Let

$$Q = \psi \left(p - \mathbf{i}W'(x)\right), \quad Q^\dagger = \psi^\dagger \left(p + \mathbf{i}W'(x)\right),$$

where $p = -\mathbf{i}\partial_x$ on wavefunctions as usual. These satisfy

$$\{Q, Q\} = 0, \{Q^\dagger, Q^\dagger\} = 0, \{Q^\dagger, Q\} = p^2 + (W'(x))^2 - [\psi^\dagger, \psi]W''(x), \tag{2.3}$$

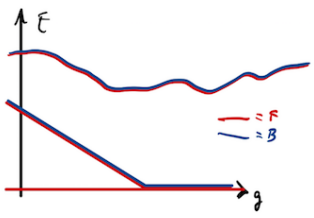
resulting in a hamiltonian $H = \frac{1}{2}\{Q, Q^\dagger\}$ the terms of which are respectively a standard kinetic term $\frac{p^2}{2}$, a potential $V = \frac{1}{2}(W'(x))^2$, and a Zeeman term. To interpret the last term in this way, regard the two states $|\pm\rangle$ (with $|+\rangle = \psi^\dagger |-\rangle$ etc) as spin eigenstates of the particle in the z-basis, so $\psi = \sigma^-, \psi^\dagger = \sigma^+$ and $[\psi, \psi^\dagger] = -\sigma^3$. Q is a symmetry generator, $[Q, H] = 0$, is automatic from (2.3).

It is easy to look for supersymmetric groundstates, much easier than solving the second-order Schrödinger equation $H\Psi(x) = E\Psi(x)$. We just have to solve the first-order equations $Q|\Psi\rangle = 0 = Q^\dagger |\Psi\rangle$. In fact, the solution is just

$$|\Psi\rangle = \int dx\, |x\rangle \left(\Psi_-(x) \otimes |-\rangle + \Psi_+(x) \otimes |+\rangle\right), \quad \Psi_\pm(x) = c_\pm e^{\pm W(x)}$$

– if these functions are normalizable.

Witten index. What happens as we vary parameters of H? (In the example, this means the shape of W; call the coupling parameter g.) States with $E > 0$ move around in pairs of opposite fermion parity. They can go to $E = 0$ in such pairs. The crucial point is that unpaired states at $E = 0$ are stuck at $E = 0$. This means that the difference in the number of odd and even $E = 0$ states, the *Witten index*



$$n_B^{E=0} - n_F^{E=0} \equiv \text{tr}(-1)^F$$

is topological, it cannot change under adiabatic variations of H.

Now I will explain the "tr" notation. Since states with $E > 0$ come in pairs with opposite fermion parity,

$$\text{tr}(-1)^F = \text{tr}(-1)^F e^{-\beta H}$$

– the states with nonzero energy cancel in pairs. But this is independent of β, so we can regard $\mathrm{tr}(-1)^F$ as a sum over all the states in the Hilbert space[20]

This is an *index* in the following sense. We can decompose the Hilbert space into odd and even eigenspaces of $(-1)^F$, $\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F$. Under this decomposition, a hermitian supercharge has the form

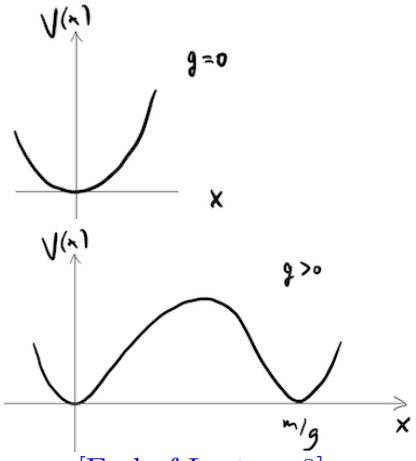
$$Q + Q^\dagger = \begin{pmatrix} 0 & M^\dagger \\ M & 0 \end{pmatrix}.$$

Odd $E = 0$ states satisfy $M^\dagger \psi_F = 0$. Even $E = 0$ states satisfy $M\psi_B = 0$. Therefore the index of the operator M is

$$\mathrm{ind}(M) \equiv \dim\ker M - \dim\mathrm{coker} M \equiv \#\{\text{solutions of } M\psi = 0\} - \#\{\text{solutions of } M^\dagger\psi = 0\} = \mathrm{tr}(-1)^F.$$

One reason to care about the Witten index is that if $\mathrm{tr}(-1)^F \neq 0$, supersymmetry is unbroken. (The converse is not true, since there could be an equal number of odd and even $E = 0$ states.) Although we defined the Witten index in the context of $D = 1$, the definition works perfectly well in QFT. You might worry about short-distance issues but you should not.

In contrast, there is a real danger at long distances in field space: $E = 0$ vacua can run off (or come in from) $x = \infty$. For example, consider the family of models with $W'(x) = mx - gx^2$. At $g = 0$ this is just a spinful particle in a harmonic oscillator potential, $V \overset{x\to\infty}{\sim} x^2$. In stark contrast, at small g, $V \overset{x\to\infty}{\sim} x^4$. A new minimum of V appears suddenly at $x \sim m/g$. If the field space is compact, there is no such issue.



[End of Lecture 8]

For purposes of generalization, an action is useful. An action which produces by Legendre transform the Hamiltonian above is

$$S[x, \psi, \bar\psi] = \int dt \left(\frac{1}{2}\dot{x}^2 + \mathbf{i}\psi^\star\dot\psi - \frac{1}{2}\left(W'(x)\right)^2 + \frac{1}{2}W''(x)[\psi^\star, \psi] \right). \tag{2.4}$$

[20] If you are happy regarding the thermal partition function $\mathrm{tr} e^{-\beta H}$ as a path integral with periodic imaginary time of period β, then the Witten index is also a natural object: it is the same path integral, but with periodic boundary conditions for the fermions – the Matsubara frequencies for the fermions are also integers.

Following Noether's theorem, this action is invariant (the variation of the Lagrangian is a total derivative) under the supersymmetry transformation generated by $Q, Q^\dagger$:

$$\begin{aligned}\delta_\epsilon x &\equiv \mathbf{i}[\epsilon Q - \bar\epsilon Q^\dagger, x] = \epsilon\psi - \bar\epsilon\psi^\star \\ \delta_\epsilon \psi &\equiv \mathbf{i}[\epsilon Q - \bar\epsilon Q^\dagger, \psi] = -\mathbf{i}\bar\epsilon(p + \mathbf{i}W'(x)) \\ \delta_\epsilon \psi^\star &\equiv \mathbf{i}[\epsilon Q - \bar\epsilon Q^\dagger, \psi] = \mathbf{i}\epsilon(p - \mathbf{i}W'(x)).\end{aligned} \tag{2.5}$$

Don't trust my signs – I am rusty with this stuff.

Side remark on superspace: these transformations can be regarded as translations on *superspace*, with coordinates $(t, \theta, \bar\theta)$. Since grassmann variables have $\theta^2 = 0$, the general real function on superspace is

$$X(t, \theta, \bar\theta) = x(t) + \theta\psi(t) - \bar\theta\psi^\star(t) + \theta\bar\theta F,$$

a superfield. If we write the action as an integral over all of superspace $S = \int dt d\theta d\bar\theta L(X(t, \theta, \bar\theta))$, it is automatically supersymmetry invariant. (The supersymmetry transformation of X is $X \to X + \epsilon(\partial_\theta + \mathbf{i}\bar\theta\partial_t)X - \bar\epsilon(\partial_{\bar\theta} - \mathbf{i}\theta\partial_t)X$ – basically a translation in θ.) Note that $\int d\theta d\bar\theta \left(+...\theta\bar\theta Z\right) = Z$ integration just picks out the highest component. For example, choosing $L = \frac{1}{2}DX\bar DX + W(X)$, with $D = \partial_\theta - \mathbf{i}\bar\theta\partial_t$ a covariant derivative, gives

$$\int d\theta d\bar\theta L(X(t, \theta, \bar\theta)) = \frac{1}{2}F^2 - W'F + \frac{1}{2}\dot x^2 + \mathbf{i}\psi^\star\dot\psi + \frac{1}{2}W''(x)[\psi^\star, \psi].$$

The field F is an *auxiliary field* – it has no derivatives and can be integrated out by solving its equation of motion $F = W'$; plugging back in gives back the action (2.4). This technology is very useful if you find yourself needing to write down supersymmetric actions, but I don't want to say more about it.

Supersymmetry and cohomology, first pass. Part of the supersymmetry algebra was $Q^2 = 0$. That means we can regard

$$\cdots \mathcal{H}_B \overset{Q}{\to} \mathcal{H}_F \overset{Q}{\to} \mathcal{H}_B \overset{Q}{\to} \cdots \tag{2.6}$$

as a complex, and consider its cohomology:

$$H^B(Q) \equiv \frac{\ker Q : \mathcal{H}^B \to \mathcal{H}^F}{\text{im } Q : \mathcal{H}^F \to \mathcal{H}^B}, \quad H^F(Q) \equiv \frac{\ker Q : \mathcal{H}^F \to \mathcal{H}^B}{\text{im } Q : \mathcal{H}^B \to \mathcal{H}^F}.$$

But now consider some nonzero-energy eigenstate $|\alpha\rangle$ of the Hamiltonian H. If it is Q-closed, act on it with the operator $1 = (QQ^\dagger + Q^\dagger Q)/(2E)$. This shows that

it is also Q-exact[21]. We conclude that $H^{B/F}(Q)$ is the space of bosonic/fermionic supersymmetric groundstates.

Sometimes there is a finer grading on the complex. This happens if there is a conserved $\mathsf{U}(1)$ charge F, $[F, H] = 0$, with F taking integer values, satisfying $[F, Q] = Q$. For some reason such a symmetry, under which the supercharge is charged, is called an *R-symmetry*. Then the complex (2.6) can be refined to

$$\dots \mathcal{H}^{p-1} \xrightarrow{Q} \mathcal{H}^{p} \xrightarrow{Q} \mathcal{H}^{p+1} \xrightarrow{Q} \dots$$

where $\mathcal{H}^p$ is the eigenspace of F with eigenvalue p. In this case, we can define a cohomology group for each p:

$$H^p(Q) \equiv \frac{\ker Q : \mathcal{H}^p \to \mathcal{H}^{p+1}}{\text{im } Q : \mathcal{H}^{p-1} \to \mathcal{H}^p}.$$

The Witten index is the Euler character of the complex

$$\text{tr}(-1)^F = \sum_{p \in \mathbb{Z}} (-1)^p \dim H^p(Q).$$

It is called *co*homology rather than homology because Q *increases* the index p. More significantly we will see that there is some reversal of arrows relative to homology.

So whenever we can define a nilpotent Q (*i.e.* $Q^2 = 0$) we can construct cohomology groups. You can expect that these groups are invariant under continuous deformations (*i.e.* topological), but we haven't shown that yet.

Given a Lorentz-invariant supersymmetric QFT in D dimensions, we can try to put it on a D-manifold X. For general curved X, there will be no global spinors and hence no Q – supersymmetry will be broken by the curvature. Witten discovered a procedure called *topological twist* to use an R-symmetry to change the spin of the supercharges, so that one of them is a scalar and hence exists on any manifold. The cohomology of this Q then produces invariants of X. To see that this data is topological, one observes that the stress tensor $T_{\mu\nu}$ (which encodes the dependence on the metric, since $T_{\mu\nu} \propto \frac{\delta S}{\delta g_{\mu\nu}}$) is of the form $T_{\mu\nu} = \{Q, \lambda_{\mu\nu}\}$ for some operator λ. This means it is Q-exact and so the Q-cohomology is independent of the metric[22].

Below we study a less fancy and more direct way to make invariants of a manifold using supersymmetry.

[21] The operator $\frac{Q^\dagger}{2E}$ is playing the role of a *homotopy operator* here. In general, if we can find an operator $K : \Omega^p \to \Omega^{p-1}$ such that $1 = dK + Kd$, then the cohomology of d is trivial; K is called a homotopy operator.

[22] To be more explicit, it means, for example, that the change in the partition function under a

By the way, the supersymmetric quantum mechanics algebra (2.1) can be realized in many-body lattice models of itinerant fermions. An interesting series of papers on such models has been written starting with this paper of Fendley, Schoutens and de Boer. The simplest possibility is to take the Hilbert space to represent the free-fermion algebra $\{c_i, c_j^\dagger\} = \delta_{ij}$, and consider $Q = \sum_i c_i^\dagger$, but this leads to a trivial hamiltonian which is just a c-number. Instead they study a system of hard-core fermions, where states with fermions occupying neighboring sites are removed from the Hilbert space, and let $Q = \sum_i c_i^\dagger P_i$, where $P_i \equiv \prod_{j \text{ next to } i}(1 - c_j^\dagger c_j)$ is the projector requiring neighboring sites to be empty. This produces an interesting Hamiltonian and a nonzero Witten index related to some interesting combinatorics problems.

Note that on a Hilbert space which is just a finite tensor product of finite-dimensional Hilbert spaces and free fermion operators, the Witten index must vanish. This is because it is the index of a finite-dimensional square matrix. It is square because for each bosonic state in such a Hilbert space, there is a fermionic state. Thanks to Bowen Shi for pointing this out.

change in the action of the form $\Delta S = \int \{Q, \lambda\}$ is

$$\delta Z = \int D\phi e^{-S} \{Q, \lambda\} = \delta_{\text{susy}} \left(\int D\phi e^{-S} \right) = 0$$

since the measure and the action are invariant under a supersymmetry transformation. More generally, any correlation function of Q-closed observables $[Q, \mathcal{O}] = 0$ will also be unchanged.

Supersymmetric nonlinear sigma model. Choose a smooth manifold $\mathcal{M}$ with Riemannian metric γ_{ij}. A nonlinear sigma model (NLSM) is a field theory of maps from a base space (which we'll take to be flat) into a target space, $\mathcal{M}$. In terms of a set of coordinates ϕ^i on $\mathcal{M}$, we can define an action for such a field theory by

$$S[\phi,\psi,\bar\psi] = \int d^D x \left(\frac{1}{2}\gamma_{ij}(\phi)\left(\partial_\mu\phi^i\partial^\mu\phi^j + \bar\psi^i \mathbf{i}\gamma^\mu D_\mu\psi^j\right) + \frac{1}{8}R_{ijkl}(\phi)\bar\psi^i\psi^k\bar\psi^j\psi^l\right). \quad (2.7)$$

The ψ^i are a collection of $n \equiv \dim\mathcal{M}$ majorana fermions[23]. Here $R_{ijkl}(\phi)$ is the Riemann curvature of the metric γ_{ij}. I emphasize that we're studying a field theory living in D-dimensional flat space. The n scalar fields ϕ^i are coordinates on $\mathcal{M}$, and the metric data γ_{ij} play the role of coupling constants. In the case where $\mathcal{M}$ is a symmetric space G/H for two Lie groups G, H, such non-linear sigma models arise frequently as effective field theories of phases with spontaneously-broken continuous symmetries, as we'll discuss further next quarter. This action (2.7) is more general in that $\mathcal{M}$ need not be a symmetric space; it is more specific in that it is supersymmetric.

The field theory with action (2.7) is supersymmetric for $D \leq 3$[24]. What is its Witten index? Or more ambitiously how many supersymmetric groundstates does it have? We'll answer this question in two different ways which illuminate each other, and lead us to discover the important mathematical ideas in the heading of this section.

Classically, any constant configuration $\phi(t)$ has zero energy. And any dependence on space $\phi(x)$ costs energy going like $1/V$, the volume of the base space. So we can lower the energy by keeping ϕ uniform in space. Because of this, we can capture the essential physics of the supersymmetric groundstates by *dimensionally reducing* the system (2.7) to $0+1$ dimensions. That is: plug in a configuration $\phi(t,x) = \phi(t)$, and do the integrals over x. This gives[25]

$$S[\phi,\psi,\bar\psi] = \frac{V}{2}\int dt \left(\gamma_{ij}(\phi)\dot\phi^i\dot\phi^j + \gamma_{ij}(\phi)\bar\psi^i\mathbf{i}\gamma^0 D_0\psi^j + \frac{1}{4}R_{ijkl}(\phi)\bar\psi^i\psi^k\bar\psi^j\psi^l\right). \quad (2.8)$$

[23]The covariant derivative on the fermion field is

$$D_\mu\psi^i \equiv \partial_\mu\psi^i + \Gamma^i_{jk}\partial_\mu\phi^j\psi^k,$$

where Γ^i_{jk} is the Christoffel connection for γ_{ij}. This says that under coordinate transformations ψ^i transforms like a tangent vector (or that $\psi_i \equiv \gamma_{ij}\phi^j$ transforms as a cotangent vector).

[24]To be Lorentz invariant and supersymmetric in higher dimensions puts constraints on the form of $\mathcal{M}$. This is because a spinor has more components in higher dimensions, so supersymmetry is more constraining. We will restrict ourselves to two-component majorana spinors. So here we are studying $D = 3, \mathcal{N} = 1$ supersymmetry. The nomenclature $\mathcal{N}$ counts the number of minimal-spinor supercharges in D spacetime dimensions. 4D $\mathcal{N} = 1$ supersymmetry requires four real supercharges.

[25]Alternatively, this action follows directly from the 0+1d superspace Lagrangian $\int d\theta d\bar\theta\gamma_{ij}(\Phi)D\Phi^i\bar D\Phi^j$, with $\Phi^i = \Phi^i(t,\theta,\bar\theta) = \phi^i + \theta\psi^i - \bar\theta\bar\psi^i + \theta\bar\theta F^i$ a real superfield as above.

Quantum mechanically the story is more interesting: the wavefunction spreads over $\mathcal{M}$. To see what happens, choose a basis for the gamma matrices ($\gamma^0 = \sigma^z$) so that the Majorana condition is $\psi = \begin{pmatrix} \psi \\ \psi^\star \end{pmatrix}$. Doing the Legendre transformation, we have $p_i = \frac{\partial L}{\partial \dot{\phi}^i} = -\mathbf{i} D_{\phi^i}$ acts as a covariant derivative on wavefunctions, and $\{\psi_i, \psi_j\} = 0, \{\psi_i, \psi_j^\star\} = \gamma_{ij}(\phi)$. The supercharges are

$$Q = \mathbf{i} \sum_i \psi^{\star i} p_i, \quad Q^\star = -\mathbf{i} \sum_i \psi^i p_i$$

and satisfy $Q^2 = (Q^\star)^2 = 0, \{Q, Q^\star\} = 2H$. (The supersymmetry algebra fixes the additive normalization of H and hence removes any possible operator-ordering ambiguity.)

By the way, this action (2.8) has a symmetry under

$$\psi^i \to e^{-\mathbf{i}\alpha} \psi^i, \bar{\psi}^i \to e^{\mathbf{i}\alpha} \bar{\psi}^i. \tag{2.9}$$

The associated conserved Noether charge is $F = \gamma_{ij} \bar{\psi}^i \psi^j$, the fermion number. Quantumly, we have $[H, Q] = 0$. (Note that this is consistent with $[F, Q] = Q, [F, Q^\star] = -Q^\star$.) F generates the transformation (2.9) in the sense that $\delta_\alpha \mathcal{O} = \mathbf{i}\alpha [F, \mathcal{O}]$.

Now let's build the Hilbert space. Start with a vacuum $|0\rangle$ of all the fermions, satisfying $\psi^i |0\rangle = 0, \forall i = 1..n$. Let's declare that this has fermion number 0: $F|0\rangle = 0$. (That is, I've fixed the ordering ambiguity in F so that quantumly $F = \gamma_{ij} \psi^{\star i} \psi^j$.) We can make many states from this vacuum by multiplying by a function of the bosonic variables $A(\phi)$. From the fermionic vacuum, we can make states with one fermion:

$$A_i(\phi) \psi^{\star i} |0\rangle$$

or two fermions:

$$A_{ij}(\phi) \psi^{\star i} \psi^{\star j} |0\rangle, \quad A_{ij} = A_{ji}.$$

we can continue until we get to states with $n = \dim \mathcal{M}$ fermions:

$$A_{i_1 \cdots i_n}(\phi) \psi^{\star i_1} \psi^{\star i_2} \cdots \psi^{\star i_n} |0\rangle,$$

which state is proportional to the 'plenum' – the state with one of each type of fermion, annihilated by all the fermionic creation operators.

The antisymmetric objects $A_{i_1 \cdots i_p}(\phi)$ are the components of *differential forms*; with p indices they are called p-forms. Let's denote by $\Omega^p(\mathcal{M})$ the space of such p-forms (we will impose some regularity conditions later). Differential forms are usually written as $A_{i_1 \cdots i_p} d\phi^1 \wedge \cdots d\phi^p$. Here $d\phi^i$ means the same thing as $\psi^{i\star}$ – it's an anticommuting

object which transforms under a coordinate change $\phi^i \to x^I(\phi)$ like $d\phi^i = dx^I \frac{\partial \phi^i}{\partial x^I}$. We conclude that the Hilbert space of the NLSM is

$$\mathcal{H} = \bigoplus_{p=0}^{n} \Omega^p(\mathcal{M})$$

the space of all p-forms on $\mathcal{M}$. The subspace Ω^p is the states with fermion number $F = p$. Upon adding just a little more structure this is called the *de Rham complex* of $\mathcal{M}$.

To see this extra structure, we ask: how does $Q = \psi^{\star j} p_j$ act on $\mathcal{H}$? The observables act on $\mathcal{H}$ as follows[26]

$$p_i = -\mathbf{i} D_{\phi_i}, \quad \psi^{\star i} = d\phi^i \wedge, \quad \psi^i = \gamma^{ij} i_{\partial_{\phi^i}}$$

where $\wedge$ is the wedge product on forms and $i_v : \Omega^p \to \Omega^{p-1}$ is the map which contracts with v on the first index: If $A = A_{ijk} d\phi^i \wedge d\phi^j \wedge d\phi^k$, then $i_v A = v^i A_{ijk} d\phi^j \wedge d\phi^k$. So on a state with fermion number q,

$$\begin{aligned} Q\left|A_q\right\rangle &\equiv Q\left(A_{i_1 \cdots i_q}(\phi) \psi^{\star i_1} \cdots \psi^{\star i_q} \left|0\right\rangle\right) && (2.10) \\ &= D_{\phi^j} A_{i_1 \cdots i_q}(\phi) \psi^{\star j} \psi^{\star i_1} \cdots \psi^{\star i_q} \left|0\right\rangle && (2.11) \\ &= \frac{1}{(q+1)!} \left(\partial_{\phi^{i_1}} A_{i_2 \cdots i_{q+1}}(\phi) \pm \text{perms}\right) \psi^{\star i_1} \cdots \psi^{\star i_{q+1}} \left|0\right\rangle && (2.12) \\ &= \left|dA_q\right\rangle . \end{aligned}$$

Here $d : \Omega^q \to \Omega^{q+1}$ is the exterior derivative. Note that the covariant derivative doesn't matter because the extra bits cancel in the antisymmetrization. The fact that $d^2 = 0$ is usually an annoying thing to show; here it is automatic from the fact that $Q^2 = 0$. This makes the Hilbert space of the NLSM, $\Omega_\bullet$, into a complex, the *de Rham* complex.

[End of Lecture 9]

In contrast, $Q^\star$ removes a fermion of type i and differentiates in the i direction – it is a divergence operator. More precisely,

$$Q^\star \left|A\right\rangle = \gamma^{j i_q} D_{\phi^j} A_{i_1 \cdots i_q}(\phi) \psi^{\star i_2} \cdots \psi^{\star i_q} \left|0\right\rangle = \left|d^\dagger A\right\rangle .$$

When $\mathcal{M}$ is oriented, this operation is the adjoint of d with respect to the inner product on forms defined by

$$(\omega_1, \omega_2) = \int_{\mathcal{M}} \omega_1 \wedge \star \omega_2$$

[26] If you have not ever seen differential forms before, take a peek at the beginning of §2.2 to get the notation.

where $\star$ is the Hodge dual operation taking p-forms to $(n-p)$-forms[27] ; in components it is

$$(\star\omega)_{i_{p+1}\cdots i_n} = \frac{\sqrt{\gamma}}{p!}\epsilon_{i_1\cdots i_n}\omega^{i_1\cdots i_p}.$$

(Indices are raised using the metric, $\omega^i \equiv g^{ij}\omega_j$.) A good coordinate-free way to think about the definition of the Hodge star is that for any η:

$$\eta^\star \wedge \star\omega = (\eta,\omega)\mathrm{vol}$$

where vol is the volume form on $\mathcal{M}$. We can find a useful expression for $d^\dagger$ in terms of $\star$ just using the definition of adjoint:

$$d^\dagger = (-1)^x \star d\star$$

where x depends on $\dim\mathcal{M}$ and the degree of the form which you can figure out. This follows from Stokes' theorem, $0 = \int_\mathcal{M} d(\omega_1 \wedge \star\omega_2)$ if $\mathcal{M}$ is closed (has no boundary). A hint is that $\star^2 = (-1)^{p(n-p)}$. Note that $(d^\dagger)^2 = 0$, too.

Then

$$H = \frac{1}{2}\{Q^\star, Q\} = \frac{1}{2}\left(dd^\dagger + d^\dagger d\right) \equiv \triangle,$$

the Laplace-Beltrami operator – the Laplacian on forms. A supersymmetric ground-state has a wavefunction which is a *harmonic* p-form, that is, it is annihilated by $\triangle$. The number of these is $b^p(\mathcal{M})$, the pth Betti number. (We'll see that this definition agrees with our previous definition in terms of homology.) Since on a p-form state, $F\left|A^{(p)}\right\rangle = p\left|A^{(p)}\right\rangle$, we have

$$\mathrm{tr}(-1)^F = \sum_{p=0}^{n}(-1)^p b^p(\mathcal{M}) = \chi(\mathcal{M})$$

the Euler characteristic of $\mathcal{M}$.

Suppose that $\mathcal{M}$ has an isometry k – this means there is an operator K on the Hilbert space of the NLSM with $[H,K] = 0 = [Q,K]$. K acts on each $\Omega^p(\mathcal{M})$ independently, and we can diagonalize it on the harmonic forms. Let $b^p_{(\kappa)}$ be the number of harmonic p-forms with eigenvalue κ of K ($KA_{i_1\cdots i_p} = \kappa A_{i_1\cdots i_p}$). Then

$$\mathrm{tr}(-1)^F K = \sum_{p=0}^{\dim\mathcal{M}}(-1)^p\kappa b^p_{(\kappa)} \equiv \mathrm{Lef}(K)$$

[27]Note that if we were considering differential forms with complex coefficients, we would define the inner product to be $(\omega_1,\omega_2) = \int_\mathcal{M}\omega_1^\star \wedge \star\omega_2$. Since the NSLM has a time-reversal symmetry, the eigenfunctions of its Hamiltonian can be taken to be real, and we don't need to worry about this right now.

is called the *Lefschetz number* of the isometry K. Nonzero Lefschetz number for some isometry also means unbroken supersymmetry, since it requires there to be a supersymmetric groundstate (*i.e.* at least one harmonic form).

Back to the exterior derivative. Since we already know $Q^2 = 0$, $\Omega^\bullet$ with the exterior derivative forms a complex: since $d^2 = 0$, the image of $d : \Omega^{p-1} \to \Omega^p$ is a subspace of the kernel of $d : \Omega^p \to \Omega^{p+1}$. So we can define the cohomology of the complex

$$H^p(\mathcal{M}) \equiv \frac{\ker\, d : \Omega^{p-1} \to \Omega^p}{\mathrm{im} d : \Omega^p \to \Omega^{p+1}} \ ,$$

the *de Rham cohomology* of $\mathcal{M}$.

The Hodge star operator gives a linear map $\Omega^p \to \Omega^{n-p}$. What is this operation physically? It's the particle-hole transformation that replaces each filled fermion level with an empty one, and vice-versa. As we'll see, this induces an isomorphism on the cohomology.

To be more explicit, notice that the NLSM in $D = 2$ has a discrete *chiral symmetry*, $\psi \to \gamma_5 \psi$ where $\gamma_5 = \prod_i^D \gamma_i$ is the product of all the gamma matrices. In the Weyl basis (where γ_5 is diagonal) with $Q = \begin{pmatrix} Q_+ \\ Q_- \end{pmatrix}$, this acts by $Q_\pm \to \pm Q_\pm$, that is, $\gamma_5 Q_\pm = \pm Q_\pm \gamma_5$. For Q_-, this is just like the algebra $\{(-1)^F, Q\} = 0$, which means just as before that $E > 0$ states are paired with one of each chirality, while $E = 0$ states can be chiral. Therefore $\mathrm{tr}\gamma^5$ is *also* topological, and also only gets contributions from $E = 0$ states. (Also if $\mathrm{tr}\gamma^5 = 0$, then supersymmetry is unbroken.) Given an isometry of $\mathcal{M}$, $\mathrm{tr}\gamma_5 K$ is also topological.

To be more explicit, the majorana basis operators $\psi_i^\star, \psi_i$ are γ^0 eigenstates, but $\{\gamma^5, \gamma^0\} = 0$, so $\gamma^5 : \psi^\star \leftrightarrow \psi$ acts by particle-hole transformation. This means furthermore that it takes the empty state $|0\rangle$ to the completely filled state $|1\rangle \equiv \psi^{1\star} \cdots \psi^{n\star} |0\rangle$, since it takes the condition $\psi |0\rangle = 0$ to the condition $\psi^\star |1\rangle = 0$. On a q-form state, then,

$$\gamma^5 \psi^{i_1\ \star} \cdots \psi^{i_q\ \star} |0\rangle = c\epsilon_{i_1 \cdots i_n} \psi^{i_{q+1}\ \star} \cdots \psi^{i_n\ \star} |0\rangle$$

it gives a $(n-q)$-form, where c is chosen so that $(\gamma^5)^2 = 1$. To compute $\mathrm{tr}\gamma^5$ then, we decompose $\Omega^q \oplus \Omega^{n-q}$ into eigenstates of γ^5 (the eigenvalues are ± 1). Calling the number of $\pm$ eigenstates $b_\pm^q$, we have

$$\mathrm{tr}\gamma_5 = \sum_{q=0}^{\lfloor n/2 \rfloor} (b_+^q - b_-^q) = \mathrm{sign}\mathcal{M},$$

the *Hirzebruch signature*.

2.2 Differential forms consolidation

[Bott and Tu, early sections, Polchinski §B.4, Nash and Sen §2.3] A p-form on a smooth manifold $\mathcal{M}$ is made from a completely-antisymmetric p-index tensor $A_{i_1\cdots i_p}$

$$A = \frac{1}{p!} A_{i_1\cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}. \tag{2.13}$$

Here dx^i are coordinate differentials, *i.e.* a basis of cotangent vectors associated with a set of coordinates on $\mathcal{M}$ ($dx^i(\partial_{x^j}) = \delta^i_j$)[28]. The set of p-forms on $\mathcal{M}$ (perhaps with some smoothness and integrability properties) is a vector space $\Omega^p(\mathcal{M})$. The coefficients can be from any field, but we will think mostly about $\mathbb{R}$. These vector spaces enjoy a product, the wedge product, which we've already written in (2.13). A good way to define the wedge product is in terms of the coordinate differentials: the wedge product $dx^{i_1} \wedge dx^{i_p}$ is completely antisymmetric and separately linear in each factor.

$$(A_p \wedge B_q)_{i_1\cdots i_{p+q}} = \frac{(p+q)!}{p!q!} A_{[i_1\cdots i_p} B_{i_{p+1}\cdots i_{p+q}]}$$

where the square brackets indicate antisymmetrization of indices, *i.e.* average over permutations weighted by $(-1)^\sigma$, the sign of the permutation. The wedge product is graded antisymmetric, meaning

$$A_p \wedge B_q = (-1)^{pq} B_q \wedge A_p.$$

The exterior derivative is a linear differential operator $d : \Omega^p \to \Omega^{p+1}$ defined by $d = dx^\nu \wedge \partial_\nu$, or more explicitly

$$(dA_p)_{i_1\cdots i_{p+1}} = (p+1)\partial_{[i_1} A_{i_2\cdots i_{p+1}]}.$$

It satisfies $d^2 = 0$ by equality of mixed partials: $[\partial_i, \partial_j] = 0$ on smooth functions.

[28] In case you are not familiar with these notions, here is brief recap to explain the notation. A tangent vector on a manifold $\mathcal{M}$ at a point p, $v \in T_p\mathcal{M}$, is of the form $v = v^i \frac{\partial}{\partial x^i}$ in terms of some local coordinates x^i. It is a differential operator in the following sense. For any function f and for any curve $x^i(t)$ through p, the rate of change of f along the curve is

$$\frac{df}{dt}|_p = \frac{dx^i}{dt} \frac{\partial}{\partial x^i} f|_p.$$

Since this is true for any $\frac{dx^i}{dt} = v^i$ and any point p and any f, the important part is the $\frac{\partial}{\partial x^i}$.

The second ingredient is that a cotangent vector is an element of the dual vector space $T^\star_p\mathcal{M}$ – an object which eats a tangent vector and gives a number. A basis for such things is given by the coordinate differentials dx^i, which satisfy $dx^i(\partial_{x^j}) = \delta^i_j$.

The main job in life of a p-form on $\mathcal{M}$ is to be integrated over p-dimensional submanifolds $X_p \subset \mathcal{M}$: $\int_{X_p} A_p$ is a coordinate-invariant number. Stokes' theorem says

$$\int_{X_p} dA_{p-1} = \int_{\partial X_p} A_{p-1}.$$

So far, the metric has not been involved. The Hodge star operation requires more structure.

To get some familiarity with the above language let's think about the case $\mathcal{M} = \mathbb{R}^3$ for a moment. Then $\Omega^0(\mathbb{R}^3)$ and $\Omega^3(\mathbb{R}^3)$ are both spanned by ordinary functions, while $\Omega^1(\mathbb{R}^3)$ and $\Omega^2(\mathbb{R}^3)$ are both spanned by vector fields – functions with a single index. On functions, $df = \partial_i f dx^i$. On 1-forms,

$$d(f_i dx^i) = (\partial_y f_z - \partial_z f_y)\, dy{\wedge}dz + (\partial_x f_y - \partial_y f_x)\, dx{\wedge}dy + (\partial_z f_x - \partial_x f_z)\, dz{\wedge}dx = \frac{1}{3!}\epsilon_{ijk}\partial_i f_j \epsilon_{ilm} dx^l{\wedge}dx^m.$$

On 2-forms

$$d(f_x dy \wedge dz + f_y dz \wedge dx + f_z dx \wedge dy) = \partial_i f_i dx \wedge dy \wedge dz.$$

So this accounts for all the classic operations of vector calculus:

$$d(\text{0-form}) = \text{gradient}, \quad d(\text{1-form}) = \text{curl}, \quad d(\text{2-form}) = \text{divergence}.$$

A classical physics context where one encounters a cohomological question is in fluid dynamics: given a vector field, say describing the flow of a fluid on some space X, when is it the gradient of a well-defined function on X? Or in electrostatics on some space X, an allowed electric field configuration must be the gradient of a scalar potential on $X \backslash$ the locations of the charges.

Here are some familiar statements written in the above language. The electromagnetic field is a 2-form on spacetime, $\mathbb{R}^4$ (I use ijk for spatial indices and $\mu\nu$ for spacetime indices):

$$F = dA = E_i dx^i{\wedge}dt + B_x dy{\wedge}dz + B_y dz{\wedge}dx + B_z dx{\wedge}dy = E_i dx^i{\wedge}dt + B_i dx^j{\wedge}dx^k \epsilon_{ijk}/2.$$

The dual field strength, in flat spacetime, is

$$\star F = -B_i dx^i \wedge dt + E_i dx^j \wedge dx^j \epsilon_{ijk}/2.$$

Maxwell's equations (away from charges) are $dF = 0, \star dF = 0$. The first is the Bianchi identity, which is automatic if A is well-defined, while the second is the equations of motion associated with the Maxwell action

$$S[A] = -\frac{1}{2e^2}\int F \wedge \star F = -\frac{1}{4e^2}\int d^D x \sqrt{g} F_{\mu\nu} F^{\mu\nu}.$$

Maxwell's equations say that both F and $\star F$ represent cohomology classes in spacetime (minus the locations of charges). Consider the simplest possible nontrivial example of a point charge at rest at the origin of coordinates. The field strength is $F = q\frac{dr\wedge dt}{r^2} = -qd\left(\frac{dt}{r}\right)$, well-defined in $\mathcal{M} \equiv \mathbb{R}^4 \setminus \mathbb{R}_t$, where we remove the origin at all times. F is clearly exact and hence represents the trivial class in $H^2(\mathcal{M})$. However, using $dr = x^i dx^i/r$ and *e.g.* $\star dx \wedge dt = dy \wedge dz$, we have

$$\star F = q\frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{r}.$$

This is a nontrivial element of $H^2(\mathcal{M})$, as you can see by integrating it over the appropriate Gaussian surface, *i.e.* the 2-sphere surrounding the particle to get $\int_{S^2} \star F = 4\pi q$. So we can interpret the charge of the particle as an element of the cohomology group $H^2(\mathcal{M})$.

To include charges more explicitly, we can add a term to the action of the form $e\oint_C A$ where C is the worldline of the charges. This is called 'minimal coupling' in the sense that a 1-form really wants to be integrated over a 1-dimensional subspace and no extra ingredients are required to do it. We could in addition include kinetic terms for the charges to make them dynamical; these terms involve the metric.

Abelian p-form gauge fields. The usual Maxwell field strength is $F_2 = dA_1$. It is invariant under gauge transformations $A_1 \to A_1 + d\lambda_0$ since $d^2 = 0$. A large family of useful generalizations of this is p-form abelian gauge fields:

$$F_{p+1} = dA_p, \quad \delta A_p = d\lambda_{p-1}.$$

Again the field strength is gauge invariant by $d^2 = 0$. An action is

$$S[A] = -\frac{1}{2g^2}\int F_{p+1} \wedge \star F_{p+1} \propto -\int \frac{\sqrt{g}}{(p+1)!} F_{\mu_1\cdots\mu_{p+1}} F^{\mu_1\cdots\mu_{p+1}}.$$

For $p = 0$, this is just $L = -\frac{1}{2}(\partial\phi)^2$, a massless scalar. The equations of motion are

$$0 = \frac{\delta S}{\delta A(x)} \propto d \star F(x).$$

In flat spacetime of enough dimensions, we can fourier transform and see that this describes a massless excitation with a spin that depends on p.

The analog of minimal coupling for a p-form gauge field is to a p-dimensional worldvolume:

$$\Delta S = e\int_{X_p} A_p$$

– this is the worldvolume of a $(p-1)$-brane, an object with $p-1$ spatial dimensions.

In spacetime dimensions $D = 2 \bmod 4$, it is consistent with the equations of motion to impose a (anti-)self-duality equation: $F_{D/2} = \pm \star F_{D/2}$, which gets rid of half the degrees of freedom. In $D = 2$ this describes a chiral scalar. In $D = 4$, this describes only one circular polarization of the photon. The Maxwell-like action, however, vanishes when imposing this condition and there is no covariant action principle for such fields. They also enjoy various gravitational anomalies.

The duality operation $dA_p = \star dA^\vee_{D-p-2}$ exchanges the Bianchi identity and the equation of motion. We can learn something by giving a path-integral derivation of the duality. The partition function (euclidean) is

$$\int [dA] e^{-\frac{1}{2g^2}\int dA\wedge\star dA} = \int \frac{[dAdBdA^\vee]}{(\star)} e^{-\frac{1}{2g^2}\int (F-B)\wedge\star(F-B)+\mathbf{i}\int B\wedge dA^\vee} = \int [dA^\vee] e^{-\frac{g^2}{2}\int dA^\vee\wedge\star dA^\vee}.$$

In the first step, we introduce $A^\vee$ as a Lagrange multiplier to impose $dB = 0$ (and $\oint_X B \in 2\pi\mathbb{Z}$ for all 2-cycles X). The middle object has a new redundancy under

$$(\star) \qquad A \to A + \Lambda, B \to B + d\Lambda \tag{2.14}$$

for an arbitrary p-form Λ; when $dB = 0$ (and B has integral periods) this can be used to set $B = 0$, giving back the first expression. In the second step, we set $A = 0$ and do the gaussian integral over B, producing a nontrivial action for $A^\vee$. (The same manipulation works for other gauge-invariant observables.) Notice that the coupling constant gets inverted. A simple example of this is $p = 0, D = 2$ which relates a compact scalar of radius $R = \frac{1}{g}$ to one with radius R – this is called T-duality. The manipulation above is described in §2.2 here for $D = 4, p = 1$, and here for $D = 2, p = 0$.

In some dimensions, there are other terms we can add to the action without any further ingredients, such as Chern-Simons terms:

$$S_{\mathrm{CS}}[A] \propto \int A \wedge F \wedge F \cdots .$$

If the ranks of the forms add up to D this integral makes sense, doesn't involve the metric, and is gauge-invariant up to a boundary term. In $D = 2 + 1$, the CS term $\int A \wedge F$ is gaussian in A and hence the theory with this term in the action is still solvable. It is also has fewer derivatives than the Maxwell term and hence is more relevant in the sense of the renormalization group. We'll come back to this later in §4.

In the 1-form case, we can study non-abelian gauge fields, which are Lie-algebra-valued 1-forms (*i.e.* the coefficients are matrices and so is the gauge parameter λ_0):

$$F_2 = dA_1 - \mathbf{i}A_1 \wedge A_1 = dA_1 - \mathbf{i}A_1^2, \quad \delta A_1 = d\lambda_0 - \mathbf{i}A_1\lambda_0 + \mathbf{i}\lambda_0 A_1.$$

There isn't a very good generalization of this to $p \neq 1$.

Hodge duality on forms. Let's uncover the close relation between cohomology and harmonic forms. First, on a closed manifold, any harmonic form is automatically closed and coclosed. This is because by integration by parts $(\omega, \triangle\omega) = \| d\omega \|^2 + \| d^\dagger \omega \|^2$ is a sum of positive terms which must vanish independently if $\triangle\omega = 0$. The Hodge theorem[29] says that there is a unique harmonic representative of each cohomology class. The essential ingredient[30] is that any differential form has an orthogonal decomposition

$$\omega = d\alpha + d^\dagger \beta + \gamma \tag{2.15}$$

with γ harmonic. (I think I will not prove this statement. It is proved in the reference in the previous footnote by constructing a Green's function for $\triangle$. It involves some analysis.) The fact that the decomposition is orthogonal is automatic since *e.g.* $(d\alpha, d^\dagger\beta) = (d^2\alpha, \beta) = 0$.

Now if ω is closed, we have

$$\| d^\dagger \beta \|^2 = (d^\dagger\beta, d^\dagger\beta) \overset{\text{orthogonal}}{=} (d^\dagger\beta, \omega) = (\beta, d\omega) = 0.$$

Therefore, if ω is closed, it must be of the form $\omega = d\alpha + \gamma$, and therefore $[\omega] = [\gamma]$, with γ harmonic. Note that the harmonic representative minimizes the L_2 norm, $\| \eta \|^2 \equiv \int \eta^\star \wedge \star\eta > 0$ (the norm induced by the inner product on forms) within the cohomology class.

We conclude that there is an isomorphism between the de Rham cohomology of $\mathcal{M}$ and the space of harmonic forms on $\mathcal{M}$. You might have thought that the definition of harmonic forms involves all this extra data involving the metric, but this isomorphism shows that at least their multiplicity is independent of that extra information.

We can see directly that the supersymmetric groundstates of the NLSM are the harmonic representatives, since they satisfy $Q\Psi = 0 = Q^\dagger\Psi$.

Now notice that the Hodge star sends harmonic forms to harmonic forms, since $\triangle\star = \star\triangle$. Therefore it is an isomorphism on cohomology.

[29] Some nice clear notes on this subject are this and this.

[30] Actually, this Hodge decomposition is not necessary to show that there is an isomorphism between cohomology classes and harmonic forms. First, to see that each cohomology class has a harmonic representative: since $[Q, H] = 0$, each cohomology class has a representative by an eigenstate of H. But as we saw, if $H|\psi\rangle = E|\psi\rangle$, with $E > 0$, then $K \equiv \frac{Q^\dagger}{2E}$ is a homotopy operator. That is, acting with the supersymmetry algebra on $|\psi\rangle$, if $|\psi\rangle$ satisfies $Q|\psi\rangle = 0$, then $|\psi\rangle = QK|\psi\rangle$ is exact.

Conversely, if $H|\psi\rangle = 0$, so the wavefunction of $|\psi\rangle$ is a harmonic form, then $0 = \langle\psi|H|\psi\rangle = \| Q|\psi\rangle \|^2 + \| Q|\psi\rangle \|^2$ and in particular $|\psi\rangle$ represents a cohomology class. Furthermore if $|\psi\rangle \neq 0$, then this class is nontrivial, *i.e.* $|\psi\rangle$ is not exact. This is because if $|\psi\rangle = Q|\alpha\rangle$, then since $[H, Q] = 0$, then $H|\alpha\rangle = 0$, too. But this implies $Q|\alpha\rangle = 0$, so that $|\psi\rangle = Q|\alpha\rangle = 0$. This argument appears in this form in this paper.

On a noncompact space, the Hodge duality is between arbitrary forms and those of compact support. The idea is just that integration is a pairing between p-forms and $n-p$ forms

$$(\omega, \eta) = \int_{\mathcal{M}} \omega \wedge \eta$$

which gives a map on cohomology, which is well-defined if one of the two forms is compactly-supported:

$$H^p(\mathcal{M}) \otimes H_c^{n-p}(\mathcal{M}) \to \mathbb{R}.$$

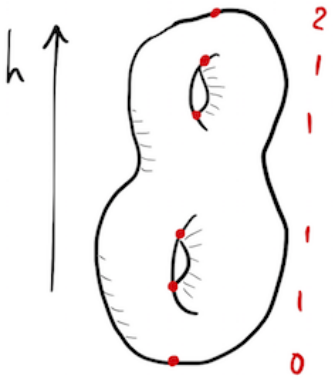
[End of Lecture 10]

2.3 Supersymmetric QM and Morse theory

Finally we return to approach number two to the groundstates of the supersymmetric NLSM. Choose a function on $\mathcal{M}$, $h(\phi)$, and add a magnetic field term (like we discussed earlier in the case of a single variable):

$$\Delta S = -\int d^D x \left(\frac{1}{2} \gamma^{ij} \partial_{\phi^i} h \partial_{\phi^j} h + \partial_{\phi^i} \partial_{\phi^j} h \bar{\psi}^i \psi^j \right).$$

That is: we add a scalar potential $V(\phi) = \frac{1}{2} |\vec{\nabla} h|^2$ and a mass matrix for the fermions $m_{ij}(\phi) = \partial_{\phi^i} \partial_{\phi^j} h$. This term is supersymmetric in $D \leq 3$ since it is $\Delta L = \int d\theta d\bar{\theta} h(\Phi(t, \theta, \bar{\theta}))$. This lifts the classical degeneracy, leaving only the critical points of h as zero-energy classical vacua. (In the example at right, where I've chosen the vertical direction of the embedding space as the function h, there are six such 'classical vacua'.) Let's assume that h has the nice property that it is critical ($\partial_{\phi^i} h = 0 \ \forall i$) only at isolated points of $\mathcal{M}$, and at those points $\partial_{\phi^i} \partial_{\phi^j} h$ has no zero eigenvalues. This is the generic situation – if it fails for some h, we can perturb it by some small thing to fix it. Such a function is called a Morse function or height function.



Let

$$\{p^A, A = 1..k\} \equiv \{\text{points } p \text{ in } \mathcal{M} \text{ where } \partial_{\phi^i} h(p) = 0, \forall i = 1..n\}$$

denote the set of classical supersymmetric vacua, the critical points of h. Quantumly there is tunneling between these vacua, which can lift them in bose-fermi pairs.

Q: Which are bosonic and which are fermionic? The definition of $(-1)^F$ is $[(-1)^F, \mathcal{O}_B] = 0, \{(-1)^F, \mathcal{O}_F\} = 0$. The overall sign is not fixed by this. We can define $(-1)^F |0\rangle = |0\rangle$.

To answer the question, consider the following toy model of a single majorana fermion in $D = 2$.

$$S[\psi] = \frac{1}{2}\int d^2x \left(\bar{\psi}\mathbf{i}\partial\!\!\!/\psi - m\bar{\psi}\psi\right).$$

Here m can have either sign, and (in the basis with $\gamma^0 = Y$), $\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$ with $\psi_{1,2} = \psi_{1,2}^\dagger$ real. With periodic boundary conditions in the spatial direction ($\psi(x+L) = \psi(x)$), we can define the fermion zero mode operators $\sigma_{1,2} \equiv \frac{1}{\sqrt{L}}\int_0^L dx\psi_{1,2}(x)$. Upon canonical quantization, these modes satisfy the algebra

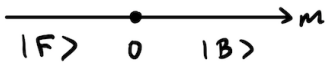
$$\{\sigma_a, \sigma_b\} = 2\delta_{ab}, \quad a, b = 1, 2.$$

In terms of $c = (\sigma_1 + \mathbf{i}\sigma_2)/2, c^\dagger = (\sigma_1 - \mathbf{i}\sigma_2)/2, \{c, c^\dagger\} = 1$, this algebra is represented by two states $c\,|0\rangle = 0, c^\dagger\,|0\rangle = |1\rangle$. Plugging in the expansion $\psi_{a=1,2}(x) = \sigma_a + \sum_{n\neq 0} e^{\frac{2\pi\mathbf{i}nx}{L}} a_n^a$, the hamiltonian on the zero-momentum modes is

$$H = -\mathbf{i}m\sigma_1\sigma_2 = m\sigma_3 = -m(-1)^F,$$

where $\sigma^{1,2,3}$ act as the Pauli matrices on this 2-state system. (The nonzero modes in x have $k = n/L$ and energy $\sqrt{k^2+m^2} > 0$ and are empty in the groundstate for all m.) We conclude that for $m > 0$, the groundstate is bosonic, while for $m < 0$ the groundstate is fermionic.

The phase diagram for this simple model looks as follows: The two phases are separated by a quantum phase transition at $m = 0$. The assumption that h is a Morse function is specifically to avoid this critical point.[31]



Now consider n such modes of masses $m_1 \cdots m_n$. Declare the groundstate with $m_1, \cdots m_n > 0$ to be bosonic. Then

$$(-1)^F\,|\text{gs}\rangle = (-1)^{\#\text{ of negative } m_i}\,|\text{gs}\rangle\,.$$

Now consider n such modes with a mass matrix m_{ij}, *i.e.* $L \ni \int m_{ij}\bar{\psi}_i\psi_j$. Note that m_{ij} is a symmetric matrix since $\{\psi_i, \psi_j\} = 0$ or $i \neq j$. By making an orthogonal

[31] Note that this simple field theory is a continuum description of the Kitaev chain. In the usual lattice description of this model, there are two Majorana modes (*i.e.* one electron mode) on each site of a chain. The Hamiltonian is $H = \sum_i \mathbf{i}\left(\gamma_i\tilde{\gamma}_i + t\tilde{\gamma}_i\gamma_{i+1}\right)$ (which is a rewriting of a mean-field Hamiltonian for a superconductor, roughly $H = \sum\left(c^\dagger c + \Delta c^\dagger c^\dagger + h.c.\right)$) and the coupling parameter $m \sim t - 1$. The two phases are distinguished by whether or not there exists a dangling Majorana mode at the ends of the chain. The model is also closely related to (the fermionic description of) the transverse field Ising chain. Thanks to Yi-Zhuang You for helpful comments.

transformation on the ψ^i and $\bar{\psi}^i$, we can diagonalize $m_{ij} = \left(O\text{diag}(m_1 \cdots m_n)O^T\right)_{ij}$ and we reduce to the previous case.

Now back to the problem of the fluctuations about the critical point p_A, where $m^A_{ij} = \frac{\partial^2 h}{\partial\phi^i \partial\phi^j}|_{p_A}$. Notice that m^A_{ij} is a symmetric matrix. Define

$$n_a \equiv \#\text{ of negative eigenvalues of } m^A_{ij}$$

to be the Morse index of the critical point p_A. We conclude that for the NLSM,

$$\text{tr}(-1)^F = \sum_A (-1)^{n_A}.$$

Previously we saw that $\text{tr}(-1)^F = \chi(\mathcal{M})$. We conclude that $\sum_A (-1)^{n_A}$ is independent of the choice of height function h. This is a basic theorem of Morse theory.

We can similarly evaluate the Lefschetz index of a symmetry K of the NLSM if we choose h to be invariant under the associated isometry k.

At this point we can study lots of examples. In the figure above of the genus-two surface, we can take the height function to be the vertical coordinate in the picture; the critical points and their Morse indices are indicated. We get $\text{tr}(-1)^F = 1-1-1-1-1+1 = -2 = 2-2g$. Next, we can take $\mathcal{M} = S^n = \{x_i | \sum_{i=0}^n x_i^2 = 1\}$, $h = x_0$ and $K : x_n \to -x_n, \psi_n \to -\psi_n, \psi_n^\star \to -\psi_n^\star$. On the homework you'll find

$$\text{tr}(-1)^F = 1 + (-1)^n, \quad \text{tr}(-1)^F K = 1 - (-1)^n.$$

We conclude that supersymmetry is unbroken in the S^n sigma model for any n. The example of $\mathcal{M} = \mathbb{CP}^N$ (it is tempting to take the height function to be $h = |z_0|$, but actually this is not a Morse function; but it can be perturbed to one) gives $\text{tr}(-1)^F = N+1$. This suggests (correctly) that there are $N+1$ groundstates.

Here is a powerful maneuver: Let Q be the supercharge for the NLSM without the perturbation by h. Given a Morse function h, define a new supercharge

$$Q_t \equiv e^{-th} Q e^{+th} \ .$$

You can check that

$$\{Q_t, Q_t^\dagger\} = 2H_t$$

where H_t is the perturbed hamiltonian. Redoing the analysis above, the exterior derivative is replaced by the conjugated exterior derivative

$$d_t \equiv e^{-th} d e^{+th}.$$

Since d_t and d are related by a similarity transformation, they have the same cohomology, independent of t. This means we can work out the cohomology (groundstates of H) by taking t large.

Near each critical point of h (minimum of $V = \frac{1}{2}|\vec{\nabla} h|^2$), the hamiltonian looks like

$$H_t \simeq \frac{1}{2}\sum_i \left(-\partial_{\phi^i}^2 + t^2 m_i^2 \phi_i^2 + t m_i [a_i^\dagger, a_i]\right) + \mathcal{O}(\phi^3) = \sum_i \left(H_i + t m_i K_i\right) + \mathcal{O}(\phi^3),$$

where m_i are the eigenvalues of $m_{ij} = \partial_{\phi^i}\partial_{\phi^j} h|$. This approximation gets better and better as t grows. Here $H_i = -\partial_{\phi^i}^2 + t^2 m_i^2 \phi_i^2$ is a harmonic oscillator, $K_i = [a_i^\dagger, a_i]$, and $[H_i, K_j] = 0$. By the above argument, this hamiltonian has a unique groundstate with groundstate energy $E_0 = t(A + \mathcal{O}(1/t))$. More precisely, the full spectrum is

$$E_{(N,k)} = t\sum_i \left(|m_i|(1 + 2N_i) + m_i k_i\right), \quad N_i = 0, 1, 2..., k_i = 0, 1.$$

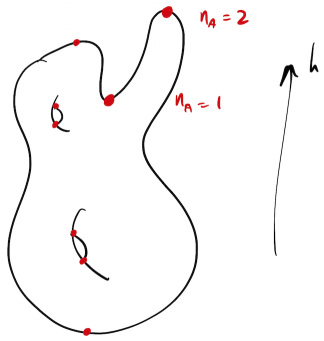
The groundstate requires $N_i = 0$ and $k_i = 1 \Leftrightarrow m_i < 0$, in which case $E = 0$. q-form states are those with $\sum_i k_i = q$.

We conclude that associated with each critical point p_A, there is a unique approximate groundstate $|\alpha_A\rangle$ with $E = 0$, which is a q-form if the Morse index of the critical point p_A is q. Only these states have a chance of being true groundstates of the full system, since other states have energy which grows with t.

But even these may not be true groundstates: states associated with different critical points may pair up by non-perturbative effects. This is a proof of a weak form of the Morse inequalities:

$$b^p \leq m^p \equiv \#\text{ of critical points of } h \text{ with Morse index } p.$$

Note that there are definitely examples of Morse functions which do not saturate this inequality. Consider for example, the function h defined at right. It has two extra critical points, relative to the more minimal Morse function shown previously for the genus-two surface. These two critical points indeed have opposite fermion number and we expect tunneling to left the pair of them to $E > 0$.



It is possible to prove a stronger form of relation between the critical points of h and the cohomology. A fancy way to describe it is to define a new chain complex: let $X^p \equiv \text{span}\{|\alpha_p\rangle \text{ of Morse index } p\}$. There is a coboundary operator $\delta : X^p \to X^{p+1}$ with $\delta^2 = 0$, and the cohomology of δ is $H^p(\mathcal{M})$. The idea is that the pairing up of Bose-Fermi pairs of classical groundstates becomes a pairing up in the sense of cohomology of δ.

This proceeds by considering the tunneling processes between the critical points. No amount of perturbation theory will connect the different classical vacua.

[End of Lecture 11]

Recall that tunneling can be understood in terms of the euclidean path integral. (See Coleman, *Uses of Instantons.*) Without fermion fields, a matrix element telling us about the mixing between perturbative groundstates A and B can be written

$$\langle B|\,\mathcal{O}e^{-HT}\,|A\rangle = \int_{\phi(0)=\phi_A,\phi(T)=\phi_B}[d\phi]e^{-S_{\rm eucl}[\phi]}\mathcal{O}[\phi] \simeq \sum_{\underline{\phi}} e^{-S[\underline{\phi}]}\mathcal{O}[\underline{\phi}]\det{}^{-\frac{1}{2}}\left(\frac{\delta^2 S}{\delta\phi^2}\right). \tag{2.16}$$

Here $\underline{\phi}$ are extrema of the euclidean action: instantons.

For the supersymmetric quantum mechanics NLSM, the bosonic part of the euclidean action is

$$S_{\text{eucl, B}}[\phi] = \frac{1}{2}\int d\tau\left(\gamma_{ij}\partial_\tau\phi^i\partial_\tau\phi^j + t^2\gamma^{ij}\partial_{\phi^i}h\partial_{\phi^j}h\right) \tag{2.17}$$

$$\frac{1}{2}\int d\tau\left|\partial_\tau\phi^i \pm t\gamma^{ij}\partial_{\phi^j}h\right|^2 \mp t\int d\tau\partial_\tau h \tag{2.18}$$

$$\geq t|h(\tau=\infty)-h(\tau=-\infty)| \equiv t|\Delta h| \tag{2.19}$$

(I defined $|v|^2 \equiv \gamma_{ij}v^iv^j$) with equality only if the first-order equation

$$0 = \partial_\tau\phi^i \pm t\gamma^{ij}\partial_{\phi^j}h \tag{2.20}$$

is satisfied. This equation describes gradient flow by the height function! We conclude that the instanton action for an instanton connecting critical points p^A and p^B is $S[\underline{\phi}^{A\to B}] = t|h(B)-h(A)|$.

Now we must talk about the fermions. Because of supersymmetry, they actually make things simpler. We replace (2.16) by

$$\langle B|\,\mathcal{O}e^{-HT}\,|A\rangle = \int_{\phi(0)=\phi_A,\phi(T)=\phi_B}[d\phi d\psi d\psi^\star]e^{-S_{\rm eucl}[\phi,\psi,\psi^\star]}\mathcal{O}[\phi,\psi,\psi^\star] \tag{2.21}$$

$$\simeq \sum_{\underline{\phi}} e^{-S[\underline{\phi}]}\det{}^{-\frac{1}{2}}(B)\int[d\psi d\psi^\star]e^{\int\bar\psi F\psi}\mathcal{O}[\underline{\phi},\psi,\psi^\star] \tag{2.22}$$

$$= \sum_{\underline{\phi}} e^{-S[\underline{\phi}]}\int d\bar\eta\mathcal{O}[\underline{\phi},\bar\eta_0]\ . \tag{2.23}$$

There are two main points here. First is that the integral over the nonzero modes of the fermions produces a determinant which cancels the bosonic determinant: this is because supersymmetry relates the operators B and F governing the quadratic terms so that given an eigenvector of B with nonzero eigenvalue λ, there is an eigenvalue of $F^\dagger F$ with eigenvalue λ. Second is that there is a fermion zeromode: $\bar\psi(t) = \bar\eta + \sum_{\lambda>0}\bar\psi_\lambda(t)a_i$. This

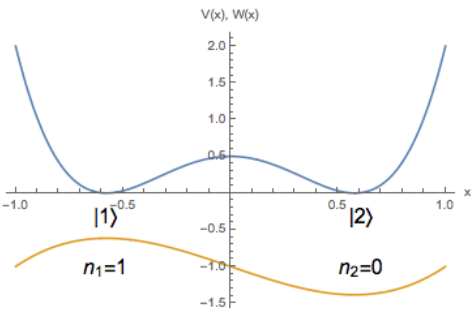
is a collective coordinate of the instanton, related by supersymmetry to the translation zeromode. There is a translation zeromode because the instanton location spontaneously breaks translations in euclidean time. Fermion zeromodes are just grassmann variables over which we must integrate in the path integral. But recall that the table of integrals for grassmann variables is $\int d\bar{\eta}\bar{\eta} = 1, \int d\bar{\eta} = 0$ – we get zero unless we soak up the fermion zeromode with the choice of operator $\mathcal{O}$.

The claim, then, is that

$$\epsilon_{AB} \equiv \langle B| Q_t |A\rangle$$

is nonzero only if the critical points p_A and p_B have Morse indices which differ by 1: $p_B - p_A = 1$, since $Q_t = \bar{\psi}p + ...$ contains a $\bar{\eta}$ in its expansion.

[Argyres' notes] Consider the special case of $\mathcal{M} = \mathbb{R}$, a single variable and $W(x) = gx^3 - x$. We noticed earlier that this model spontaneously breaks supersymmetry because neither $|-\rangle\, e^{-W(x)}$ nor $|+\rangle\, e^{+W(x)}$ is normalizable at both ends. But classically there are two degenerate zero-energy vacua, $|1,2\rangle$. These two states can't directly mix, since they have different fermion number (Morse index).



Yet somehow they must be lifted non-perturbatively. The resolution is that $|1\rangle$ mixes with excited states above $|2\rangle$. More directly, though, they pair up in the following sense. Let $|0_\pm\rangle$ be the two degenerate $E > 0$ true groundstates. The final groundstate energy is

$$0 < E_0 = \langle 0_+| H |0_+\rangle = \frac{1}{2}\langle 0_+| \{Q, Q^\dagger\} |0_+\rangle = \frac{1}{2}\langle 0_+| QQ^\dagger |0_+\rangle \geq \frac{1}{2}|\underbrace{\langle 0_-| Q^\dagger |0_+\rangle}_{\equiv\epsilon}|^2.$$

In the third step we assumed WLOG that $Q|0_+\rangle = 0$ (so $|0_-\rangle = Q^\dagger|0_+\rangle$), and in the last step we inserted a resolution of unity.

$$\epsilon = \langle 0_-| Q^\dagger |0_+\rangle \propto \langle 1|0_-\rangle \langle 0_-|Q^\dagger|0_+\rangle \langle 0_+|2\rangle \tag{2.24}$$

$$= \lim_{T\to\infty} e^{\frac{E_0 T}{\hbar}} \langle 1| e^{-H(T/2-t_0)} Q^\dagger e^{-H(T/2+t_0)} |2\rangle \tag{2.25}$$

$$= \int_{x(-\infty)=x_1}^{x(\infty)=x_2} DxD\psi D\bar{\psi}\; e^{-S/\hbar} Q^\dagger(t_0) \tag{2.26}$$

Here we approximate the state $|1,2\rangle$ by a delta function at the minimum $x_{1,2}$. Note that $|1\rangle$ only overlaps with $|0_-\rangle$ (and $|2\rangle$ with $|0_+\rangle$) by fermion number conservation.

The gradient flow equation (2.20) in this case is $\dot{\underline{x}} = \pm W'(\underline{x})$, and the instanton action is $\Delta W = W(x_2) - W(x_1)$. Now consider the fluctuations about the instanton

solution:

$$S\left(x=\underline{x}+\delta x,\psi=\delta\psi,\bar{\psi}=\delta\bar{\psi}\right)=\Delta W+\frac{1}{2}\int d\tau\left(\delta x B\delta x+\delta\bar{\psi}F\delta\psi\right)$$

with

$$F=\partial_\tau+W'',B=-\partial_\tau^2+W'''W'+(W'')^2=F^\dagger F$$

where $F^\dagger\equiv-\partial\tau+W''$. You can see that if $B\xi=\lambda\xi$ with $\lambda\neq0$, then $\eta\equiv F\xi/\sqrt{\lambda}$ satisfies $F^\dagger\eta=\sqrt{\lambda}\xi, F\xi=\sqrt{\lambda}\eta$. This seems to imply that

$$\epsilon\stackrel{?}{=}e^{-\Delta W/\hbar}\left(\frac{\det F}{\det\frac{1}{2}B}Q^\dagger(t_0)+\mathcal{O}(\hbar)\right)=e^{-\Delta W/\hbar}\left(Q^\dagger(t_0)+\mathcal{O}(\hbar)\right).$$

There are two problems with this: it depends on the arbitrary time t_0, and it is a grassmann number.

Surprisingly the solutions of these problems are related. The solution is that both F and B have zeromodes: $\delta x=\dot{\underline{x}}$ has $F\delta x=(\partial_\tau-W'')\dot{\underline{x}}=(\partial_\tau-W'')W'=W''W'-W''W'=0$. B has a zeromode because the instanton breaks translation symmetry; The zeromode of F is its superpartner. A good way to describe this is in terms of the superfield configuration of the instanton:

$$\underline{X}(t,\theta,\bar{\theta})=\underline{x}(t)-\bar{\theta}\eta\dot{\underline{x}}(t).$$

It satisfies $[H,\underline{X}]\neq0,[Q^\dagger,\underline{X}]\neq0$ but $[Q,\underline{X}]=0$; so there is a zeromode of $\delta x=\delta t\dot{\underline{x}}$ (which just shifts the time t_1 when the instanton goes from x_1 to x_2) and of $\delta\bar{\psi}=\bar{\eta}\dot{\underline{x}}$, but there is no $\delta\psi$ zeromode because $F^\dagger u=(-\partial_\tau-W'')u=0$ has no normalizable solutions.

These zeromodes mean that the saddle point is not isolated, but comes in a family. We must integrate over these collective coordinates of the instanton:

$$\epsilon=\int d\bar{\eta}\int_{-T/2}^{T/2}dt_1e^{-\Delta W}\left(Q^\dagger(t_0)|+\mathcal{O}(\hbar)\right) \tag{2.27}$$

$$e^{-\Delta W/\hbar}\int d\bar{\eta}\int dt_1\left(\bar{\eta}\dot{\underline{x}}(t_1-t_0)\left(\dot{\underline{x}}-W'\right)+\mathcal{O}(\hbar)\right) \tag{2.28}$$

$$e^{-\Delta W/\hbar}\left(\int_{x_1}^{x_2}dx(W'-W')+\mathcal{O}(\hbar)\right) \tag{2.29}$$

$$e^{-\Delta W/\hbar}\mathcal{O}(\hbar). \tag{2.30}$$

This has resolved the two problems above. Annoyingly, the first nonzero contribution comes at one-loop about the instanton. See this paper for the details of the calculation; it's a little complicated but there is a simple answer.

The expression for the fabled coboundary operator is

$$\delta\,|A\rangle = \sum_{B, p_B = p_A + 1} e^{-t(h(B)-h(A))} n(A,B)\,|B\rangle$$

where

$$n(A,B) = \sum_{\text{paths of steepest descent from } A \text{ to } B} (\pm 1)$$

and the sign comes from a comparison of the orientations on $T_A\mathcal{M}$ and $T_B\mathcal{M}$ induced by the associated forms.

So far I've had in mind the case where the dimension n of the target manifold is finite. If we are brave, we can consider infinite dimensional examples. For example, take $\mathcal{M}$ to be the space of gauge fields on a 3-manifold X, with gauge group G. Then a 1-form on this space is $\delta A^A(x)$ – the exterior derivative is the field variation, like in the equations of motion. But the product is the antisymmetric wedge product, so we can identify this with a fermionic field, $\psi^A(x)$. (Here A is a Lie algebra index.).

Then we can choose as the height function (actually functional) – and this is why 3d is special – the Chern-Simons functional:

$$h[A] = S_{\text{CS}}[A] = \frac{k}{4\pi}\int_X \text{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right).$$

The Hamiltonian is defined to be $H = \{Q, Q^\dagger\}/2$, with $Q = e^{-h}\delta e^h$. The purpose of $h[A]$ here is that its critical points are flat connections on X, $0 = F^A$. The analog of the gradient flow equation between critical points (using the metric $\|\,\delta A\,\|^2 = -\int_X \delta A \wedge \star_3 \delta A$) is

$$\partial_t A = \frac{\delta S_{\text{CS}}[A]}{\delta A} = \star_3 F. \tag{2.31}$$

Interpreted as an equation for a gauge field on the 3+1d spacetime $\mathbb{R}\times X$, this is the self-dual Yang-Mills equation $0 = F^+ \propto F + \star_4 F$ in the gauge where $A_0 = 0$. Miraculously, it is a covariant equation in 3+1d.

The groundstates of H are representatives of *Floer homology*, which gives invariants of the 3-manifold X. Moreover, because of (2.31), the resulting groundstate Hilbert space can be interpreted as the Hilbert space for Donaldson theory quantized on X. See here for more.

2.4 Global information from local information

[Bott and Tu §4] Let me explain a bit more why the name *co*homology. A smooth map $f : M \to N$ gives a map on functions $f^\star : \Omega^0(N) \to \Omega^0(M)$ defined by $f^\star(g) \equiv g \circ f$, the *pullback*. Note the reversal of arrows. More generally, such a map $f : M \to N$ gives a pullback map on forms of arbitrary rank:

$$f^\star : \Omega^p(N) \to \Omega^p(M)$$

which moreover is a chain map on the de Rham complexes – it commutes with the exterior derivative d: $[f^\star, d] = 0$. In terms of local coordinates y_i for N it is

$$f^\star \left(g_{i_1 \cdots i_p} dy^{i_1} \wedge \cdots dy^{i_p} \right) = g_{i_1 \cdots i_p} \circ f df_{i_1} \wedge \cdots df_{i_p}$$

where $f_i \equiv y_i \circ f = f^\star(y_i)$ are some (maybe good) coordinates on M. The fact that this commutes with d follows from the chain rule.

A fancy description of what we've just shown is that $\Omega^\bullet$ is a "contravariant functor from the category of smooth manifolds (and smooth maps) to the category of commutative differential graded algebras (and homomorphisms)". The word 'contravariant' indicates that the arrows get reversed, and this is the origin of the 'co'.

Poincaré Lemma. As an application of the pullback, let's prove a fundamental result about de Rham cohomology: the cohomology of $\mathbb{R}^n$ is the same as that of a point: $H^q(\mathbb{R}^n, \mathbb{R}) = \mathbb{R}\delta^{q,0}$.

To warm up, consider $H^\bullet(\mathbb{R})$. Any element of $\Omega^\bullet(\mathbb{R})$ is $f_0(t) + f_1(t)dt$. This is closed if $f_0' = 0$, *i.e.* if f_0 is constant. The constant function is definitely not d of something so it generates $H^0(\mathbb{R}) = \mathbb{R}$. Any 1-form $\omega = f_1(t)dt$ is closed, since there are no 2-forms. But $g(t) \equiv \int_0^t f_1(t')dt'$ satisfies $dg = \omega$, so all 1-forms are exact and $H^1(\mathbb{R}) = 0$.

Now for the general story. Without extra effort, we can prove a more general result: for any M, $H^\bullet(M \times \mathbb{R}) = H^\bullet(M)$. Consider the following maps:

$$\begin{array}{ccc} M \times \mathbb{R} & & \Omega^\star(M \times \mathbb{R}) \\ {\scriptstyle s}\uparrow \quad \downarrow{\scriptstyle \pi} & & {\scriptstyle s^\star}\uparrow \quad \downarrow{\scriptstyle \pi^\star} \\ M & & \Omega^\star(M) \end{array}$$

Here $\pi(x,t) \equiv x$ is the projection which forgets about $\mathbb{R}$, and $s(x) \equiv (x,0)$ is the 'zero-section', just the inclusion of M. From the definitions, $\pi \circ s = 1$, so $s^\star \circ \pi^\star = 1$, but $s \circ \pi \neq 1$ so $\pi^\star \circ s^\star \neq 1$ on $\Omega^\bullet$. However, $\pi^\star \circ s^\star$ is indeed the identity on cohomology, and we conclude that $H^\bullet(M \times \mathbb{R}) = H^\bullet(M)$.

To see this, we construct a homotopy operator $K : \Omega^p(M \times \mathbb{R}) \to \Omega^{p-1}(M \times \mathbb{R})$, which satisfies

$$1 - \pi^\star \circ s^\star = (-1)^{q-1} (dK - Kd) . \tag{2.32}$$

The signs don't matter, but as in our previous encounters with homotopy operators, this means that the LHS acts trivially on the cohomology. To construct K note that any form on $M \times \mathbb{R}$ is

$$\pi^\star \phi_0 f_0(x,t) + \pi^\star \phi_1 f_1(x,t) dt$$

for $\phi_{0,1} \in \Omega^\bullet(M)$. K takes this to $0 + \pi^\star \phi_1 \int_0^t dt' f_1(x,t')$, and you can check (2.32).

Now suppose we have a space $M = U \cup V$ divided into two open sets U, V. The inclusion maps define a sequence:

$$0 \leftarrow U \cup V \xleftarrow{i} U \amalg V \underset{i_V}{\overset{i_U}{\leftleftarrows}} U \cap V \leftarrow 0 \tag{2.33}$$

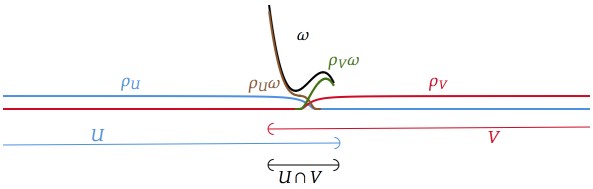
where $U \amalg V \equiv \{(0,u) | u \in U\} \cup \{(1,v) | v \in V\}$ is the disjoint union. This induces the following exact sequence, the Mayer-Vietoris sequence, on the de Rham complexes:

$$0 \to \Omega^\bullet(U \cup V) \xrightarrow{i^\star} \Omega^\bullet(U) \oplus \Omega^\bullet(V) \xrightarrow{i_U^\star - i_V^\star} \Omega^\bullet(U \cap V) \to 0. \tag{2.34}$$

The pullback $i^\star$ of the inclusion is a restriction map on forms.

Here is the proof that the Mayer-Vietoris sequence is exact: We need to show that given $\omega \in \Omega^q(U \cap V)$ we can write it as $\omega = u - v$ with $u \in \Omega^q(U), v \in \Omega^q(V)$. The key idea is the notion of a *partition of unity* on M. This is a set of smooth functions $\{\rho_\alpha\}_{\alpha \in I}$ such that

- $\sum_\alpha \rho_\alpha = 1$
- every point in M has a neighborhood where $\sum_\alpha \rho_\alpha$ is a finite sum of nonzero terms.



If the index set I is finite the second condition is trivially true. I is some label set. If we take I to be our labels on open sets, so that the support of ρ_α is a subset of U_α, this is called a *partition of unity subordinate to the open cover*.

For our simple example with two open sets $M = U \cup V$, $\rho_U + \rho_V = 1$, and any form is

$$\omega = \rho_U \omega + \rho_V \omega = \rho_U \omega - (-\rho_V \omega).$$

Here $\rho_U \omega \in \Omega^q(V)$ since ρ_U vanishes (and hence is well-defined) on $V \setminus U \cap V$. Similarly $-\rho_V \omega \in \Omega^q(U)$ and we are done.

Just as with homology (but with an overall reversal of arrows), an exact sequence of chain maps on complexes induces a long exact sequence on the cohomology.

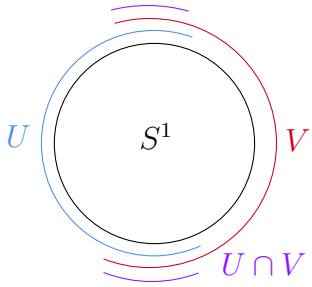
$$
\begin{array}{ccccc}
 & & H^{q+1}(U\cup V) & \longrightarrow & H^{q+1}(U)\oplus H^{q+1}(V) \longrightarrow H^{q+1}(U\cap V) \\
d^\star & \nearrow & & & \\
H^q(U\cup V) & \longrightarrow & H^q(U)\oplus H^q(V) & \longrightarrow & H^q(U\cap V)
\end{array}
$$

In terms of a partition of unity subordinate to the open cover, an explicit expression for the connecting homomorphism is

$$
d^\star[\omega] = \begin{cases} [-d(\rho_V \omega)] & \text{on } U \\ [d(\rho_U \omega)] & \text{on } V \end{cases}.
$$

which has support on $U \cap V$.

This long-exact Mayer-Vietoris sequence is a useful device for computing cohomology. Consider the example of the circle covered by two patches:



$$
\begin{array}{ccccccc}
 & & S^1 = U\cup V & & U \amalg V & & U\cap V \\
H^1(S^1) & & & \longrightarrow 0 & & \longrightarrow 0 & \\
 & \nearrow & & & & & \searrow^{d^\star} \\
H^0(S^1) & & \ker\delta & \longrightarrow \mathbb{R}\oplus\mathbb{R} & & \xrightarrow{\delta} \mathbb{R}\oplus\mathbb{R} & \\
 & \searrow & & & & & \nearrow \\
 & & & & 0 & &
\end{array}
$$

Here we've used the Poincaré lemma[32] in the form $H^q(\text{ball}, \mathbb{R}) = \mathbb{R}\delta^{q,0}$ to fill in the right two columns. This determines the left column: $H^0(S^1) = \ker\,\delta$ and $H^1(S^1) = \text{coker}\delta$. The map δ acts by $\delta(\omega, \tau) = (\tau - \omega, \tau - \omega)$ which has rank one. Hence we recover $H^0(S^1, \mathbb{R}) \cong H^1(S^1, \mathbb{R}) \cong \mathbb{R}$.

[End of Lecture 12]

[32] Actually what we figured out above was $H^\bullet(\mathbb{R}^n)$, not $H^\bullet(\text{ball})$. We'll prove that these are the same after we discuss the notion of homotopy equivalence.

2.5 Homology and cohomology

Cohomology with compact support.

First we must introduce the notion of the de Rham complex with compact support:

$$\Omega_c^\bullet(\mathcal{M}) \equiv \{\text{forms on } \mathcal{M} \text{ with compact support}\}.$$

Its cohomology groups are $H_c^\bullet(\mathcal{M})$. Consider for example $H_c^\bullet(\mathbb{R})$. f is closed says $f' = 0$, but there are no constant functions of compact support on $\mathbb{R}$, so $H_c^0(\mathbb{R}) = 0$. If the support of f is a subset of the interval (a, b) (for some a, b), then

$$\int_{\mathbb{R}} df = \int_{-\infty}^{\infty} dx \frac{df}{dx} = \int_a^b dx \frac{df}{dx} = f(b) - f(a) = 0.$$

Moreover, ω is exact, $\omega = df$ (for f with compact support) iff $\omega \in \Omega_c^1(\mathbb{R})$ integrates to zero. To see the converse, note that the latter statement means that $f(x) = \int_{-\infty}^x \omega$ has compact support. Therefore, $H_c^1(\mathbb{R}) = \Omega_c^1(\mathbb{R})/\ker\left(\int\right) = \mathbb{R}$.

The Poincaré Lemma for compactly supported cohomology says $H_c^q(\mathbb{R}^n) = \delta^{n,q}\mathbb{R}$, and more generally $H_c^{q+1}(M \times \mathbb{R}) \simeq H_c^q(M)$ for any M. See Bott & Tu page 39 for the proof.

There is something funny about compactly supported cohomology: given a map $f : M \to N$ the pullback $f^\star$ of $\omega \in \Omega_c^\bullet(N)$ is not necessarily compactly supported. Consider for example $\pi : M \times \mathbb{R} \to M$ – the pullback of a compactly-supported form on M has support on all of $\mathbb{R}$. So despite appearances $\Omega_c^\bullet$ is *not* a contravariant functor from the category of smooth manifolds to that of graded algebras.

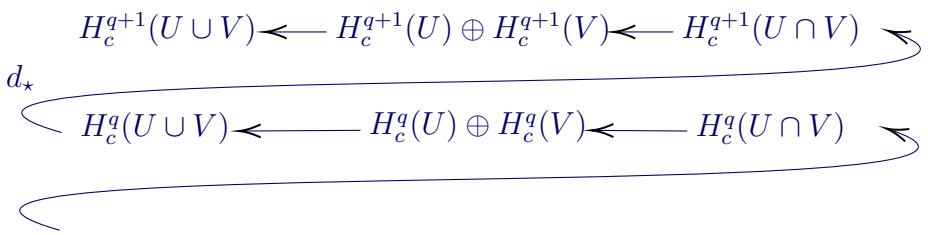
But actually there is a simpler and useful way to think of it as a functor: it is a *covariant* functor under *inclusions* of open sets. What I mean is given $i : U \to M$ an inclusion, then $i_\star : \Omega_c^\bullet(U) \to \Omega_c^\bullet(M)$ extends $\omega \in \Omega_c^\bullet(U)$ by zero to a (manifestly compactly-supported) form on M. Very simple.

Now the Mayer-Vietoris inclusions (2.33) give the following exact sequence (which is backwards relative to the Mayer-Vietoris sequence on the ordinary cohomology):

$$\begin{array}{ccccc} 0 \leftarrow \Omega_c^\bullet(U \cup V) & \overset{\text{sum}}{\leftarrow} & \Omega_c^\bullet(U) \oplus \Omega_c^\bullet(V) & \overset{(-i_{U\star}, i_{V\star})}{\leftarrow} & \Omega_c^\bullet(U \cap V) \leftarrow 0 \qquad (2.35) \\ & & (-i_{U\star}\omega, i_{V\star}\omega) & \mapsto & \omega. \qquad (2.36) \end{array}$$

The arrows are all backwards relative to (2.34). The idea for showing that (2.35) exact is also simple: $\Omega_c^\bullet(U \cup V) \ni \omega = \rho_U\omega + \rho_V\omega$. The first term is now an element of $\Omega_c^\bullet(U)$ and the second term is an element of $\Omega_c^\bullet(V)$. As usual, this short exact sequence

implies a long exact sequence on (the compactly-supported) cohomology, and again the arrows are backwards:



Pairings. The wedge product on forms induces a product structure on the de Rham cohomology. On an oriented compact manifold M, this provides a pairing between k-forms and $(n-k)$-forms, $\int_M \omega_k \wedge \eta_{n-k}$, which induces a non-degenerate pairing on cohomology. (If M is non-compact, the pairing is between cohomology and compactly-supported cohomology.) A non-degenerate pairing between two vector spaces A and B is the same as saying that A is isomorphic to the dual space of B:

$$H^k(M)^\star \cong H_c^{n-k}(M). \tag{2.37}$$

(See Bott and Tu p. 44 for a proof of non-degeneracy. It uses the five-lemma that you proved on the homework.)[33] This relation (2.37) is yet another thing called Poincaré duality.

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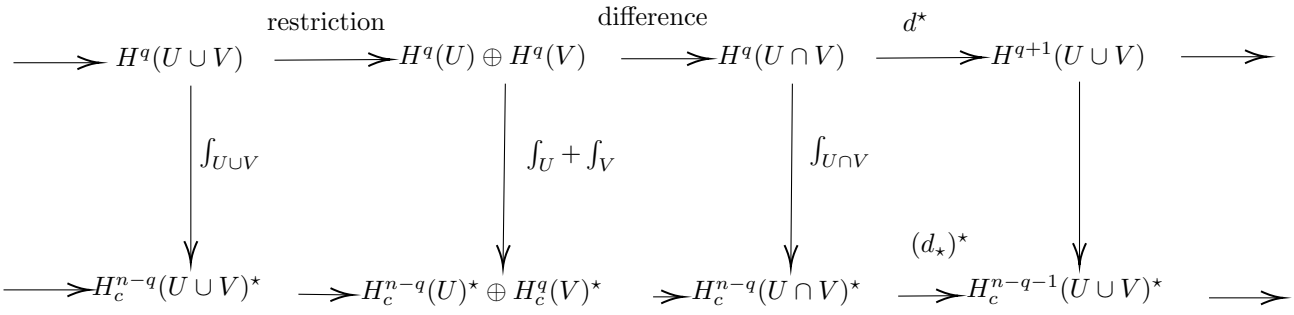
[33] Let me say a little bit about the argument.

Now we want to show that the pairing $H^q(M)\otimes H_c^{n-q}(M)\to\mathbb{R}$ by $(\omega,\eta)\mapsto\int_M\omega\wedge\eta$ is nondegenerate, so that $\eta\in H_c^{n-q}(M)$ gives an element of $H^q(M)^\star$ (the vector space of homomorphisms from $H_q(M)$ to $\mathbb{R}$), by $\omega\mapsto\int_M\omega\wedge\eta\in\mathbb{R}$. To see this, first start with the case where $M=U\cup V$ as in our Mayer-Vietoris discussion and show that the following diagram commutes



Notice that taking $\star$ reverses the direction of the arrows, so in the bottom row the rank of the forms is decreasing. Now two out of three of the vertical maps are isomorphisms by the Poincaré lemmas. Then the 5-lemma implies that the third one is, too. The extension for general $\mathcal{M}$ then proceeds by an induction on the number of open sets in the open cover. Suppose the Poincaré duality works for any manifold with a good cover by p or fewer open sets (*good* cover means the open sets and their intersections are each homeomorphic to a ball). Then consider $M=U_0\cup U_1\cdots\cup U_p$. $V\equiv(U_0\cup\cdots\cup U_{p-1})\cap U_p$ has a good cover by p open sets, namely $\{U_0\cap U_p, U_1\cap U_p,\cdots U_{p-1}\cap U_p\}$. Now by the induction assumption, Poincaré duality holds for $U_0\cup\cdots\cup U_{p-1}$ and for $U\equiv U_p$, for V,

There is also a nice pairing between p-forms α and p-cycles μ: we can integrate the p-form over the p-cycle to get a number:

$$\begin{array}{ccc} H^p(X) \otimes H_p(X) & \to & F \\ (\alpha, \mu) & \mapsto & \int_\mu \alpha \end{array} .$$

(Here F is the ring over which we defined the differential forms, which I'm going to assume is $\mathbb{R}$ from now on.) Since α is closed, the answer depends only on the homology class $[\mu]$ and not on the choice of representative:

$$\int_{\mu+\partial\nu} \alpha = \int_\mu \alpha + \int_{\partial\nu} \alpha = \int_\mu \alpha + \int_\nu d\alpha$$

by Stokes' Theorem. Similarly, if we add to α something exact, $\alpha \to \alpha + d\beta$ the result is unchanged (also by Stokes' Theorem: $\int_\mu d\beta = \int_{\partial\mu} \beta = 0$), since μ has no boundary.

Finally, on an oriented manifold, we can associate with each closed submanifold S of dimension k an element $[\eta_S] \in H^{n-k}(M)$, its Poincaré dual (yes, this name is overloaded). If $i : S \to M$ is the inclusion map, then for all $\omega \in H^k(M)$

$$\int_S i^\star \omega \equiv \int_M \omega \wedge \eta_S. \tag{2.38}$$

Here $i^\star : H^k(M) \to H^k(S)$ is the pullback map, which I'll define momentarily. This equation defines η_S as an element of $H^k(M)^\star \cong H^{n-k}(M)$. So for each $[S] \in H_k(M)$ we have an $[\eta_S] \in H^{n-k}(M)$, and we conclude[34]

$$H_k(M) \cong H^{n-k}(M).$$

In particular, this shows that the Betti numbers defined by homology and by cohomology $b_{n-k}(M) = b_k(M) = b^k(M) = b^{n-k}(M)$ are the same.

---

and for $U \cap V$, which fit in the diagram above, so it also holds for M by the 5-lemma.

If M is actually compact, then (2.37) still holds and we can drop the c subscript.

[34] On a noncompact manifold, there is some ambiguity in what we might mean by the form Poincaré dual to S. This is because $\int_S \cdot$ defines a linear function on k-forms on M, and therefore by the version of Poincaré duality in (2.37) it specifies an element of $H_c^{n-k}(M)$. This element is not necessarily the same as η_S defined in (2.38).

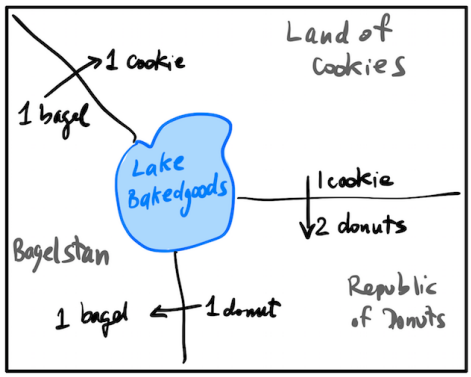
2.6 Cech cohomology

[I highly recommend the great book by Bott and Tu for this and many of our other topics. I believe the book *Topology Illustrated* by Peter Saveliev also has a nice discussion of this topic, but I don't have a copy.]

Suppose you specify a set of relative preferences along some set of axes. Maybe you prefer high to low, hot to cold, wet to dry. Now suppose there is some space where each point is labelled by each of these characteristics (height, temperature, dampness). Is there a preference function on this space that can be maximized to decide where you want to live?

This is an analog of the question: given a vector field on some space, is there a function whose gradient it is?

Another realization of the same issue is arbitrage. For example, suppose there are three countries arranged in a triangle, each of which uses a different currency. There are exchange rates across each of the three pairs of borders. If these rates are chosen poorly (as in the example at right), an enterprising person can generate wealth for herself by going around in a circle (in the correct direction) exchanging currency.



In this exposition, Maldacena uses this example as an analogy to explain gauge theory. The value one generates by going in a loop is like magnetic flux. The further point (obscured by my example where baked goods are used as currency, since baked goods have intrinsic value) is that the value of a given unit of currency is arbitrary, so currency exchanges are like gauge transformations. The intrinsic value of baked goods (at least if everyone agreed about it) plays the role of a Higgs field, making it locally obvious that someone is doing something dumb in the above picture. This local obviousness is like a mass for the EM field.

This is an alternative cohomology theory which can reproduce the data associated with manifolds we discussed before but also applies to such more general situations. The simplest version of the idea is to think about *locally constant* functions on patches. Locally constant means that on each connected component of its domain, the function takes a constant value. Cover the manifold X with open sets U_α. These open sets intersect in e.g. $U_{\alpha\beta} \equiv U_\alpha \cap U_\beta$, and triple-intersections $U_{\alpha\beta\gamma} \equiv U_\alpha \cap U_\beta \cap U_\gamma$ and so on. Define C^k to be the vector space of A-valued locally-constant functions on the disjoint

union of the $(k+1)$-overlaps for some abelian group A:

$$C^k \equiv \{\text{locally constant functions, } \coprod_{\alpha_0 \cdots \alpha_k} U_{\alpha_0 \cdots \alpha_k} \to A\}.$$

So an element of C^0 just assigns an element of A to each U_α. Then there is an analog of the boundary map (actually coboundary map, since it goes in the other direction) which makes this into a chain complex, $\delta : C^k \to C^{k+1}$. It is defined as a difference of restrictions, just as in the Mayer-Vietoris sequence, as follows. For example, given $f : U_{\alpha\beta} \to A$ on all double-overlaps, this defines a function $f : U_{\alpha\beta\gamma} \to A$ on all triple-overlaps just by restriction. The coboundary map $\delta : C^0 \to C^1$ is

$$(\delta f)_{\alpha\beta} = f_\alpha - f_\beta.$$

The idea is that δf checks agreement, *i.e.* whether or not f can be regarded as a function on the union. The map for $C^1 \to C^2$ is:

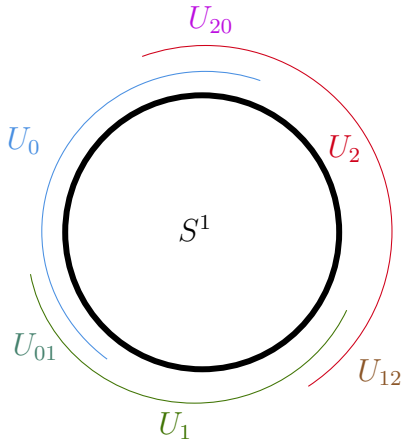
$$(\delta f)_{\alpha\beta\gamma} = f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} = f_{\alpha\beta} + f_{\beta\gamma} - f_{\alpha\gamma}.$$

Note that we define $f_{\alpha\gamma} \equiv -f_{\gamma\alpha}$. There is a similar definition for general k, so that $\delta^2 = 0$, and the cohomology of the complex is well-defined, and again is a topological invariant (actually the same data as above). The definition for general k is

$$(\delta f)_{\alpha_1 \cdots \alpha_{k+1}} = \sum_i (-1)^i f_{\alpha_1 \cdots \widehat{\alpha_i} \cdots \alpha_{k+1}}$$

where $\widehat{\alpha}$ indicates that α is missing. In this expression, we've chosen an (arbitrary) order for the subsets.

Cech cohomology is very simple to actually calculate in reasonable examples. Consider a cover of a circle by three patches U_0, U_1, U_2, with overlaps U_{01}, U_{12}, U_{20}. The space of 0-cochains is $C^0 = \{\omega_\alpha, \alpha = 0, 1, 2 | \omega_\alpha \text{ is constant on } U_\alpha\} = A^3$, while the space of 1-cochains is $C^1 = \{\eta_{\alpha\beta}, \alpha, \beta = 0, 1, 2 | \eta_{\alpha\beta} \text{ is constant on } U_{\alpha\beta}\} = A^3$. There are no triple-overlaps (and because the space is one-dimensional, there can be no triple-overlaps in any open cover) so $C^2 = 0$.



The coboundary map $\delta : C^0 \to C^1$ acts by $(\delta\omega)_{\alpha\beta} = \omega_\alpha - \omega_\beta$. The Cech complex is

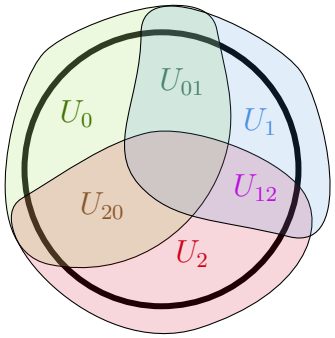
$$0 \to A^3 \xrightarrow{\delta} A^3 \to 0$$

with

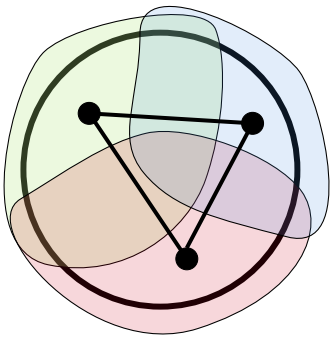
$$\delta = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$

More, explicitly $H^0(S^1) = \ker(\delta) = \{\omega_0 = \omega_1 = \omega_2\} = A$. And $H^1(S^1) = A^3/\mathrm{im}(\delta) = A$. A 1-cocycle $\eta = (\eta_{01}, \eta_{12}, \eta_{20})$ is a coboundary if $\eta_{01} + \eta_{12} + \eta_{20} = 0$. So a generator of $H^1(S^1)$ is of the form $(g, 0, 0)$ where $A = \langle g \rangle$.

Not all covers of a manifold will give the same answer. A *good cover* is one for which every intersection $U_{\alpha_1 \cdots \alpha_k}$ is topologically a ball. So the example above is a good cover of the circle. An example of good cover of the 2-sphere has four open sets. One is the northern hemisphere. The other three each cover an enlarged one-third pie-slicing of the southern hemisphere, as at right.



A good cover of $\mathcal{M}$ is associated with a cell decomposition of $\mathcal{M}$: associated a 0-cell to each open set, if $U_{\alpha\beta}$ is non-empty, a 1-cell connects the 0-cells α and η. If $U_{\alpha\beta\gamma}$ is non-empty, we fill in the face of the triangle $\alpha\beta\gamma$ with a 2-cell. Keep going. There is a close relation between δ and the boundary map for this cell complex. On the homework you can work out the Cech homology for the 2-sphere and you will see close parallels with the homology of the tetrahedron.



[End of Lecture 13]

I apologize that I didn't explain Cech cohomology by starting with a physical system that realizes it. An example of a place in physics where Cech cohomology is used to good effect is this paper by Witten and Bagger, where they show that only certain non-linear sigma models may be coupled to $D = 4, \mathcal{N} = 1$ supergravity – the target space must be a Hodge-Kähler manifold.

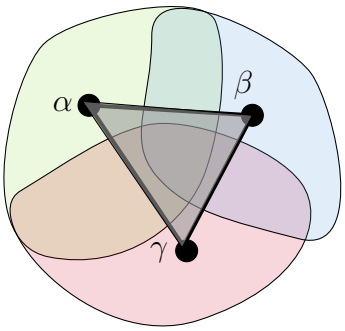
But actually, consider the following physical system. Take a good open cover of some manifold. Associate to each intersection of two open sets $U_{\alpha\beta}$ a qubit, $\mathrm{span}\{|\sigma_{\alpha\beta} = 0, 1\rangle\}$ (or more generally, we attach an element of an abelian group A to each double overlap).

We can regard the basis labels $\sigma_{\alpha\beta} = -\sigma_{\beta\alpha}$ as a 1-cochain in $C^1(\mathfrak{U}, A)$.

Now let $H = -\sum_{U_\alpha} A_\alpha - \sum_{U_{\alpha\beta\gamma}} B_{\alpha\beta\gamma}$. Here the operators A and B are defined in the given basis by

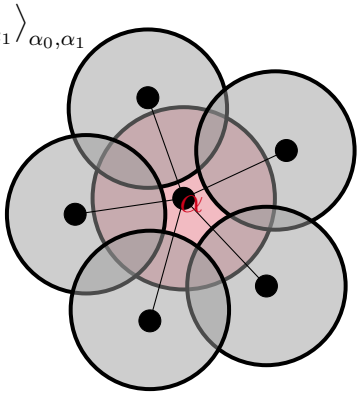
$$B_{\alpha\beta\gamma} \left|\{\sigma\}\right\rangle = (-1)^{(\delta\sigma)_{\alpha\beta\gamma}} \left|\{\sigma\}\right\rangle$$

(where recall that $(\delta\sigma)_{\alpha\beta\gamma} = \sigma_{\alpha\beta} + \sigma_{\beta\gamma} + \sigma_{\gamma\alpha}$) is the plaquette operator associated to the triangle $\alpha\beta\gamma$. A state $\left|\{\sigma\}\right\rangle$ which satisfies $B = 1$ means σ is a cocycle.



$$A_\alpha \left|\{\sigma\}\right\rangle = \left|\{\sigma_{\alpha_0\alpha_1} + \delta_{\alpha,\alpha_0} - \delta_{\alpha,\alpha_1}\}_{\alpha_0,\alpha_1}\right\rangle \left|\{\sigma_{\alpha_0\alpha_1} + (\delta\lambda(\alpha))_{\alpha_0\alpha_1}\right\rangle_{\alpha_0,\alpha_1}$$

is the star operator associated with the site α. In the last expression, $\lambda(\alpha)$ is a 0-cochain which is only nonzero on the set U_α. (The signs don't actually matter because I'm talking about the special case of $A = \mathbb{Z}_2$.) You can see they commute because of $\delta^2 = 0$. This is the same as the toric code on the cell complex constructed from the open cover.



A nice thing about the idea of Cech cohomology is that we can take the coefficients to live pretty much anywhere; we'll use this in the following discussion.

Proof of equivalence of de Rham and Cech cohomology. Theorem: If $\mathfrak{U}$ is a good cover of M, then $H^\bullet_{\mathrm{dR}}(M) \cong H^\star_\vee(\mathfrak{U}, \mathbb{R})$, the Cech cohomology with real coefficients.

Here's an outline of the proof. First consider the following sequence of inclusions, as in the discussion of the Mayer-Vietoris sequence for two open sets:

$$M \leftarrow \coprod_\alpha U_\alpha \overset{i_0}{\underset{i_1}{\leftleftarrows}} \coprod_{\alpha_0\alpha_1} U_{\alpha_0\alpha_1} \lll \coprod_{\alpha_0\cdots\alpha_2} U_{\alpha_0\alpha_1\alpha_2} \cdots$$

These maps induce a generalized Mayer-Vietoris sequence on $\Omega^\bullet$ (of which (2.34) is the special case with two open sets):

$$0 \to \Omega^\bullet(M) \xrightarrow{r} \bigoplus_\alpha \Omega^\bullet(U_\alpha) \xrightarrow{\delta} \bigoplus_{\alpha\beta} \Omega^\bullet(U_\alpha\beta) \xrightarrow{\delta} \bigoplus_{\alpha\beta\gamma} \Omega^\bullet(U_{\alpha\beta\gamma}) \to \cdots$$

For example, the inclusion map $i_\alpha : U_{\alpha\beta\gamma} \to U_{\beta\gamma}$ leads to the restriction maps

$$\delta_\alpha : \Omega^\bullet(U_{\beta\gamma}) \to \bigoplus_\alpha \Omega^\bullet(U_{\alpha\beta\gamma}).$$

The definition of δ on the collection of all cochains alternates signs (again in some arbitrary ordering of the open sets): $\delta \equiv \sum_i (-1)^i \delta_{\alpha_i}$, *i.e.*

$$(\delta\omega)_{\alpha_0\cdots\alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \omega_{\alpha_0\cdots\widehat{\alpha_i}\cdots\alpha_{p+1}}.$$

The RHS is a form on the total intersection $U_{\alpha_0\cdots\alpha_{p+1}}$. With these signs, $\delta^2 = 0$ so it is a complex. But moreover, it is exact. The idea of the proof is that a partition of unity subordinate to $\mathfrak{U}$, $\{\rho_\alpha\}$, gives a homotopy operator $K : \bigoplus \Omega^\bullet(U_{\alpha_0\cdots\alpha_p}) \to \bigoplus \Omega^\bullet(U_{\alpha_0\cdots\alpha_{p-1}})$: Given a p-cochain, $\omega \in \bigoplus \Omega^\bullet(U_{\alpha_0\cdots\alpha_p})$, we define $(K\omega)$ on $U_{\alpha_0\cdots\alpha_{p-1}}$ by

$$(K\omega)_{\alpha_0\cdots\alpha_{p-1}} = \sum_\alpha \rho_\alpha \omega_{\alpha\alpha_0\cdots\alpha_{p-1}}.$$

It satisfies $K\delta + \delta K = 1$, so if ω is closed, then $\omega = \delta(K\omega)$, it is also exact.

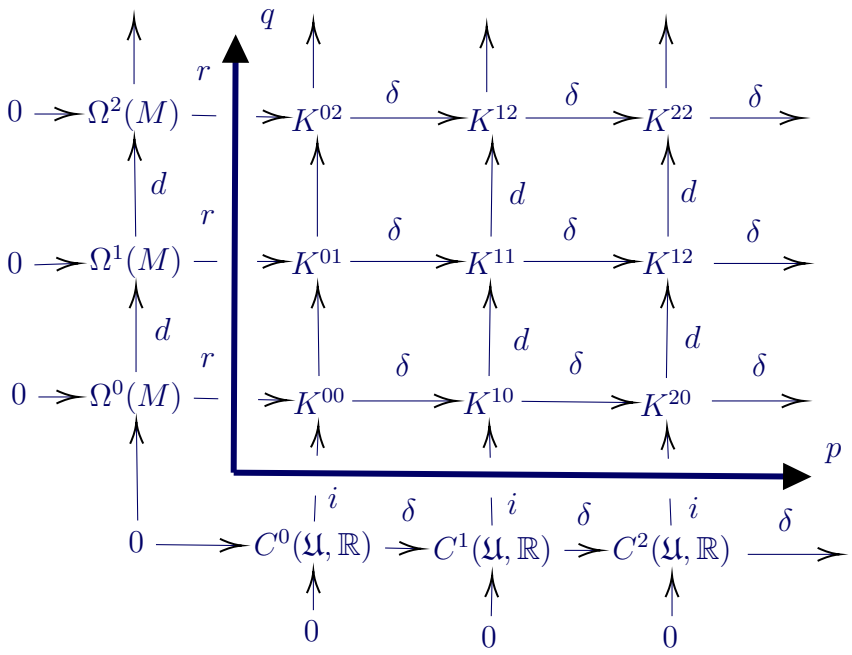
Note the strong similarity between the general Mayer-Vietoris exact sequence and the Cech complex! The key difference is that the elements in the Cech complex are *locally-constant* functions, which are incapable of smoothly going to zero. This means that there's no such thing as multiplying them by a partition of unity. So there's no homotopy operator in that case.

Now let

$$K^{p,q} \equiv C^p(\mathfrak{U}, \Omega^q) \equiv \oplus_{\alpha_0\cdots\alpha_p} \Omega^q(U_{\alpha_0\cdots\alpha_p})$$

denote the space of q-form valued p-cochains for the open cover $\mathfrak{U}$. We can think of this as taking the coefficient group of the cochains to be the space of q-forms! Then how can we resist considering the following (augmented) *double-complex*. The rows are

just the Mayer-Vietoris sequences:

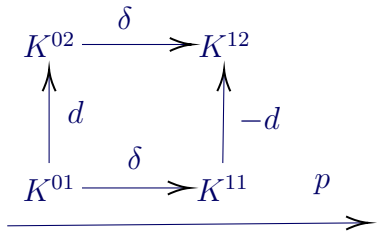


The left-most column is the de Rham complex, describing differential geometry of forms on M. The bottom-most row is the Cech complex, which is purely combinatorial data about the open cover $\mathfrak{U}$ of M; it is attached by the inclusions of the locally-constant functions on any U into the continuous functions on U. Each of its entries is the kernel of the bottom-most d operator (constant functions). Inside the box is an unholy mixture of the two; horizontal maps are all coboundary operators δ, and vertical maps are all exterior derivatives d.

From any such double complex we can make a new single-complex with coboundary operator

$$D \equiv \delta + (-1)^p d.$$

This squares to zero because δ and d commute (by construction of δ) and because of the minus sign (which inserts π-flux through the squares of the double complex):



Here I appeal to a **Lemma:** If the rows of the double complex are exact, then the cohomology of $D = \delta + (-1)^p d$ is the same as the cohomology of the leftmost column.

The proof of this is not so bad, but I don't think I can explain it better than Bott and Tu did on page 96-97.

So, exactness of the general Mayer-Vietoris sequence says the rows are exact, and therefore

$$H^\bullet_{\mathrm{dR}}(M) \cong H^\bullet_D(C^\star(\mathfrak{U}, \Omega^\bullet)).$$

If furthermore $\mathfrak{U}$ is a good cover (so that any non-empty intersections are contractible), then the columns are also exact. This follows from the Poincaré lemma, $H^q_{\mathrm{dR}}(\text{ball}) = \delta^{q,0}\mathbb{R}$. So we conclude that

$$H^\bullet_\vee(\mathfrak{U}, \mathbb{R}) \cong H^\bullet_D(C^\star(,\Omega^\bullet))$$

also.

This equivalence implies many nice facts about both sides: it shows that the de Rham cohomology is finite-dimensional if there is a finite cover. It shows that the Cech cohomology is independent of the choice of good cover, and is just a property of M.

2.7 Local reconstructability of quantum states

One feature of topologically ordered groundstates is that the reduced density matrices on disks do not determine the global groundstate.

This idea is spiritually very similar to the perspective of Cech cohomology – if there is nontrivial $H^1(X)$, then the data on patches of X can be fit together in more than one way.

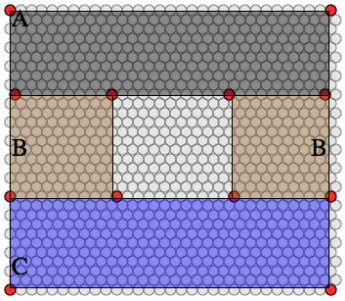
The only minor difference is that in the former case we are talking about density matrices – these are supposed to be the analogs of cochains. But we can't just add density matrices – they don't live in an abelian group. Rather they live in a convex set. There is some mathematical theory waiting to be constructed.

While we're waiting for this theory to be developed (the leading candidate for the role of the passive voice here is surely Alexei Kitaev), there is a nice quantum information theoretic way to measure the obstruction to reconstructability of a given state. It is called the *topological entanglement entropy* (TEE), and is simply a sum of von Neumann entropies of subregions:

$$I(A:C|B) \equiv S_{AB} + S_{BC} - S_B - S_{ABC}.$$

There is a quantum information theorem that says (see *e.g.* the book by Petz): if $I(A:C|B) = 0$ then the density matrix ρ_{ABC} (such a state is called a *quantum Markov state*) can be uniquely reconstructed from its marginals ρ_{AB}, ρ_{BC}.

Consider the choice of ABC at right. In gapped groundstates we expect that for each region A, the entropy satisfies an area law, $S(A) = |\partial A|\Lambda - \gamma b_0(\partial A)$, where $b_0(\partial A)$ is the number of components of the boundary of A. The area-law contributions cancel out pairwise (notice that the corners cancel too). All that is left is $I(A : C|B) = 2\gamma \geq 0$. The last inequality is the statement of strong subadditivity of the von Neumann entropy.



So the TEE should be regarded as a symptom of the existence of some nontrivial (appropriate generalization of) cohomology.

There are three more important things to be said about the TEE [Kitaev-Preskill Levin-Wen]:

1. It is independent of deformations of the regions which preserve the topology of each part.

2. It is independent of changes of the state that keep the correlation length small compared to the sizes of the regions.

3. It can be related to the anyon data for the associated topological order. In particular it is $\gamma = \log \sum_a d_a^2$, where a runs over all the anyon types and d_a are their quantum dimensions.

First note that

$$-2\gamma = -I(A : C|B) = -\left(S_{AB} + S_{BC} - S_B - S_{ABC}\right)$$

can be grouped as $(S_{ABC} - S_{BC}) - (S_{AB} - S_B)$. The first term is the entropy gained by closing the top of BC, while the second term is the entropy gained by closing the top of B. Since these only differ by what's going on at the bottom, if the entanglement is all short-ranged, they should be the same (and more the same as the regions get larger compared to the correlation length). On the other hand, if there is string condensation, strings can wind around ABC and decrease the first term.

To see the first two statements, let D be the complement of ABC, so that $ABCD$ is in a pure state. Now suppose we deform the boundary of C a bit so that it absorbs some of D. The term $S_{AB} - S_B$ shouldn't care about this because it only involves distant regions. Since $ABCD$ is pure we can write the other term as $S_{ABC} - S_{BC} = S_D - S_{AD}$. Now we expect that $\Delta S_D - \Delta S_{AD} = 0$ – appending A shouldn't affect the change in entropy of S_D from moving the distant boundary with C.

A similar argument can be made for the triple-intersections of regions – one of the regions is always far away and therefore not involved.

Now consider a deformation of the local Hamiltonian whose groundstate we are talking about (or we could talk about changes in the state itself by acting with local unitaries). Any change that acts completely within one of the regions doesn't change the entropies. To see the effect of a change that happens near a junction, just move the junction using the previous result, so that the change is now completely within a region.

3 Quantum Double Model and Homotopy

[Hatcher, beginning] First some definitions. A *homotopy* is a family of maps $f_t : X \to Y, t \in I$ (I is the interval) such that

$$f : \begin{array}{ccc} X \times I & \to & Y \\ (x,t) & \mapsto & f_t(x) \end{array}$$

is continuous. (In the following everything in sight is assumed to be continuous.) Two maps $f_{0,1} : X \to Y$ are said to be *homotopic* if there exists a homotopy f_t with the obvious boundary conditions. In this case we will write $f_0 \simeq f_1$.

An important class of examples is the following. A *deformation retraction* of X into $A \subset X$ is a homotopy from $f_0 = \mathrm{id} : X \to X$ to $f_1 =$ a *retraction* $r : X \to X$ with $r(X) = A$ and $r^2 = r$ like a projector. For example, we can find a deformation retraction of a disk X to a point A, or an annulus X to a circle A, or a disk with two holes to ∞ or θ or two circles attached by a line segment, like eyeglasses:  .

There are retractions which are not deformation retractions, such as from $X =$ two points to $X =$ one point, or from $X =$ annulus to $X =$ one point.

An important definition: X is *homotopy equivalent* to Y (or X and Y have the same homotopy type or $X \simeq Y$) if there exist $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are homotopic to the identity map.

For example, if X deformation retracts to $A \subset X$ via $f : X \times I \to X$ with $r : X \to A$ the retraction and $i : A \to X$ the inclusion, then $ri = \mathbb{1}$, and $ir \simeq \mathbb{1}$ by the homotopy f. Therefore $X \simeq A$. So the two-hole disk and ∞ and θ and the eyeglasses are all homotopy equivalent. Another example is $\mathbb{R}^n$ and the n-dimensional open ball.

As the name suggests, homotopy equivalence is an equivalence relation, as you can check. Deformation retraction is not. $X \simeq Y$ iff there exists Z which deformation retracts to X or Y. If $X \simeq$ a point, then we say X is *contractible*.

3.1 Notions of 'same'

There are many possible notions of when two spaces are 'the same'. They differ by what structure we care about. For example, if we are interested in spaces with metrics, we regard two spaces to be equivalent if they are related by an invertible isometry – a smooth map which preserves the metric. If we are just interested in doing calculus, two smooth manifolds are equivalent if they are related by an invertible smooth map (diffeomorphism). If we are only doing topology, continuous maps – homeomorphisms

are enough.

Homotopy vs homeomorphism. Homotopy equivalence is yet another notion of equivalence, which depends only on continuity, but is weaker than homeomorphism. There are manifolds which are homotopy equivalent but not homeomorphic: an example of such a pair is the 3d lens spaces $L_{1/7}$ and $L_{2/7}$, which can be defined as quotients of S^3 like $\mathbb{RP}^3$. Later I plan to talk about a physically-motivated invariant which can distinguish them, called *torsion*, with various adjectives in front of it. (Annoyingly, it is not the same as torsion homology.) A much simpler example is given above[35]: a ball and a point are homotopy equivalent by the deformation retraction, but they have different dimension. The dimension of a manifold is a homeomorphism invariant, but not a homotopy invariant.

There are still other equivalence relations we might care about. For example, we could regard two n-manifolds Y_0 and Y_1 as equivalent if there exists some $(n+1)$-manifold X with $\partial X = Y_1 - Y_0$. Such an X is called a *bordism* between $Y_{0,1}$, and Y_0 is said to be bordant to Y_1. A closed manifold is a bordism from the empty manifold to itself. Not all 0-dimensional manifolds are bordant to each other: the number of points is conserved mod two, since a 3-string junction is not a manifold. However, a circle is bordant to any number of circles by a disk or an annulus or a pair of pants. The existence of the creamy filling of a donut with g handles means that any Riemann surface is bordant to nothing: bordism equivalence doesn't even preserve the euler character. It does preserve it mod two, however, since an unoriented surface is not the boundary of a 3-manifold. This notion plays an important role in the spacetime definition of topological field theory. Maybe we'll come back to that.

[End of Lecture 14]

3.2 Homotopy equivalence and cohomology

[Bott and Tu p. 35] Here we will show that if two manifolds are homotopy equivalent, then they have the same de Rham cohomology. The proof follows mostly from our discussion of the Poincaré Lemma, *i.e.* the isomorphism between $H^\bullet_{dR}(M)$ and $H^\bullet_{dR}(M\times\mathbb{R})$ around (2.32).

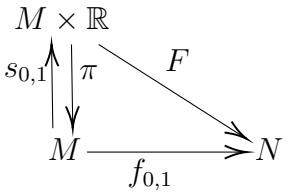
Lemma: Homotopic maps induce the same map on cohomology.
Proof: A homotopy between $f_0, f_1 : M \to N$ is a continuous map

$$F : M \times \mathbb{R} \to N \quad \text{with} \quad F(x,t) = f_t(x) \text{ for } t = 0, 1.$$

[35] Thanks to Meng Zeng for reminding me.

Let $s_{0,1} : M \to M \times \mathbb{R}$ be the 0-section and 1-section, $s_t(x) \equiv (x, 1) \in M \times \mathbb{R}$. Then $f_t = F \circ s_t$ so

$$f_t^\star = (F \circ s_t)^\star = s_t^\star \circ F^\star. \tag{3.1}$$

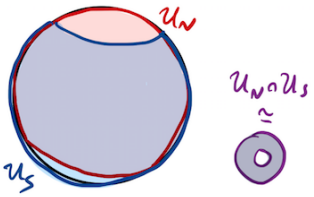


But now recall our earlier discussion: we could have used either s_0 or s_1 in that discussion, since the choice of origin was completely arbitrary; this means that either $s_0^\star$ or $s_1^\star$ is an inverse of $\pi^\star$ on the cohomology, so they are equal, $s_0^\star = s_1^\star$ on $H^\bullet(N)$, and therefore (3.1) says $f_0^\star = f_1^\star$. ■

An immediate consequence of this is that if $X \simeq Y$ then $H^\bullet_{dR}(X) \simeq H^\bullet_{dR}(Y)$. $X \simeq Y$ means there exist $f : X \to Y$ and $g : Y \to X$ with $g \circ f \equiv f_0$ homotopic to $\mathbb{1} = f_1$. But this means $f^\star \circ g^\star = \mathbb{1}$ on the cohomology of X, and $g^\star \circ f^\star = \mathbb{1}$ on the cohomology of Y so $f^\star = (g^\star)^{-1}$ is an isomorphism.

In particular, if A is a deformation retract of X, then $H^\bullet(X) \simeq H^\bullet(A)$. An example we've used already is that the Poincaré Lemma about $H^\bullet(\mathbb{R}^n)$ also tells us $H^\bullet(n\text{-ball})$ and $H^\bullet(\text{point})$.

As another example, we now know that $H^\bullet(\text{annulus} \equiv S^{n-1} \times I) = H^\bullet(S^{n-1})$. We can use this, with the Mayer-Vietoris sequence, to compute $H^\bullet(S^n)$ iteratively. Cover S^n with two patches – U_S = everything but the north pole, and U_N = everything but the south pole; $U_S \cap U_N$ is an annulus, which is homotopy equivalent to S^{n-1}.



There is one minor wrinkle: in using the pullback all over the place above, I've assumed that the maps f and g inducing the homotopy equivalence are smooth. What if X and Y are only homotopy equivalent via continuous but not smooth maps? This doesn't happen: every continuous map $f : X \to Y$ is homotopic to a C^∞ map. This is Bott and Tu proposition 17.8.

3.3 Homotopy equivalence and homology

[Hatcher p. 133] Now we wish to show that homotopy-equivalent manifolds also have the same homology. (This is the argument that leads to the name 'homotopy operator'.) This is actually a stronger statement than the previous subsection, since it also incorporates torsion, which is invisible to the deRham theory.

First, a map $f : X \to Y$ induces a map on the chains, $f_\sharp : \Omega_\bullet(X) \to \Omega_\bullet(Y)$. To see this, think of a q-cell as a continuous map $\sigma_q : \Delta_q \to X$ including the cell into the manifold (Δ_q is an ideal q-cell, such as q-simplex or a q-cube, I^q, and this is called the

characteristic map of the cell). Then

$$f_\sharp(\sigma) = f \circ \sigma : \Delta_q \to Y.$$

Here we are using a particular cell decomposition of Y, which is compatible with the one induced (from the decomposition of X) by these maps. Since we've proved subdivision invariance of the homology, we lose no generality here[36]. The induced map on chains is a chain map: $f_\sharp \partial = \partial f_\sharp$. Therefore, there is an induced map on homology $f_\star : H_\bullet(X) \to H_\bullet(Y)$. (That is, $H_\bullet$ is a covariant functor from manifolds to abelian groups.)

Lemma: If $f_{0,1} : X \to Y$ are homotopic, $f_0 \simeq f_1$, then $f_{0\star} = f_{1\star} : H_\bullet(X) \to H_\bullet(Y)$, the induced maps on homology are the same.

Main result: If $f : X \to Y$ is a homotopy equivalence $X \simeq Y$, then $f_\star : H_\bullet(X) \to H_\bullet(Y)$ is an isomorphism.

Proof of main result: f is a homotopy equivalence means that there is a map $g : Y \to X$ with $f \circ g \simeq \mathbb{1}$ and $g \circ f \simeq \mathbb{1}$. $(fg)_\star = f_\star g_\star$ and $\mathbb{1}_\star = \mathbb{1}$, so $f_\star g_\star = \mathbb{1} = g_\star f_\star$ on homology.

Proof of lemma: Given a homotopy map $F : X \times I \to Y$, we can construct a homotopy operator $K : \Omega_q(X) \to \Omega_{q+1}(Y)$ such that $K\partial + \partial K = f_{0\sharp} - f_{1\sharp}$ on $\Omega_q(X)$. This immediately implies that a q-cycle $\alpha \in \Omega_q(X)$ ($\partial\alpha = 0$) satisfies

$$\partial(K\alpha) = (f_{0\sharp} - f_{1\sharp})(\alpha)$$

and hence $[f_{0\sharp}] = [f_{1\sharp}] \in H_{q+1}(Y)$, *i.e.* $f_{0\star} = f_{1\star}$.

So who is K? The idea is just that a cell decomposition of X implies a cell decomposition of $X \times I$. If we want the cells to be simplices we have to do some annoying further subdivision to cut it up into triangles. (Nearly all the effort and complication in Hatcher's proof of his Theorem 2.10 is devoted to this.) Since we don't care about that, we can just define the image of a q-chain σ in the cell decomposition of X

$$K(\sigma) = F(\sigma \times I),$$

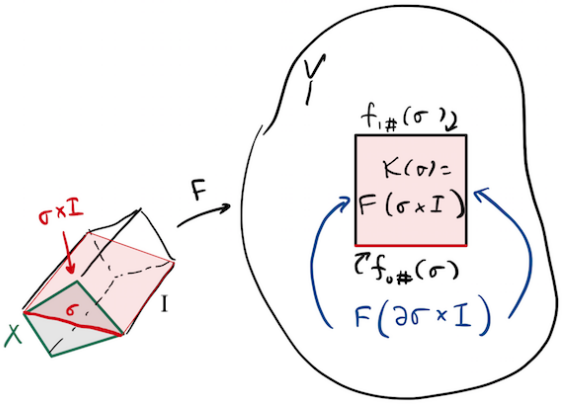
a $(q+1)$-chain in the (compatible) cell decomposition of Y.

[36] This irritation can be avoided by using what is called *singular homology*. The idea there is to define a complex from all possible continuous maps from simplices into X. This has the drawback that it's not obviously finite-dimensional, but can be proved to be equivalent to the cellular homology we studied before; see Hatcher §2.1 for more.

As you can see in the figure at right, it indeed satisfies

$$\partial K(\sigma) = K(\partial\sigma) + f_{1\sharp}(\sigma) - f_{0\sharp}(\sigma).$$

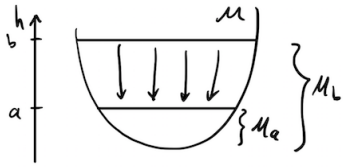
The first term comes from the 'vertical' boundaries along the homotopy, and the last two terms come from the top and bottom. (If we did subdivide $X \times I$, all the internal boundaries of $K(\sigma)$ must be defined to cancel out.) ■



3.4 Morse theory and homotopy equivalence

Since we spent some time learning about Morse theory earlier, I don't feel bad about a brief digression here using it to learn about the homotopy type of manifolds.

Suppose h is a Morse function on some smooth manifold M. Let $M_a \equiv h^{-1}([-\infty, a])$. If $h^{-1}([a,b])$ is compact and contains no critical points of h then $M_a \simeq M_b$.



Here's why: put a metric γ_{ij} on M. Define the gradient, an operation which makes a vector field from a scalar field, by

$$\gamma_{ij}\nabla^i f Y^j \equiv \left\langle \vec{\nabla} f, \vec{Y} \right\rangle \equiv df(\vec{Y})$$

(at each point in M) for any vector field Y on M. Let $\vec{X} \equiv -\vec{\nabla}h/\|\vec{\nabla}h\|$ ($\|\vec{Y}\| \equiv \sqrt{\langle Y, Y\rangle}$). This is a unit vector field, well-defined away from critical points of h (and in particular on $h^{-1}([a,b])$ under the hypotheses above); it points in the direction of fastest decrease of h. Flowlines of $\vec{X}$ give a deformation retraction from M_b to M_a. ■

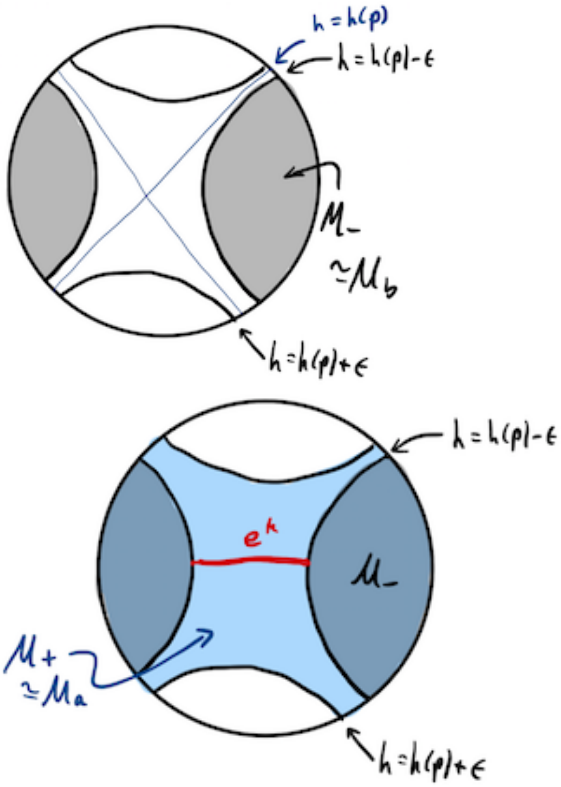
Here's a slightly more ambitious statement: Suppose $h^{-1}([a,b])$ is compact and contains a single (non-degenerate) critical point of Morse index k. Then $M_b \simeq M_a \cup e^k$. The RHS is a space obtained by attaching a k-cell to M_a.

Here's the idea: as in our discussion of the NLSM, there exist coordinates on M near the critical point p which diagonalize the hessian $\partial^2 h|_p$, where

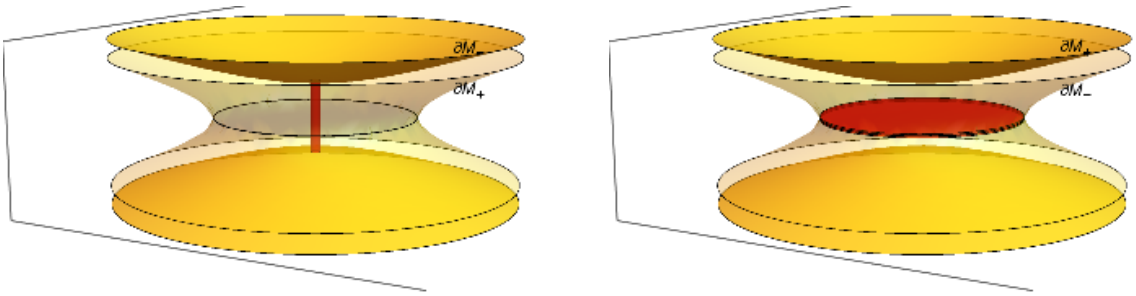
$$h = h(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2 + \mathcal{O}(x^3).$$

Consider the level sets of this function that lie just above and just below the critical value $h = h(p)$. At the critical value, the level set is not a smooth manifold. (The case $n = 2, k = 1$ is depicted at right. In the general case, the level sets near the critical point are the hyperboloids of rotation obtained by rotating this picture in both $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$ (the negative and positive eigenspaces of the hessian).) If we define $M_- \equiv h^{-1}(h(p) \leq \epsilon) \simeq M_b$ and $M_+ \equiv h^{-1}(h(p) \leq \epsilon) \simeq M_a$, we can see from the picture that we must attach a k-cell to M_- to get M_+.

A consequence of the existence of (generic!) Morse functions is that any compact manifold is homotopy equivalent to a finite cell complex.



Here are pictures of the cases $n = 3$ and $k = 1, 2$ respectively:



In the left picture, the inside of ∂M_- is filled to make M_-; we attach to this the red 1-cell and obtain a region homotopic to M_+ which is the inside of ∂M_+. In the right picture, the outside of ∂M_- is filled to make M_-; we attach to this the red 2-cell and obtain a region homotopic to M_+ which is the region between the two components of ∂M_+.

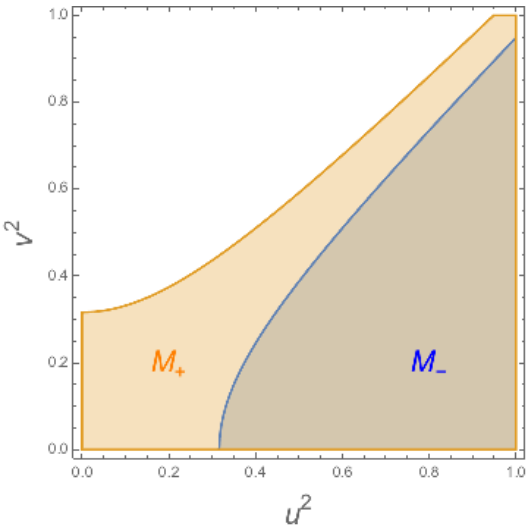
We can plot the general case in terms of

$$u^2 \equiv \sum_{i=1}^{k} x_i^2, \quad v^2 \equiv \sum_{i=k+1}^{n} x_i^2,$$

in terms of which

$$M_\pm \sim \{-u^2 + v^2 \leq \pm\epsilon\}.$$

Each point in this picture is a $S^{k-1} \times S^{n-k-1}$. Near the critical point, the Morse function has has an approximate $\mathsf{SO}(k) \times \mathsf{SO}(n-k)$ symmetry.



3.5 Homotopy groups

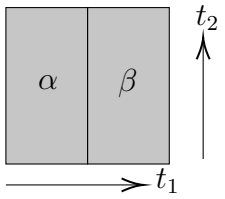
Let X be a topological space with a *base point*, $p \in X$, just a point in X that we like for some reason. The homotopy groups of X are:

$$\pi_q(X) \equiv \text{homotopy classes of maps} : (I^q, \partial I^q) \to (X, p).$$

Here $I^q \equiv I \times I \times \cdots I$ is the inside of the q-dimensional cube, homotopic to a q-ball, and its boundary is the unit cube, homotopic to a $(q-1)$-sphere. Since all points in ∂I^q map to the same point in X, an equivalent definition would consider maps from $I^q/\partial I^q \simeq S^q$ taking the north pole to the base point.

For $q > 0$, π_q is a group under the following product operation: Given $\alpha, \beta : (I^q, \partial I^q) \to (X, p)$ (so that $[\alpha], [\beta] \in \pi_q(X)$), define $[\alpha][\beta] = [\alpha \star \beta]$ where $\alpha \star \beta$ is the map

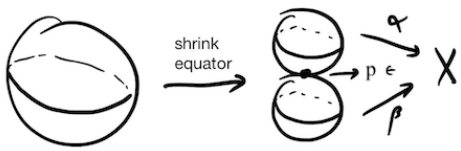
$$(\alpha \star \beta)(t_1, \cdots, t_q) = \begin{cases} \alpha(2t_1, t_2, \cdots, t_q) & , 0 \leq t_1 \leq \frac{1}{2} \\ \beta(2t_1 - 1, t_2, \cdots, t_q) & , \frac{1}{2} \leq t_1 \leq 1 \end{cases}$$



(3.2)

(I draw the picture for $q = 2$. All the black lines map to the base point.)

If instead we were using maps from S^q, we first map S^q to two S^qs attached at their north poles by shrinking the equator to a point, then map the top sphere by α and the bottom sphere by β.

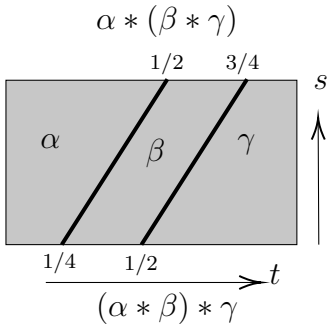


For $q = 0$, $\pi_0(X) = \text{maps} : \text{point} \to X/\simeq \equiv [\text{point}, X]$ is not in general a group. Rather $\pi_0(X)$ is the set of path components of X. (For the special case where $X = G$ is a Lie group, $\pi_0(G) = G/G_0$, where G_0 is the component of G containing the identity element; this is a group.)

Basic facts about homotopy groups:

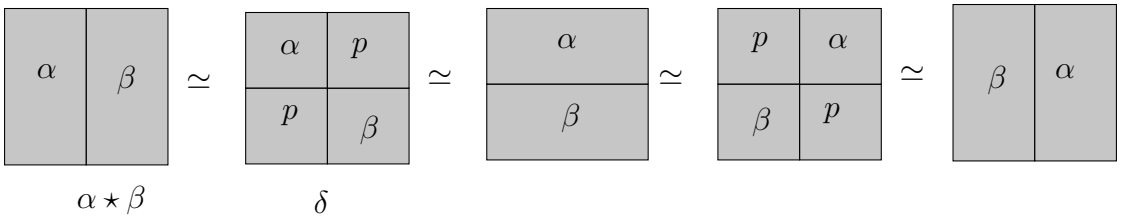
1. $\pi_q(X)$ is a group. The identity operation is the constant map to the base point. The inverse is $[f^{-1}(t_1 \cdots t_q)] = [f(1 - t_1, t_2, \cdots t_q)]$.

   The product is associative in the sense that $\alpha * (\beta * \gamma) \simeq (\alpha * \beta) * \gamma$ are homotopy equivalent. Here is the homotopy:



2. $\pi_q(X)$ is abelian for $q > 1$. $\pi_1(X)$, called the *fundamental group of* X, is special in that it can be non-abelian.

   Proof of statement 2:



   $\alpha \star \beta$ in the first step is the map in (3.2). Here the definition of the map δ:

$$\delta(t_1, \cdots, t_q) = \begin{cases} \alpha(2t_1, 2t_2 - 1, \cdots, t_q) & , 0 \le t_1 \le \frac{1}{2}, \frac{1}{2} \le t_2 \le 1 \\ \beta(2t_1 - 1, 2t_2, \cdots, t_q) & , \frac{1}{2} \le t_1 \le 1, 0 \le t_2 \le \frac{1}{2} \\ p, \text{otherwise} \end{cases}$$

   – the points in the lower left and upper right all map to the base point. ■

3. If $X \simeq Y$ then $\pi_q(X) \cong \pi_q(Y)$.

4. $\pi_q(X \times Y) = \pi_q(X) \times \pi_q(Y)$. Even simpler than the Kunneth formula.

   Basic fact 4 follows from the fact that any map $I^q \to X \times Y$ is of the form (f_x, f_y) with $f_x : I^q \to X, f_y : I^q \to Y$. And it is a group homomorphism since $(f_x, f_y) \star (g_x, g_y) = (f_x \star g_x, f_y \star g_y)$.

5. Let $\Omega_p X \equiv \{\text{continuous maps} : (I^1, \partial I^1) \to (X, p)\} \equiv$ the *loop space* of X. So the definition of $\pi_1(X,p)$ is just $\pi_0(\Omega_p(X))$. For $q > 2$ also[37], $\pi_{q-1}(\Omega_p X) = \pi_q(X,p)$. The idea is that a representative of $\pi_q(X)$, $f : I^q \to X$ can be viewed instead as a map $I^{q-1} \to \Omega X$ just by picking a slice of I^q.

   A corollary of this statement is that $\pi_1(\Omega X)$ is always abelian.

6. Like homology, π_q is a covariant functor from the category of topological spaces (and continuous maps) to the category of groups (and group homomorphisms). To see this, consider a map $\phi : (X, x_0) \to (Y, y_0)$. Given a representative of $\pi_q(X)$, $\alpha : (I^q, \partial I^q) \to (X, x_0)$, we can use ϕ to make a representative of $\pi_q(Y)$, namely $\phi \circ f : (I^q, \partial I^q) \to (Y, y_0)$. So we can define an induced map on the homotopy groups
$$\phi_\star[\alpha] \equiv [\phi \circ f].$$
   This is a group homomorphism in the sense that $\mathbb{1}_\star = \mathbb{1}$, $\phi \circ (\alpha \star \beta) = (\phi \circ \alpha) \star (\phi \circ \beta)$ and given also $\psi : (Y, y_0) \to (Z, z_0)$, we have $\psi_\star \circ \phi_\star = (\psi \circ \phi)_\star$.

   One consequence of this is the obvious-sounding statement (basic fact 3) that homotopy-equivalent spaces have the same homotopy groups (Hatcher Proposition 1.18). If $f : X \to Y$ and $g : Y \to X$ are the relevant maps then the induced map $f_\star : \pi_q(X, x_0) \to \pi_q(X, f(x_0))$ is an isomorphism with inverse $g_\star$.

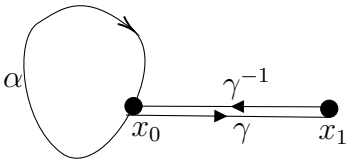
7. Making the choice of base point explicit, $\pi_q(X,p) \cong \pi_q(X,p')$ (*i.e.* they are isomorphic groups) if X is path connected. So we don't need to make the choice of base point explicit.

[37] This is a teleological footnote using ideas we'll develop below to prove this statement more. Please ignore it on a first pass. I learned it from Bott and Tu and here. The following sequence (for X path connected) defines a fiber bundle
$$\Omega X \to PX \to X \tag{3.3}$$
called the *path fibration* where $PX \equiv \{\text{maps } \mu : I \to X, \mu(0) = p\}$ (p is the base point). The first map is just inclusion ($\Omega X = \{\mu \in PX | \mu(1) = p\}$), and the second map is the projection $\pi(\mu) = \mu(1)$.

The statement that (3.3) is a fiber bundle means the second map in (3.3) satisfies the 'homotopy lifting property' or 'covering homotopy property' (this elaborate-seeming statement is explained on page 198-199 of Bott and Tu; the fact that it holds for $\pi : PX \to X$ is simple when X is path-connected). This means the short-exact sequence (3.3) induces a long-exact sequence on the homotopy groups
$$\cdots \to \pi_q(\Omega X) \to \pi_q(PX) \to \pi_q(X) \to \pi_{q-1}(\Omega X) \to \cdots$$
And finally PX is contractible because each path can be deformation retracted to the base point, or as it says in the link above: "the picture is that of sucking spaghetti into one's mouth".

So the exactness of the long-exact sequence means $\pi_q(X) \cong \pi_{q-1}(\Omega X)$.

About the dependence on the base point (item 7): Suppose that X is path connected. A path γ from x_0 to x_1 induces a map on the loop spaces $\Omega_{x_0}X \to \Omega_{x_1}X$ by $\alpha \to \gamma \star \alpha \star \gamma^{-1}$ (where γ^{-1} means the path γ traversed backwards).



This induces a map

$$\gamma_\star : \pi_{q-1}(\Omega_{x_0}X, \overline{x_0}) \to \pi_{q-1}(\Omega_{x_1}X, \overline{x_1})$$

where $\overline{x_t}$ is the constant map to x_t. But this is the same as a map

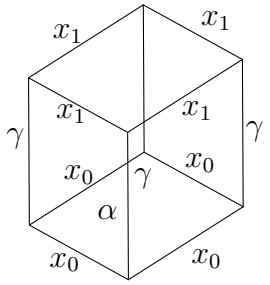
$$\gamma_\star : \pi_q(X, x_0) \to \pi_q(X, x_1).$$

This map is an isomorphism, with $(\gamma^{-1})_\star = (\gamma_\star)^{-1}$.

More explicitly, for $[\alpha] \in \pi_q(X, x_0)$ define a homotopy

$$F : I^{q+1} = I^q \times I \to X$$

as follows. For $u \in I^q$, we want $F(u, 0) = \alpha(u)$ and $F(u, t) = \gamma(t), \forall u \in \partial I^q$. This defines the map on all but one face of ∂I^{q+1}. Now I quote a theorem ('the box principle of obstruction theory') that such an F can be extended to all of I^{q+1}, and we define $[F(u, 1)] = \gamma_\star[\alpha]$.



If we take $x_0 = x_1$, this defines an action of $\pi_1(X, x_0)$ on $\pi_q(X, x_0)$ describing the result of moving the base point around in a non-contractible loop. Its nontriviality measures something about how much the choice of base point matters. Bott and Tu Prop. 17.6.1 shows that

$$\pi_q(X, x_0)/\pi_1(X, x_0) \cong [S^q, X]$$

where the quotient on the LHS is by the action defined above, and the RHS is homotopy classes of maps from S^q to X without any notion of base point ('free homotopy'). This is not a group. The map from the LHS to the RHS is just inclusion of base-point-preserving maps into the set of all maps.

What's special about I_q or spheres? It is actually possible to define homotopy groups of maps from other spaces. But in general the set of homotopy classes of maps from one space to another is not a group.

Higher homotopy groups are hard to compute. Even for spheres $\pi_q(S^n)$ are not all

known for large-enough q and n. For small q, n here is the table:

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	$\mathbb{Z}_2{}^2$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{24}\times\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{120}\times\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^5$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{72}\times\mathbb{Z}_2$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_{60}$	$\mathbb{Z}_{24}\times\mathbb{Z}_2$	$\mathbb{Z}_2^3$
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{120}$	$\mathbb{Z}_2^3$
S^8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{120}$

(from here; that page also has a larger table).

Fundamental group. Let's focus on $q = 1$, the fundamental group, for a bit. In this case the product is simpler to understand: a representative of $\pi_1(X)$ is just a closed path in X starting and ending at the base point. The product of two paths is just one followed by the other, with the parameter rescaled so that total duration is still 1.

There is information in $\pi_q(X)$ that is not present in the homology. For example, $\pi_1(X)$ has more information than $H_1(X)$. Here is an example of a space X with trivial $H_1(X, \mathbb{Z})$ (such a space is called 'acyclic') but nontrivial $\pi_1(X)$. Take a figure eight and glue in two 2-cells whose boundaries are a^5b^{-3} and $b^3(ab)^{-2}$ where a and b are the two loops. The cell complex is then

$$0 \to \mathbb{Z}^2 \xrightarrow{M} \mathbb{Z}^2 \xrightarrow{0} \mathbb{Z} \to 0$$

with

$$\partial_2 = M = \begin{pmatrix} 5 & -2 \\ -3 & 1 \end{pmatrix} . \quad \text{Since } \det M = -1$$

there is not even any torsion first homology. But

$$\pi_1(X) = \langle a, b | a^5b^{-3} = 1, b^3(ab)^{-2} = 1 \rangle = \langle a, b | a^5 = b^3 = (ab)^2 \rangle = I^\star \tag{3.4}$$

the binary icosahedral group, a double cover of the icosahedral group (the symmetry group of the icosahedron and dodecahedron) $I \cong A_5$, also known as the alternating group on (*i.e.* the even permutations of) five elements, $S_5/\mathbb{Z}_2$. Under the surjection $I\star \to I$ a maps to a $2\pi/5$ rotation through center of a pentagon, and b maps to a $2\pi/3$

rotation through a vertex. The double cover of I arises by the inclusion $I \subset \mathsf{SO}(3)$, and is induced by the double cover $\pi : \mathsf{SU}(2) \to \mathsf{SO}(3)$.

Why is (3.4) the answer for $\pi_1(X)$? Well, π_1 of a bouquet of circles – q circles attached at a point (which we take to be the base point) – is the *free group* on q elements. Each circle provides a generator, and there are no relations. A bouquet of q circles is a deformation retract of $\mathbb{R}^2 \setminus \{q \text{ points}\}$ (or $\mathbb{R}^3 \setminus \{q \text{ lines}\}$),  , so this is also π_1 of the latter spaces.

This free group on ≥ 2 elements is a deeply horrible object. The elements are all words made of the letters $a_1, a_2 \cdots a_q$ and $a_1^{-1}, a_2^{-1} \cdots a_q^{-1}$, and the only relation is that you can cancel an a_i and an a_i^{-1} if they are right next to each other. It contains copies of itself as a subgroup. Ick.

When we glue in a 2-cell we introduce a relation in π_1 according to the gluing map; this gives (3.4).

This example is related to the existence of the Milnor homology sphere M – a 3-manifold with the same homology as a sphere, but different homotopy groups. M can be defined as $S^3/I^\star$ (with $I^\star$ the binary icosahedral group as above). Therefore $\pi_1(M) = I'$. There is a lot to say about this space.

General Fact: $H_1(X, \mathbb{Z})$ is the abelianization of $\pi_1(X)$, *i.e.*

$$H_1(X, \mathbb{Z}) = \pi_1(X)/[\pi_1(X), \pi_1(X)]$$

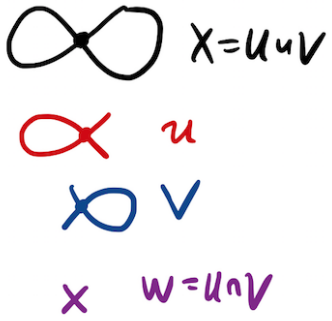
where $[G, G] \equiv \langle ghg^{-1}h^{-1}, g, h \in G \rangle$ is the commutator subgroup of G, the subgroup generated by (multiplicative) commutators of elements of G. This is not hard to see: the whole difference between π_1 and H_1 is that in the latter we keep track of the order in which the closed loops are traversed. Modding out by commutators is erasing exactly this information. [For a more explicit proof, see Hatcher Theorem 2A.1.]

van Kampen Theorem. [Justin Roberts' knot knotes has a very nice discussion with a proof sketch] Here is an analog of the Mayer-Vietoris idea for the fundamental group. Let $X = U \cup V$, two open sets, and let $W \equiv U \cap V$. Denote $\pi_1(Y) \equiv \langle s_Y | r_Y \rangle$ for $Y = U, V, W$, so s_Y is a set of generators of $\pi_1(Y)$ and r_Y is a set of relations. Let $i^{U,V}$ be the inclusion maps of W into U and V. Then

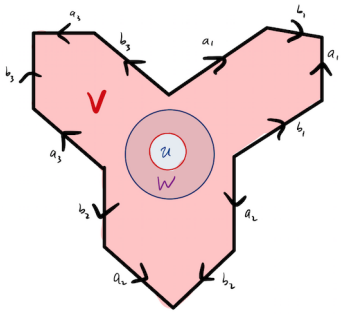
$$\pi_1(X) = \left\langle s_U \cup s_V | r_U \cup r_V \cup \{i^U_\star(g) = i^V_\star(g)\}_{g \in s_W} \right\rangle .$$

That is: a set of generators of $\pi_1(X)$ is just those of U and those of V. This double-counts the generators on the overlap. The theorem says that it's enough to add one relation for each generator of the fundamental group of the overlap.

Example 1: Cut a bouquet of two circles ($S^1 \wedge S^1$, two circles attached at a point) into two open sets U and V where each of U and V removes a single point from one of the circles. The overlap can be deformed to an X which is contractible. $\pi_1(U)$ and $\pi_1(V)$ each have one generator, and there are no relations, so $\pi_1(S^1 \wedge S^1)$ is the free group on two elements.



Example 2: Consider a genus-g Riemann surface Σ_g, described as a polygon with $2g$ sides, with the identifications given in §1.3. Let U be a disk inside the polygon and V a little more than its complement. Then $U \cap V$ is an annulus, homotopy equivalent to a circle, with $\pi_1(S^1) = \langle g \rangle$. The inclusion of g into U maps it to $i^U_\star(g) = 0$, since every loop in U is trivial. The inclusion of g into V is homotopic (in V) to $\prod_{i=1}^g [a_i, b_i]$ with $[a, b] \equiv aba^{-1}b^{-1}$. We conclude that $\pi_1(\Sigma_g) = \langle \{a_i, b_i\}_{i=1}^g | \prod_{i=1}^g [a_i, b_i] \rangle$.



Example 3: The answer we found above for π_1 of the acyclic space X can also be obtained by these methods. It is best to do it in two steps, first removing both disks and then removing just one.

Cellular approximation. Notice that all of these definitions can be applied to the case where we approximate M by a cell complex. The idea is that any path is homotopic in M to a sequence of (oriented) 1-cells.[38]

Let Δ be a path-connected cell complex, with a base point $p \in \Delta_0$, a 0-cell. One way to specify a group is by giving generators and relations: $G = \langle \{g\} | \{r\} \rangle$ is the group whose elements are all products of g (and the identity element e) modulo the relations $r = e$. To Δ we can associate the following group:

$$G_\Delta \equiv \left\langle \{g_\ell\}_{\ell \in \Delta_1} \middle| \{\prod_{\ell \in \partial\sigma} g_\ell = e\}_{\sigma \in \Delta_2} \right\rangle$$

– a generator for every 1-cell, and a relation for every 2-cell.

There are two ways to describe the relationship between G_Δ and $\pi_1(\Delta)$. The more direct is just to say that π_1 is the following subgroup

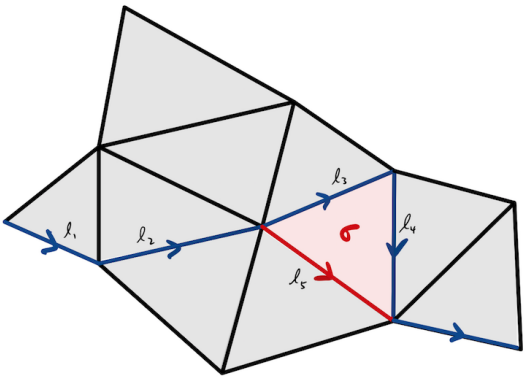
$$\pi_1(\Delta) \cong \{\text{words in } G_\Delta \text{ starting and ending at } p\} \subset G_\Delta.$$

[38] For a more precise discussion of the statements here, I commend your attention to Theorem 13.4 on p. 138 of Bott and Tu.

The idea is just that the relations $\prod_{\ell\in\partial\sigma} g_\ell = e$ are exactly the discrete remnants of the homotopy relations. In the figure at right, the group element

$$\cdots g_{\ell_1} g_{\ell_2} g_{\ell_3} g_{\ell_4} \cdots = g_{\ell_1} g_{\ell_2} g_{\ell_5} \cdots$$

as a consequence of the relation imposed by the 2-cell σ. You can see that these are the group elements associated with two homotopic paths.



Under some further assumptions on the cell complex, there is also the following "Calculating Theorem" (this is the name given to it by Nash and Sen):

$$\pi_1(\Delta) \cong G_\Delta / G_L \tag{3.5}$$

where L is a contractible 1d subcomplex containing all of the 0-cells. The idea here is that we set to 1 all the group elements associated to links in L. Then any edge is equivalent to a path of edges starting from the base point and ending at the base point by adjoining a path through L. Such an L always exists[39], but may be empty. For example, for the cellulation of Σ_g by a single 2-cell, $2g$ 1-cells, and just one 0-cell, p, we can take L to be empty, since any path starts at the base point by default. If we subdivide by adding a new 0-cell in the middle of w with an edge going to each vertex of the polygon, we must choose the new edges to constitute L.

[End of Lecture 16]

Higher homotopy groups and homology. For general q, there is a natural homomorphism

$$\begin{array}{rccc} i: & \pi_q(X) & \to & H_q(X) \\ & [f] & \mapsto & f_\star(u) \end{array}$$

where u is a generator of $H_q(S^q)$. For better or worse, this map is neither injective nor surjective.

One more fact about the relation between homotopy groups and homology, however, is the Hurewicz isomorphism theorem: The first nontrivial homotopy and homology groups of a path-connected manifold occur in the same dimension q. ($q = 0$ doesn't count.) If $q > 1$ then they are isomorphic. (In symbols: if $q > 1$, and $\pi_k(X) = 0$ for $1 \le k < q$, then $H_q(X) = 0$ for $1 \le k < q$ and $H_q(X) = \pi_q(X)$.)

[39] Let L be the 1d contractible subcomplex containing the most 0-cells. (This always exists for a finite complex.) If L didn't contain all the 0-cells, say $b \in \Delta_0 \setminus L_0$, and the edge $\langle ab\rangle \in \Delta_1$, then $L \cup \langle ab\rangle \cup \{b\}$ is bigger but still contractible.

Consider the case $q = 2$. Then the claim is that $H_2(X) \doteq H_1(\Omega X) = \pi_1(\Omega X)$ (since $\pi_1(\Omega X) = \pi_2(X)$ is abelian), and therefore $H_2(X) = \pi_2(X)$. The first equality, with the $\doteq$, follows from $H_1(X) = 0$ but I will not explain it here. There is a general proof on page 225 of Bott and Tu which uses a spectral sequence and induction from the above case.

This theorem implies that $\pi_q(S^n) = \delta^{n,q}\mathbb{Z}$ for $q \leq n$.

3.6 The quantum double model

[Kitaev, quant-ph/9707021, §4,5] (Yes, the same paper.) This is a good place to pause with the mathematical development to introduce the quantum double model, the basic non-abelian generalization of the toric code. Like the $\mathbb{Z}_{N>2}$ toric code, it is defined on an arbitrary oriented cell complex, Δ. The Hilbert space is

$$\mathcal{H} = \prod_{0-\text{cells}} R_{\text{Reg}}$$

where

$$R_{\text{Reg}} \equiv \text{span}\{|g\rangle\,, g \in G\}$$

is the regular representation of a finite group G, which we'll call the *gauge group*. To the edge with opposite orientation we associate $|g^{-1}\rangle$.

The Hamiltonian looks just like that of the toric code (up to an additive constant)

$$H = \sum_{v\in\Delta_0} (1 - A_v) + \sum_{w\in\Delta_2} (1 - B_p)\,.$$

The star and plaquette operators A and B take some explaining, and are defined as follows, in terms of some useful operators acting on the regular rep. Let

$$L^g_+ \equiv \sum_{h\in G} |gh\rangle\langle h|, \quad L^g_- \equiv \sum_{h\in G} |hg^{-1}\rangle\langle h|$$

implement the left and right action of the group (a bit like X), and let

$$T^g_+ \equiv |g\rangle\langle g|, \quad T^g_- \equiv |g^{-1}\rangle\langle g^{-1}|$$

be projectors onto particular group elements (diagonal in the preferred basis, like Z). Note that $T^g_- = T^{g^{-1}}_+$. As you can verify, they satisfy

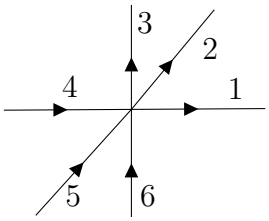
$$L^h_+ T^g_+ = T^{hg}_+ L^h_+, \qquad L^h_- T^g_+ = T^{gh^{-1}}_+ L^h_- \tag{3.6}$$

and similar relations for other combinations of $\pm$. It is sometimes useful to denote $\bar{g} \equiv g^{-1}$ to minimize marks on the page.

$$A_v = \frac{1}{|G|}\sum_{g\in G} A_v^g \equiv \frac{1}{|G|}\sum_{g\in G}\prod_{\ell\in\partial^\dagger(v)} L^g_{s(\ell)}(\ell)$$

where the $s(\ell) = \pm$ for each link is chosen according to the sign with which v appears in $\partial\ell$.

For example, for the cubic lattice with each edge chosen to point north or east or up,

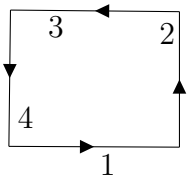


$$A_v^g = L_+^g(1)L_+^g(2)L_+^g(3)L_-^g(4)L_-^g(5)L_-^g(6).$$

$$B_w = \sum_{\prod_{i=1}^{|\partial w|} g_i = e}\prod_{\ell\in\partial w} T^{g_\ell}_{s(\ell)}(\ell)$$

where the sign is $s(\ell) = \pm$ according to whether or not the reference orientation of ℓ agrees with how it appears in ∂p. If ∂w contains k links, the sum is over all collections of $g_1\cdots g_k$ whose product (in the order in which they are traversed by ∂w) is the identity $e \in G$.

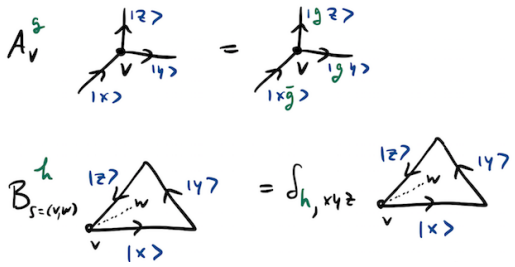
For example, on the cubic lattice with the above convention



$$B_w = \sum_{g_1g_2g_3g_4=e} T_+^{g_1}(1)T_+^{g_2}(2)T_-^{g_3}(3)T_-^{g_4}(4).$$

Both A_v and B_w are projectors, so their eigenvalues are $0, 1$. In the case when G is abelian, they are related to the usual star and plaquette operators by[40] $A \to \frac{1}{2}\left(1 + A^{\rm TC}\right)$ and $B \to \frac{1}{2}\left(1 + B^{\rm TC}\right)$.

In summary, the action of the terms in the Hamiltonian on a group-element-basis state is at right. The Hamiltonian involves only the group-average of the A^gs and only $B^{h=e}$.



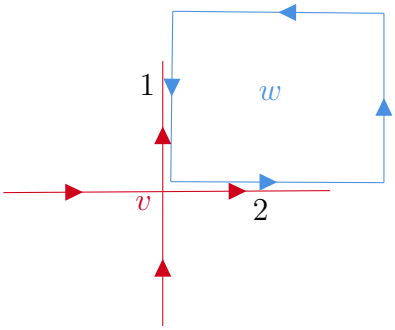
[40] Actually if I want to say that L is analogous to X and T is analogous to Z, I have reversed our convention for the TC that the star operators involve Z. My apologies. On a 2d complex, this mishap can be repaired by simply drawing the pictures on the dual lattice.

All the A_vs and B_ws commute. This follows from the relations (3.6): For example for the star and plaquette sharing the two links $1, 2$, then

$$A_v^h B_w = \cdots L_+^h(1)L_+^h(2) \sum_{g_1g_2g_3g_4=e} T_-^{g_1}(1)T_+^{g_2}(2)\cdots \tag{3.7}$$

$$= \sum_{g_1g_2g_3g_4=e} T_-^{g_1h^{-1}}(1)T_+^{hg_2}(2)\cdots L_+^h(1)L_+^h(1) \tag{3.8}$$

$$= \sum_{g_1'g_2'g_3g_4=e} T_-^{g_1'}(1)T_+^{g_2'}(2)\cdots L_+^h(1)L_+^h(1) = B_w A_v^h \tag{3.9}$$



with $g_1' \equiv g_1h^{-1}, g_2' \equiv hg_2$.

If you are interested in messing with the definition of the quantum double model to try to find other solvable lattice models associated with non-abelian groups, an important observation is that unlike the toric code where all the ingredients in the star operators commute, the L's do not commute: $[L_+^g, L_+^h] \neq 0$ if G is non-abelian. ($[L_+^g, L_-^h] = 0$, however.) This makes it quite a bit harder to generalize this model.

So the groundstates satisfy $B_w |\Psi\rangle = |\Psi\rangle = A_v |\Psi\rangle$ for all v, w. You can already see a connection between these groundstates and $\pi_1(\Delta)$ via the 'calculating theorem', (3.5): to each edge we associate a group element, but because of the condition $B_w = 1$, we keep only those configurations where the product of group elements around each plaquette is the identity, $\prod_{\ell\in\partial\sigma} g_\ell = e$. One difference is that these group elements are chosen from G, which comes with its own extra relations.

To understand this space of groundstates better, we'll develop a little bit more technology. A question we can address first, though, is the excitations.

Elementary excitations. As in the toric code there are two classes of excitations, generalizing the e and m particle in the 2d case. The analog of the e particle is a vertex that fails to satisfy the star condition. This means that rather than satisfying $A_v |\Psi\rangle = |\Psi\rangle$, a state with such an excitation transforms nontrivially under the individual A_v^g. In fact, we can find states with a particle excitation at v that transform as a *representation* R^a of G under the action of A_v^g:

$$A_v^g |\Psi_i^a\rangle = (D^a(g))_{ij} |\Psi_j^a\rangle \tag{3.10}$$

where $(D^a(g))_{ij}$ are the representation matrices for some representation of G. In particular

$$(D^a(g))_{ij} (D^a(h))_{jk} = (D^a(gh))_{ik} . \tag{3.11}$$

Notice that associated with this particle at v is a whole Hilbert space, a representation of G, rather than just a single state as in the abelian case.

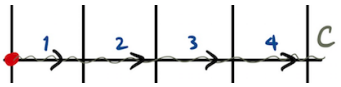
To construct the state explicitly, choose a string C ending at the vertex v. (It has another end, of course, but let's send it off to infinity and ignore it for simplicity. The particle at the other end has to transform in the conjugate rep $R^{\bar{a}}$.) We're going to make the analog of the string operator $\prod_{\ell \in C} Z_\ell$, that commutes with the toric code Hamiltonian except at the endpoints v where it fails to commute with the star operator $\prod_{\ell \in \partial^\dagger(v)} X_\ell$. To do so it's useful to think about what is the Z operator. One way to write it in the $\mathbb{Z}_N$ case is $Z = \sum_{g \in \mathbb{Z}_N} D(g)|g\rangle\langle g|$ where $D(g)$ is a nontrivial representation of $\mathbb{Z}_N$, which is 1 dimensional because $\mathbb{Z}_N$ is abelian. To generalize this to an arbitrary group, let

$$Z^a_{ij} \equiv \sum_{g \in G} \left(D^a(g)\right)_{ij} |g\rangle\langle g|$$

– the analog of σ^z is a matrix of operators in transforming in some representation of G, and there is an analog for each representation. I will take a to label an irrep.

Now instead of just multiplying the string of Zs, we matrix multiply them:

$$W^a(C)_i \equiv Z^a_{ij}(1) Z_{jk}(2) Z_{kl}(3) \cdots$$



where $1, 2, 3$ label links along the curve C and the repeated indices are summed. Including the other end of the string, the group representation property (3.11) implies

$$W^a(C)_{if} \equiv Z^a_{ij}(1) Z_{jk}(2) Z_{kl}(3) \cdots Z_{mf}(N) \tag{3.12}$$

$$= \sum_{g_1, \cdots, g_N} \left(D^a\left(g_1 g_2 \cdots g_N\right)\right)_{if} |g_1, \cdots, g_N\rangle\langle g_1, \cdots, g_N| \tag{3.13}$$

where N is the number of links in the curve C.

To see why the matrix multiplication is a good idea, consider the algebra of W with the terms in H. It commutes with the plaquettes B_w because it is diagonal in the $|g\rangle$

basis. Now act with A_{v_1} where v_1 is a vertex somewhere along the curve C:

$$A_{v_1} W^a(C)_{if} = \frac{1}{|G|} \sum_{h\in G} \cdots L_-(n)^h L_+^h(n+1) \cdots \sum_{g_n, g_{n+1}} D^a\left(g_n g_{n+1}\right)_{i_n i_{n+1}} |g_n\rangle\langle g_n| \otimes |g_{n+1}\rangle\langle g_{n+1}| \cdots \tag{3.14}$$

$$= \frac{1}{|G|} \sum_{h} \cdots \sum_{g_n, g_{n+1}} D^a\left(g_n g_{n+1}\right)_{i_n i_{n+1}} |g_n h^{-1}\rangle\langle g_n| \otimes |h g_{n+1}\rangle\langle g_{n+1}| \cdots \tag{3.15}$$

$$= \frac{1}{|G|} \sum_{h} \cdots \sum_{\tilde{g}_n \equiv g_n h^{-1}, \tilde{g}_{n+1} \equiv h g_{n+1}} D^a\left(\tilde{g}_n \tilde{g}_{n+1}\right)_{i_n i_{n+1}} |\tilde{g}_n\rangle\langle \tilde{g}_n h| \otimes |\tilde{g}_{n+1}\rangle\langle h^{-1}\tilde{g}_{n+1}| \cdots \tag{3.16}$$

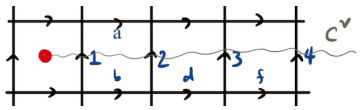
$$= \cdots \sum_{\tilde{g}_n \equiv g_n h^{-1}, \tilde{g}_{n+1} \equiv h g_{n+1}} D^a\left(\tilde{g}_n \tilde{g}_{n+1}\right)_{i_n i_{n+1}} |\tilde{g}_n\rangle\langle \tilde{g}_n| \otimes |\tilde{g}_{n+1}\rangle\langle \tilde{g}_{n+1}| \cdots \frac{1}{|G|} \sum_{h\in G} \cdots L_+^h(n) L_-^h(n+1) \tag{3.17}$$

$$= W^a(C)_{if} A_{v_1} \tag{3.18}$$

(where the $\cdots$ are factors that commute). If we act instead with A_v (the vertex at the end of the curve) we'll find (3.10). This works in any dimension.

The analog of the m particle is a codimension-two excitation around which the product of group elements is $g_1 \cdots g_N = g \neq e$. To insert this flux, we need the analog of $\prod_{\ell \perp \check{C}} X_\ell$. But we can't just act with

$$\prod_{\ell \perp \check{C}} L_+^g(\ell)$$



because that would mess up the plaquette operators. (Acting with $L_+^g(1)L_+^g(2)$ replaces $\bar{g}_1 g_b g_2 \bar{g}_a = e$ with $\bar{g}_1 \bar{g} g_b g g_2 \bar{g}_a \neq e$ – there's a g_b in the way preventing the g and $\bar{g}$ from eating each other.) Instead we must consider

$$V_{\check{C}} = L_+^g(1) \sum_{g_b} T_+^{g_b}(b) L_+^{g_b^{-1} g g_b}(2) \sum_{g_d} T_+^{g_d}(d) L_+^{(g_b g_d)^{-1} g g_b g_d}(3) \cdots .$$

This is special to two dimensions. Otherwise there are more plaquettes to worry about messing up.

In Kitaev's paper he presents 'ribbon operators' which create the general pointlike excitation in the 2d quantum double model; like the ϵ particle, these are combinations of the two things above. Maybe I should mention that the name 'quantum double model' comes from the fact that the general excitation is an irrep of an algebra called the quantum double (or Drinfeld double) of the group G, which had been studied

previously. This is the algebra realized by the operators $A_s^g B_s^h$ acting on a *site*, $s = (v, w)$ which means a vertex and a face that it touches.

$$A_s^g A_s^{g'} = A_s^{gg'},\ B_s^g B_s^{g'} = \delta_{gg'} B_s^g,\quad A_s^g B_s^h = B_s^{ghg^{-1}} A_s^g,\quad (A_s^g)^\dagger = A_s^{g^{-1}},\quad (B_s^g)^\dagger = B_s^{g^{-1}}.$$

This connection between the excitations and the irreps of this thing is special to 2d.

3.7 Fiber bundles and covering maps

Based on our experience with homology and cohomology, you might expect that means that an 'exact sequence of spaces' like

$$0 \to F \xrightarrow{i} E \xrightarrow{\pi} B \to 0 \tag{3.19}$$

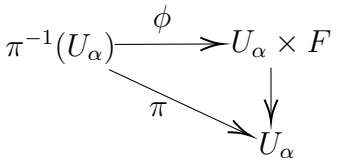
will induce a long exact sequence on their homotopy groups

$$\cdots \to \pi_q(F) \xrightarrow{i_*} \pi_q(E) \xrightarrow{\pi_*} \pi_q(B) \xrightarrow{\partial} \pi_{q-1}(F) \to \cdots \to \pi_1(B) \xrightarrow{\partial} \pi_0(F) \xrightarrow{i_*} \pi_0(E) \xrightarrow{\pi_*} \pi_0(B) \to 0. \tag{3.20}$$

In general this is not quite true. But with some extra assumptions on the sequence of continuous maps (3.19) it is. The extra assumption says that E is a *fiber bundle*; B is the base, and $F = \pi^{-1}(b_0)$ is the fiber. I mention this here also because this notion will play an important role in the interpretation of the quantum double model.

Part of the assumption is that a neighborhood every fiber $\pi^{-1}(U)$ is homeomorphic to $U \times F$. Such a map π is called a *covering map*.

The further condition for E to be a fiber bundle is that for each open set U_α in a cover of B, the diagram at right commutes. The vertical map is just forgetting about the fiber.



These maps $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ are then called *local trivializations*, analogous to local coordinates on a manifold. For the pathologists among you: try to come up with an example of a covering map which does not produce a fiber bundle; I don't want to do it.

A *section* of a fiber bundle is a map $s : B \to F$ with $\pi \circ s = \mathbb{1}$.

Transition functions. Now on the double-overlaps $U_{\alpha\beta}$ of an open cover of B, we have maps

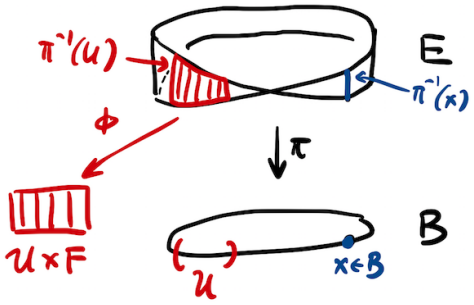
$$\phi_\alpha \circ \phi_\beta^{-1} : U_{\alpha\beta} \times F \to U_{\alpha\beta} \times F.$$

These are called transition functions. They lie in a subgroup of the group of homeomorphisms of the fiber F called the *structure group* of the bundle.

Of course a product manifold, like $T^2 = S^1 \times S^1$, is a fiber bundle, but a trivial one, where the transition functions can all be chosen to be the identity. (If we were keeping

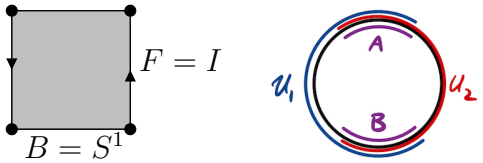
track of more information, such as the *complex structure* on the torus, the boundary conditions which identify S^1 with S^1 by a shift gives a nontrivial operation called a Dehn twist.)

Example: Mobius band. Take $B = S^1$ and $F = I$. If we impose boundary conditions that the orientation of F reverses when we go around the circle, we get the Mobius band.



Cover $B = S^1$ with two open sets $U_{1,2}$. They overlap in $U_{12} = A \cup B$, with two components. The nontrivial transition functions are

$$\phi_{12}(x) = \begin{cases} 1 & \text{if } x \in A \\ g & \text{if } x \in B \end{cases}$$



where g is the orientation-reversal of the fiber.

A very similar example is obtained by replacing the fiber by S^1; this is the Klein bottle.

Example: Hopf bundle. Take $E =$ the unit quaternions, or $\mathsf{SU}(2) \simeq S^3$, and $F = S^1$, the unit complex numbers inside the unit quaternions. Taking $S^3 \subset \mathbb{C}^2 = \{(z_0, z_1)\}$, the base is

$$\left(S^3 = \{|z_0|^2 + |z_1|^2 = 1\} \subset \mathbb{C}^2\right) / (z_0, z_1) \sim e^{\mathbf{i}\alpha}(z_0, z_1)$$

which is $\mathbb{CP}^1 \simeq S^2$. This is just the Bloch sphere of normalized pure states of a qubit, and the projection map is just forgetting the overall phase of the wavefunction. In stereographic projection $S^2 \simeq \mathbb{C} \cup \{\infty\}$, the projection map is $\pi : (z_0, z_1) \to z_0/z_1$, where the range of the map is $B \simeq S^2$ is the Riemann sphere (complex plane union the point at infinity). In polar coordinates ($r_0^2 + r_1^2 = 1$ defines the S^3) the projection is $\pi(r_0 e^{\mathbf{i}\theta_0}, r_1 e^{\mathbf{i}\theta_1}) = \frac{r_0}{r_1} e^{\mathbf{i}(\theta_0 - \theta_1)}$. Fixed $\rho = r_0/r_1$ is a $T^2 \subset S^3$ which degenerates at $\rho = \pm\infty$ to two linked circles. A visualization from Wikipedia is at right.

Another way to present the Hopf bundle projection, which arises all the time in physics is:

$$\pi : \begin{cases} \mathbb{C}^2 \to \mathbb{R}^3 \\ S^3 \to S^2 \\ z \mapsto z^\dagger \vec{\sigma} z \end{cases}$$

The top row applies to general $z = (z_0, z_1)$, and the middle row applies to the subspace where $|z_0|^2 + |z_1|^2 = 1$. We can make local sections of this bundle by finding the ± 1 eigenvectors of $\hat{n} \cdot \vec{\sigma}$.

One reason to care about the Hopf bundle, besides its ubiquity in theoretical physics, is that it gives relations between homotopy groups of spheres. The exact homotopy sequence is

$$\cdots \pi_q(S^1) \to \pi_q(S^3) \to \pi_{q-1}(S^2) \to \pi_{q-1}(S^1) \to \cdots \tag{3.21}$$

Fact: $\pi_q(S^1) = \mathbb{Z}\delta^{q,1}$ for $q \geq 1$. We conclude from (3.21) that $\pi_q(S^3) \cong \pi_q(S^2)$ for $q \geq 3$. In particular $\pi_3(S^2) = \mathbb{Z}$ is generated by the Hopf projection itself.

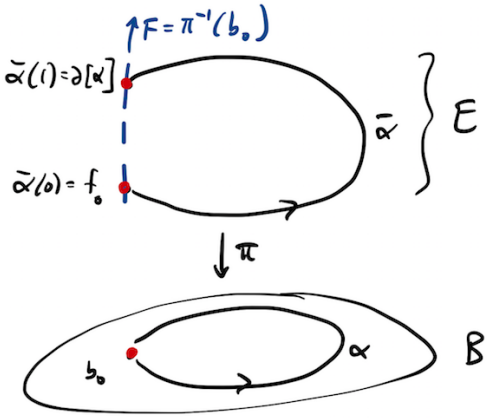
Universal cover. How do we know the homotopy groups of the circle? One way is to write $S^1 = \mathbb{R}/\mathbb{Z}$. Now we appeal to the following
General Fact: if $X = C/G$ and C is simply connected ($\equiv \pi(C) = 0$) then $\pi_1(X) = G$. (If you are not happy with this level of detail, Hatcher has a long section on $\pi_1(S^1)$.) Of intermediate generality between a quotient and a fiber bundle is the existence of a covering map $\pi : C \to X$. If C is simply connected, then the space C is called the *universal cover* of X. For example, we saw in §1.3 that a Riemann surface Σ_g of genus $g \geq 1$ can be made by taking a disk and making identifications along its boundary. Since the disk is simply connected, it is the universal cover of Σ_g, and

$$\pi_1(\Sigma_g) = \left\langle a_i, b_i | a_1^{-1} b_1^{-1} a_1 b_1 a_2^{-1} b_2^{-1} a_2 b_2 \cdots a_g^{-1} b_g^{-1} a_g b_g = 1 \right\rangle$$

where the one relation comes from the disk filling in the boundary. (By this notation I mean the group generated by the list of things before the $|$, modulo the list of relations after the $|$.) For $g = 1$, this says that a and b commute, which they'd better since $\pi_1(T^2) = \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z} \times \mathbb{Z}$ is abelian. For $g > 1$, $\pi_1(\Sigma_g)$ is non-abelian. You can see that its abelianization is $\mathbb{Z}^{2g}$ in agreement with our previous result for $H^1(\Sigma_g)$.

Brief and insufficient words about the connecting homomorphism. [Bott and Tu p. 209] I said that for fiber bundles there *is* a long exact sequence on the homotopy groups, but what is the mysterious map $\partial : \pi_q(B) \to \pi_{q-1}(F)$ in (3.20)?

The idea is that a map $\alpha : I^q \to B$ can be lifted to a map $\tilde{\alpha} : I^q \to E$, but it does not necessarily end at the base point of E, since the base point of B can have many pre-images in E, only one of which is the base point in E. Regard $F = \pi^{-1}(b_0)$ as the fiber over the base point. First the case $q = 1$: the lift of α can be chosen so that $\bar{\alpha}(0)$ is the base point in F. Then $\partial[\alpha] = [\bar{\alpha}(1)]$.



For general q, the properties of a fiber bundle guarantee that α can be lifted so that $\bar{\alpha}(t_1, \cdots t_{q-1}, 0)$ lifts to the constant map to the base point of F. Then its image under the connecting homomorphism is

$$\partial[\alpha] = [(t_1 \cdots t_{q-1}) \mapsto \bar{\alpha}(t_1 \cdots t_{q-1}, 1)].$$

α in the same homotopy class have the same image. The keyword is 'covering homotopy property'.

One final comment about covering maps: If $\pi : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is a covering map, then $\pi_\star : \pi_q(\tilde{X}, \tilde{x}_0) \to \pi_q(X, x_0)$ is an isomorphism for $q \geq 2$. The idea is that every map $S^n \to X$ lifts to a map $S^n \to \tilde{X}$ for $q \geq 2$. This is Hatcher Prop. 4.1 on page 342, and also follows from the 'covering homotopy property'. In the case of $q = 1$, the induced map is merely injective, and embeds π_1 of the covering space as a subgroup of $\pi_1(X)$ (Hatcher Prop. 1.31) – it's just the subgroup of loops which lift to closed loops in $\tilde{X}$ (unlike the one α in the figure above). There is therefore a correspondence between covers of X and subgroups of $\pi_1(X)$.

[End of Lecture 17]

3.8 Vector bundles and connections

Vector bundles. If the fiber F of a fiber bundle is a vector space of dimension r, and moreover the transition functions act linearly, then E is a vector bundle. That is, the transition functions of a vector bundle can be regarded as maps

$$g_{\alpha\beta} : U_{\alpha\beta} \to \mathsf{GL}(r, \mathbb{R})$$

where r is the *rank* of the vector bundle. In physics we usually care about cases where the transition functions live in a compact subgroup; this doesn't change the mathematical story much.

An example is the tangent bundle of a manifold M, where the fiber over a point p is $T_pM \simeq \mathbb{R}^n$, with $n = \dim M$. The structure group in general is $\mathsf{O}(n)$; for an oriented manifold the transition functions lie in $\mathsf{SO}(n)$. The structure group of the tangent bundle is called the holonomy group of the manifold.

A vector bundle is specified by its transition functions on overlaps $U_{\alpha\beta}$. For example, an n-sphere can be covered by just two patches with a single overlap homotopic to S^{n-1}. A vector bundle on S^n with structure group G is therefore specified, up to homotopy, by an element of $\pi_{q-1}(G)$. (If $\pi_0(G)$ is nontrivial, distinct vector bundles correspond to elements of $\pi_{q-1}(G)/\pi_0(G)$.)

If we choose the fibers to be a complex vector space $\mathbb{C}^n$, with transition functions in $\mathsf{GL}(n,\mathbb{C})$ we get a complex vector bundle.

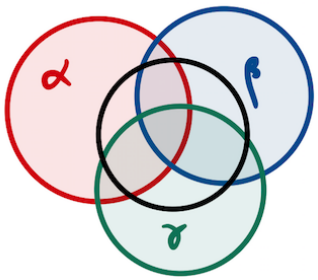
Anything we can do to vector spaces, we can do to the fibers of a vector bundle, and hence we can make new bundles from old in many ways, such as direct sum, or quotient by a sub-bundle.

Starting from a given set of transition functions (a principal G-bundle), we can make various a new vector bundle for each representation $D(g)$ of G, just by replacing the fibers by the carrier space of the representation, and the transition functions by $g_{\alpha\beta}(x) \to D(g_{\alpha\beta}(x))$; these are called associated bundles.

On triple overlaps, the transition functions satisfy the cocycle condition

$$g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1 \quad \text{on} \quad U_{\alpha\beta\gamma}. \tag{3.22}$$

It arises by walking in a contractible loop starting in patch U_α through U_β and then U_γ. This is automatic from the definition as $g_{\alpha\beta} = \phi_\alpha\phi_\beta^{-1}$.



Another way to make a vector bundle is to *start* with a collection of $g_{\alpha\beta} : U_{\alpha\beta} \to \mathsf{GL}(n,\mathbb{R})$ on double-overlaps, satisfying the cocycle condition (3.22). Then define E as the quotient of the disjoint union $\coprod_\alpha U_\alpha \times \mathbb{R}^n$ by $(x,v) \sim (x, g_{\alpha\beta}(x)v)$ on $U_{\alpha\beta}$. The same bundle obtains if we change $g_{\alpha\beta}$ by $g_{\alpha\beta} \to f_\alpha g_{\alpha\beta} f_\beta^{-1}$ for some homomorphisms $f_\alpha : G \to G$. So you see that there is a relation between vector bundles and the "1st Cech cohomology of the open cover with values in G". Except that we haven't defined (and I don't know how to define) what that is for G non-Abelian! When G is abelian this is just a true statement.

Connections. Suppose we wanted to attach a vector space V_x over each point x in spacetime to make a vector bundle. On each vector space we have an action of G, by $\Phi_\alpha(x) \mapsto \Lambda_{\alpha\beta}(x)\Phi_\beta(x)$, where $\Phi_\alpha(x) \in V_x$ (where α, β are some indices on some

representation of G). Here $\Phi_\alpha(x)$ is a section of the vector bundle in question.

Suppose we would like to do physics in a way that is independent of the choice of basis for this space, at each point. We would like to be able to compare $\Phi(x)$ and $\Phi(y)$ (for example to make kinetic energy terms for a field $\Phi(x)$) in a way that respects these independent rotations. To do this, we need more structure: we need a *connection* (or *comparator*) W_{xy}, an object in G which transforms like

$$W_{xy} \mapsto \Lambda(x) W_{xy} \Lambda^{-1}(y), \tag{3.23}$$

so that $\Phi^\dagger(x) W_{xy} \Phi(y)$ is invariant. The connection between two points W_{xy} may depend on how we get from x to y. We demand that $W(\emptyset) = \mathbb{1}$, $W(C_2 \circ C_1) = W(C_2)W(C_1)$ and $W(-C) = W^{-1}(C)$, where $-C$ is the path C taken in the opposite direction.

All of these definitions work perfectly well (in fact better) if our space is chopped up into a cell complex. Then we can associate a connection variable W_ℓ to each 1-cell; it transforms as

$$\left(W_{\langle ij \rangle}\right)_{\alpha\beta} \to \Lambda_{\alpha\gamma}(i) \left(W_{\langle ij \rangle}\right)_{\gamma\delta} \left(\Lambda^{-1}\right)_{\alpha\gamma}(j). \tag{3.24}$$

A system with these Ws as its degrees of freedom – a group element attached to each link, modulo the gauge transformations (3.24), is a lattice gauge theory.

For a moment consider the (complicated) case where spacetime is continuous and G is continuous. Then we can consider the connection between nearby points, and we can define using W a covariant derivative D

$$D\Phi(x) = \lim_{\Delta x \to 0} \Delta x^{-1} \left(W(x, x+\Delta x)\Phi(x+\Delta x) - \Phi(x)\right)$$

and extract from it a gauge field, A, in terms of which $W_C = P e^{\mathbf{i}e \int_C A}$, where P means path-ordering of the exponential. Locally, A is a Lie-algebra-valued one-form. It transforms under gauge transformations (3.23) as

$$A \to A^\Lambda = \Lambda^{-1} \left(A - d\right) \Lambda. \tag{3.25}$$

I've included a conventional factor of e, the charge of the electron (actually it would be $\frac{e}{\hbar c}$ if we weren't working in natural units). Its field strength is a Lie-algebra-valued two-form $F_{\mu\nu} = [D_\mu, D_\nu]$, describing the local path dependence of the connection. It transforms homogeneously: $F \to \Lambda^{-1} F \Lambda$.

Consider for a moment the abelian case in three or more spatial dimensions. Two paths give group elements differing by

$$W_{C'} = W_C e^{\mathbf{i}e \int_{C'-C} A} \overset{\text{Stokes}}{=} W_C e^{\mathbf{i}e \int_S F} \tag{3.26}$$

where $\partial S = C - C'$ is a surface. But which surface? Two different choices of S differ by $S' - S = \partial V$, some 3-volume. The difference in the phase is

$$e^{\mathbf{i}e\int_{S'-S} F} \overset{\text{Stokes}}{=} e^{\mathbf{i}e\int_V dF}.$$

In order for our operation (3.26) of writing the path-dependence of the phase in terms of the flux to be well-defined, we must have

$$e^{\mathbf{i}e\int_V dF} = 1 \quad \Leftrightarrow \quad e\int_V dF \in 2\pi\mathbb{Z}.$$

But $dF = \star j_m$ is the magnetic charge current density, so $\int_V dF = 4\pi g$ is the total magnetic charge in the volume V. This is Dirac quantization: $eg = e\int_V dF = e\oint_{\partial V} F = e4\pi g \in 2\pi\mathbb{Z}$.

Example: Dirac (Wu-Yang) monopole = Hopf bundle. Let's use these ideas to find a configuration of the electromagnetic field in 3-space with a magnetic monopole at the origin, *i.e.* satisfying $\vec{\nabla}\cdot B = 4\pi g\delta^3(x)$. The LHS of this equation is the time component of the 1-form $\star dF = \star d^2 A$, so A must not be globally well-defined or else this would vanish. A way out is to cover space with patches. Actually, all the action happens on the unit sphere surrounding the monopole (that is, $\mathbb{R}^3 \setminus \{0\} \simeq S^2$), so let's just think about that. We cover this two-sphere with two patches U_N and U_S consisting of everything but the south and north poles respectively. The overlap deformation retracts to the equator. On U_N and U_S respectively we take the gauge potential to be[41]

$$A^N = g(1-\cos\theta)d\varphi, \quad A^S = g(-1-\cos\theta)d\varphi = A^N - 2gd\varphi = A^N + \mathbf{i}g_{NS}^{-1}dg_{NS}$$

where

$$g_{NS}(\theta,\varphi) \equiv e^{\mathbf{i}2g\varphi}$$

is a function on $U_N \cap U_S$. Notice that $F = dA^N = dA^S = g\sin\theta d\theta\wedge d\varphi$ are both proportional to the volume form on the 2-sphere, consistent with the demand that magnetic flux is coming out from the origin in a spherically-symmetric way. g_{NS} has two names here: Mathematically, it is the transition function for a complex vector bundle of rank one between our two patches on the 2-sphere. Physically it is a function parameterizing a gauge transformation (3.25) between two choices of gauge for the vector potential.

What is the structure group G of the vector bundle in question? g_{NS} is single valued under $\varphi \cong \varphi + 2\pi$ and $G = \mathsf{U}(1)$ iff

$$4\pi g \in 2\pi\mathbb{Z}. \tag{3.27}$$

[41] I'm using polar coordinates on the unit sphere where $x = \sin\theta\cos\varphi, y = \sin\theta\sin\varphi, z = \cos\theta$.

This is the Dirac quantization condition. Notice that we didn't actually say anything about quantum mechanics. We just demanded that there was some $\mathsf{U}(1)$-valued transition function connecting the gauge potentials on the two patches. Quantum mechanics comes in because the phase of the wavefunction of a charged particle transforms under a gauge transformation by multiplication by the transition function; if this isn't single-valued, the wavefunction is not well-defined. This is the physical reason we want the structure group to be $\mathsf{U}(1)$ and not $\mathbb{R}$.

This conclusion leads to *flux quantization*: A vector bundle with a compact structure group has quantized fluxes, $\oint_S \frac{F}{2\pi} \in \mathbb{Z}$, where S is any compact 2d submanifold of B. In this example, we have

$$\begin{aligned} \oint_{S^2} \frac{F}{2\pi} &= \frac{1}{2\pi}\left(\int_{H_N} dA^N + \int_{H_S} dA^S\right) \overset{\text{Stokes}}{=} \frac{1}{2\pi}\oint_{\text{equator}} \left(A^N - A^S\right) \\ &= -\frac{1}{2\pi}\oint_{\text{equator}} \mathbf{i} g_{NS}^{-1} dg_{NS} = \frac{1}{2\pi}\int_0^{2\pi} 2g d\varphi = 2g \overset{(3.27)}{\in} \mathbb{Z} \end{aligned} \tag{3.28}$$

where H_N and H_S are the north and south hemispheres, which lie respectively in U_N and U_S and which have $\partial H_{N/S} = \pm\text{equator}$.

Another crucial general conclusion from this relation is that the periods of F (its integrals over closed cycles) are determined entirely by the transition functions of the bundle, and not by any of the extra structure of the connection. More generally, for a line bundle (complex vector bundle of rank one) on a more general manifold with good open cover $\mathfrak{U}$, there will be multiple double-overlaps. This construction

$$\frac{\mathbf{i}}{2\pi} d\log g_{\alpha\beta} \in C^1(\mathfrak{U}, \Omega^1)$$

instead produces a 1-form valued Cech 1-cocycle, which is both d- and δ-closed. Our earlier analysis of the Cech-deRham complex shows that this means that there exists a global closed 2-form with integer periods called the first Chern class of the line bundle.

Another use of these same transition functions is: we can use them to make a bundle over S^2 whose fiber is $\mathsf{U}(1) \simeq S^1$. We already know an example of such a bundle, namely the Hopf bundle. In fact, that bundle has the same transition functions. One reason we know this is that the first Chern class uniquely characterizes a line bundle (complex vector bundle of rank one) over $S^2 \simeq \mathbb{CP}^1$. This is because the transition function on the overlap of the two patches is a map $S^1 \to S^1$ and $\pi_1(S^1) = \mathbb{Z}$, and the Chern class equals this number.

To see it more explicitly, let's explicitly build local sections of the Hopf bundle $\pi : z \to z^\dagger \vec{\sigma} z$. Given $\hat{r} \in S^2 \subset \mathbb{R}^3$, let $\rho \equiv \frac{1}{2}(1 + \hat{r}\cdot\vec{\sigma})$. Since $\hat{r}^2 = 1$, this is a pure-state density matrix for a qubit, $\rho = |z\rangle\langle z|$ for some normalized state vector

$|z\rangle = z_0 |0\rangle + z_1 |1\rangle$ (*i.e.* the matrix elements are $\rho_{\alpha\beta} = z_\alpha z_\beta^\dagger$). This state is just the eigenstate of $\hat{r} \cdot \vec{\sigma}$ with eigenvalue $+1$. Here's one expression for its components:

$$z = s^N(\theta, \varphi) = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{\mathbf{i}\varphi} \end{pmatrix} \quad \text{on} \quad U_N. \tag{3.29}$$

This expression has the drawback that it is singular at the south pole, $\theta = \pi$, where φ is not a good coordinate. Another expression is

$$z = s^S(\theta, \varphi) = \begin{pmatrix} \cos\frac{\theta}{2} e^{-\mathbf{i}\varphi} \\ \sin\frac{\theta}{2} \end{pmatrix} \quad \text{on} \quad U_S. \tag{3.30}$$

This one is singular at the north pole, $\theta = 0$. As $\vec{r}$ varies around the S^2 in fact we *cannot* give an expression for the components of z which is globally well defined. They are sections of a non-trivial vector bundle. On the overlap of the two patches, the two sections are related by

$$s^N = e^{\mathbf{i}\varphi} s^S$$

from which we conclude that the transition functions of the Hopf bundle are indeed $g_{NS} = e^{\mathbf{i}\varphi}$, the same as the minimal-charge ($2g = 1$) Dirac monopole.

In this language there are a few other nice things to say. From these sections we can directly build the Dirac monopole connection as follows.

The form

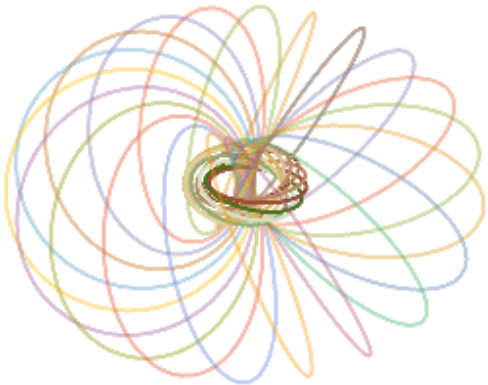
$$z^\dagger dz = \frac{\mathbf{i}}{2}(d\psi - \cos\theta d\varphi)$$

is a nice $\mathsf{U}(2)$-invariant 1-form on the $S^3 \subset \mathbb{C}^2$.

Here ψ is a third coordinate on S^3 defined by

$$\begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = e^{\mathbf{i}\psi/2} \begin{pmatrix} \cos\frac{\theta}{2} e^{\mathbf{i}\psi/2} \\ \sin\frac{\theta}{2} e^{\mathbf{i}\varphi/2} \end{pmatrix}. \tag{3.31}$$

The hopf projection forgets $\psi \in [0, 4\pi)$, which only appears in the overall phase of z. The sections $s^{N/S} : S^2 \to S^3$ embed S^2 into S^3. Actually a simple way to write them is just $\psi = \pm\varphi$, as you can see by comparing (3.31) with (3.30) and (3.29).



In the figure at right I plot a collection of Hopf fibers in stereographic projection. Along each circle, $\psi \in [0, 4\pi)$. The inner torus is fixed $\theta = \pi/6$ and various $\varphi \in [0, 2\pi)$; the outer torus is $\theta = 3\pi/4$.

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But then the pullbacks of the sections are maps $s_\star^{N/S} : \Omega^1(S^3) \to \Omega^1(U_{N/S} \subset S^2)$, and in fact

$$s_\star^{N/S}(z^\dagger dz) = A^{N/S}.$$

This strategy of making a connection from (local) sections works in general.

A nice pedagogical discussion of many simple avatars of the Hopf bundle in physics is here.

Hopf invariant. There is an insane amount of things to say related to the Hopf bundle. One that I can't resist mentioning in anticipation of discussing Chern-Simons theory is the *Hopf invariant*. Given a smooth map $f : S^3 \to S^2$, we can ask about the pullback of a generator of $H^2_{dR}(S^2)$. Since $H^3_{dR}(S^3) = 0$, it must be exact:

$$f^\star \alpha = d\omega, \text{ for some 1-form } \omega \in \Omega^1(S^3).$$

The Hopf invariant,

$$H[f] \equiv \int_{S^3} \omega \wedge d\omega$$

is independent of the choice of $\omega \to \omega + d\lambda = \omega'$, because

$$\int \omega \wedge d\omega - \int \omega' \wedge d\omega' = \int (\omega - \omega') \wedge d\omega \overset{\text{IBP}}{=} 0.$$

The same construction works for $f : S^{2n-1} \to S^n$; the resulting invariant is made from $f^\star \alpha = d\omega, \omega \in \Omega^{n-1}(S^{2n-1})$ by $H[f] = \int_{S^{2n-1}} \omega \wedge d\omega$. For odd n, $H[f] = 0$ since $\int \omega \wedge d\omega = \int \frac{1}{2} d(\omega \wedge \omega) = 0$ by Stokes.

Finally, homotopic maps $f \simeq g$ have the same Hopf invariant $H[f] = H[g]$. The proof is in Bott and Tu around page 227.

The Hopf invariant has a simple interpretation as the *linking number* between $f^{-1}(p)$ and $f^{-1}(q)$, any two circles in S^3 arising as pre-images of distinct points in S^2. For the Hopf map, this number is $H[\pi] = 1$, as you can see in the picture of the fibers above.

---

[42] Note that stereographic projection of a point $p \in S^n \subset \mathbb{R}^{n+1}$ (so $\sum_{i=1}^{n+1} p_i^2 = 1$) maps it to the point in $\mathbb{R}^n$ with coordinates $x_i = \frac{2p_i}{1+p_{n+1}}$, $i = 1..n$. So the south pole where $p_{n+1} = -1$ goes to ∞, the north pole goes to 0, and points on the equator go to themselves. The inverse map is

$$p_i = \frac{2x_i}{1+x^2}, \qquad p_{n+1} = \frac{1-x^2}{1+x^2}.$$

These automatically satisfy $\sum_{i=1}^{n+1} p_i^2 = 1$. Beware an awful typo on page 133 of Nash and Sen that cost me several minutes of my life.

For bundles of larger rank, there are more Chern classes and the story of constructing the Chern classes directly from the transition functions is a little bit more complicated; see the last pages of Bott and Tu. It is easier just to find a connection and compute its field strengths, $\mathrm{tr}F, \mathrm{tr}F\wedge F, \cdots$.

[End of Lecture 18]

I can't resist mentioning a 4d analog of the above business (3.28), since it came up earlier. (**Warning:** the signs and factors in this discussion are not to be trusted.) We saw above that a G-bundle on S^n is specified by the transition functions on the equator $g: S^{n-1}\to G$. Here's another way to think about it: S^4 is $\mathbb{R}^4$ plus the point at infinity. Let's put a connection on a G-bundle on $\mathbb{R}^4$ for some simple compact Lie group G and demand that at infinity, $F\to 0$ (so we can include the point at infinity as a smooth point)[43]. This requires that $A\overset{r\to\infty}{\to} g^{-1}dg$ for some map $g: S^3\to G$. (Such configurations ('instantons') are candidates for saddle points of the euclidean path integral in Yang-Mills theory with gauge group G.) For the case of $G=\mathsf{SU}(2)$ some examples of g are $g^0(x)=\mathbb{1}$ and

$$g^1(x)=\frac{x_4\mathbb{1}+\mathbf{i}\vec{x}\cdot\vec{\sigma}}{r}$$

and $g^\nu(x)=(g^1(x))^\nu$. The map g^1 generates $\pi_3(\mathsf{SU}(2))=\langle[g^1]\rangle$. In fact for any simple compact Lie group there is an integer family of such maps: $\pi_3(G)=\mathbb{Z}$. One can say that the map can be deformed to a map whose image lies in an $\mathsf{SU}(2)$ subgroup[44].

But now consider the following quantity analogous to c_1 for the Hopf bundle:

$$\int_{\mathbb{R}^4}\frac{\mathrm{tr}F\wedge F}{16\pi^2}=\int_{\mathbb{R}^4}d\mathrm{CS}(A)\overset{\text{Stokes}}{=}\int_{S^3}\mathrm{CS}(A=g^{-1}dg)=\frac{1}{12\pi^2}\int_{S^3}\mathrm{tr}\left(g^{-1}dg\wedge g^{-1}dg\wedge g^{-1}dg\right)=\nu\in\mathbb{Z}. \tag{3.32}$$

Here[45] $\mathrm{CS}(A)=\frac{1}{16\pi^2}\mathrm{tr}\left(AdA+\frac{2}{3}A\wedge A\wedge A\right)$ is the Chern-Simons 3-form. In the case of $G=\mathsf{SU}(2)$, the penultimate expression is an integral representation of the winding

[43] A better way to describe it, which makes contact with the other description of bundles on S^4 is to say that we cut out a ball B^4 around the point at infinity; g here is the transition function on the overlap between the B^4 coordinate patch surrounding the origin and the B^4 surrounding the point at infinity, analogous to $g_{NS}=\frac{x+\mathbf{i}y}{r}$ for the Dirac/Hopf bundle.

[44] That $\pi_3(G)=\mathbb{Z}$ for any simple Lie group is proved in this paper and in Milnor's *Morse Theory* book. The proof uses Morse theory on ΩG. The Morse function is the length of the path (with respect to the G-invariant metric), so the critical points are geodesics. The key point is that because of special properties of a Lie group, the Morse index of all the critical points is even. By the results of §3.4, this means ΩG has a cell decomposition with only even-dimensional cells, so its odd-dimensional homology groups vanish and there is no torsion. If G is simply-connected and simple, we can apply the Hurewicz result to conclude that $\mathbb{Z}=H_2(\Omega G,\mathbb{Z})=\pi_2(\Omega G)=\pi_3(G)$. (If G is not simply connected, we can do this for its universal cover $\tilde{G}$, and then use $\pi_3(\tilde{G})=\pi_3(G)$.)

[45] Be warned that the numerical prefactors depend a bit on normalization choices, such as the representation in which we choose to define the trace.

number of the map $S^3 \to S^3$ and the last step can be checked by brute force for the representative maps above. But for arbitrary compact simple G, this integral still extracts the element of $\pi_3(G)$.

This is another example of the claim that the Chern classes ((3.32), which is called the *instanton number*, is c_2, the second Chern class, in this case) are determined by the transition functions. I described the calculation in (3.32) here for a bundle on $\mathbb{R}^4$ with nice boundary conditions at infinity. The same calculation computes the instanton number of a bundle on S^4 with connection A^N on the northern patch and on the southern patch $A^S = g^{-1}(A^N - d)g$. As in (3.28), the difference of Chern-Simons forms (in the previous case, A with $dA = F$ is the Chern-Simons 1-form) again integrates to the winding number.

For more, see the discussion around page 289 of Coleman's *Aspects of Symmetry*.

3.9 The quantum double model and the fundamental group

Now I want to describe a geometric point of view on groundstates of the quantum double model (in any dimension).

Consider a cell complex Δ which triangulates some path-connected space X. A state of the quantum double model on the links (1-cells) in the group-element basis (the one described above) can be regarded as a (discretized) connection on X – it associates to any path (in Δ) an element of the group G (called the holonomy of the path), obtained by multiplying the group elements for each link in the path: $U(C) = \prod_{\ell \in C} g_\ell$.

A connection is *flat* if the holonomy is independent of local deformations of the path: $U(C) = U(C + \partial\Sigma)$, where Σ is some 2-surface. This is precisely the condition that the plaquette operators give the 1 on the state: the flux around every plaquette equals the identity.

Two connections $U(C_{x,y})$ and $U'(C_{x,y})$ (here $C_{x,y}$ is a path from x to y) are regarded as equivalent if they are related by a gauge transformation

$$U'(C_{x,y}) = g_x U(C_{x,y}) g_y^{-1}.$$

This is the action of the star operators.

Here is an attempt to be more explicit about the action of the star operators on the connection: A groundstate of quantum double model, is a state of the form

$$|\Psi\rangle = \sum_{\{g\}} \Psi[\{g\}]\, |\{g\}\rangle$$

where $|\{g\}\rangle$ is the state of the links in the group-element basis, *i.e.* the regular representation of G on each link. The plaquette condition $B_p|\Psi\rangle = |\Psi\rangle$ says $\Psi[\{g\}] = 0$ for any configuration $\{g\}$ which is not flat. The star condition $A_s|\Psi\rangle = |\Psi\rangle$ says that $|\Psi\rangle$ is an equal-weight superposition over the orbits of $\prod_v A_v^{h_v}$, which acts on the link ij (labelled by its endpoints) by the operator $L_+^{h_i}(ij)L_-^{h_j}(ij)$ (notice that these commute!) and therefore takes

$$|\{g_{ij}\}_{ij}\rangle \mapsto \left|\{h_i g_{ij} \bar{h}_j\}_{ij}\right\rangle . \tag{3.33}$$

This is the action of the gauge transformation on the connection that I wrote before: $U(x,y) \to g_x U(x,y) g_y^{-1}$.

Groundstates of the quantum double model are averages over the orbits of the star operators. So groundstates $|\Psi\rangle$ of the quantum double model on X correspond to flat G-connections on X modulo gauge equivalence.

This is just some geometric language describing the groundstates. Possibly slightly more useful is the following further fact: flat connections on X are in 1-1 correspondence with representations of $\pi_1(X)$ in G, *i.e.* group homomorphisms from $\pi_1(X)$ to G.

In one direction, a flat connection directly gives a map $\pi_1(X) \to G$, since for each loop in X, it associates a group element. Flatness means the group element is the same for any loop in the same homotopy class. The group law of loops is just composition so it is preserved by the map. Hence this is a group homomorphism.

In the other direction, given a representation $\rho : \pi_1(X) \to G$, it is clear how to associate to each loop an element of G. Less obvious is how to associate a group element to an *open* curve to make a connection. We do this in two steps. First, we make a flat G-bundle – this means a vector bundle whose transition functions in G are *constants* on the double-overlaps. Second, we use the fact that a flat G-bundle admits a flat connection. To show this second statement, given a flat atlas $\{U_\alpha, \phi_\alpha\}$ (ϕ_α are the local trivializations), on each U_α we define a flat connection by declaring that ϕ_α is parallel, $D_\mu \phi_\alpha = 0$, that is

$$A_\mu^{(\alpha)} = -\frac{\partial_\mu \phi_\alpha}{\phi_\alpha}.$$

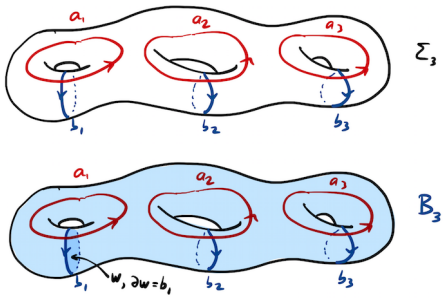
Because $g_{\alpha\beta}$ are constant, $0 = \partial_\mu g_{\alpha\beta}$, these definitions agree on the double-overlaps. (In fact the converse is also true: a vector bundle with a flat connection admits a flat atlas, obtained by parallel transporting a basis of the fiber above an arbitrary point.)

To do the first step: Regard the universal cover $\tilde{X} \to X$ as a principal $\pi_1(X)$-bundle over X. There exists a cover of X such that the (constant) transition functions $g_{\alpha\beta}$ generate $\pi_1(X)$. Then the bundle we want is the associated bundle with transition functions $\rho(g_{\alpha\beta})$. Or more explicitly, if V is the carrier space of the rep of $\pi_1(X)$, then the bundle is $\tilde{X} \times V/\pi_1(X) \to X$ with the identification $f(\gamma x) = \rho(\gamma) f(x) \in V$ for

$\gamma \in \pi_1(X)$. Here I'm using the fact that a point in $\tilde{X}$ can be labelled by (x, γ), a point in X and an element of $\pi_1(X)$ to specify which sheet of the cover it lives on.

To be more precise, flat connections modulo gauge equivalence correspond to representations of $\pi_1(X)$ in G modulo overall conjugation in G, since $W(C_{xx})$ for a closed loop still transforms under gauge transformations at the site x $W(C_{xx}) \to g_x W(C_{xx}) g_x^{-1}$. Here x is the base point. ■

Now let's consider two examples: 1) $X_1 = \Sigma_g$, a Riemann surface of genus g,
2) $X_2 = B_g$, the genus g handlebody, *e.g.* the region of 3-space whose boundary is Σ_g in the picture at right.
Flat connections on B_g are a subset of those on Σ_g, namely the one where the holonomy around the contractible cycles b_i (the ones filled in by the creamy filling of the pastry) is equal to the identity. Using the correspondence above, this is because the contractible curves b_i generate a subgroup of $\pi_1(\Sigma_g)$ and the representations of $\pi_1(B_g)$ are those of $\pi_1(\Sigma_g)$ which are trivial on this subgroup. In the quantum double on B_g, this is enforced by the fact that in B_g the curves b_i are the boundaries of a collection of plaquettes w.



So just as the groundstates of the toric code on X encode the homology groups of X, the groundstates of the quantum double model on X encode the fundamental group of X. In the case of the toric code, this insight allowed us to use physics to understand various things about homology groups (in particular: subdivision invariance, behavior under change of coefficients, and the notion of relative homology). Here are a few comments about the analogous questions for the fundamental group. I will not be nearly as explicit as in the homology case, partly because I haven't figured it all out.

Subdivision invariance and entanglement renormalization. Something not a priori obvious from the 'calculating theorem' is that the result for π_1 is independent of retriangulations of the complex. As in the case of the toric code and homology, we can give a satisfying physics proof of this fact using entanglement renormalization. The analogs of the control-X unitaries are the controlled left- and right-multiplication:

$$\mathsf{CL}_{12} \equiv \sum_{g \in G} T_+^g(1) L_+^g(2), \qquad \mathsf{CR}_{12} \equiv \sum_{g \in G} T_+^g(1) L_-^g(2).$$

Analogous to the rules for conjugating products of Pauli operators by CX like $\mathsf{CX}\,(X \otimes \mathbb{1})\,\mathsf{CX} = X \otimes X$, the operators participating in the quantum double Hamiltonian satisfy rules

such as

$$\mathsf{CL}_{12}\left(L_+^g(1)\otimes \mathbb{1}(2)\right)\mathsf{CL}_{12}^{-1} = L_+^g(1)\otimes L_+^g(2).$$

Here is a detailed description, written by Jin-Long Huang (note that he draws the pictures on the dual lattice). Unlike the abelian case, the order of operations matters here, and not all combinations lead to something pretty. Also unlike the abelian case, here the operation only gives a statement about the groundstates, not about the full Hamiltonian.

Relation to homology. We might expect that one could derive the relation that H_1 is the abelianization of π_1 by a higgsing, analogous to our discussion of change of coefficients in homology. In the language of the quantum double model, this could be accomplished by perturbing the hamiltonian so that e excitations in certain representations R proliferate. We'd like to choose R to transform trivially under an abelian subgroup of G. (Alternatively (1) include some matter degrees of freedom living on the 0-cells, and transforming in the representation R of G, and at the same time (2) modify the star term to also generate the transformation of the matter field, and (3) perturb the Hamiltonian so that the matter fields condense.) Let me know if you see how to push this through.

Relative homotopy. It is also possible define a notion of relative homotopy. Given $Y \subset X$ a closed subspace containing the base point p, the relative homotopy group is

$$\pi_q(X,Y,p) \equiv \pi_{q-1}\left(\text{paths from } p \text{ to } Y\right)$$

or slightly more explicitly

$$\pi_q(X,Y,p) = \{\alpha : (I^q,\partial I^q) \to (X, p \text{ or } Y)\}/\simeq . \tag{3.34}$$

What I mean by this is: all of the faces of ∂I^q get mapped to the base point as usual, except for the bottom face of ∂I^q, which gets mapped anywhere into Y. By the 'bottom face' of ∂I^q I mean $\{t_1 = 0\}$. So $\alpha|_{t_1=0} : (I^{q-1},\partial I^{q-1}) \to (Y,p)$. The product is defined as usual for $q > 1$, but $\pi_1(X,Y)$ is not a group.

The inclusion map $i : Y \to X$ induces a map $i_\star : \pi_k(Y,p) \to \pi_k(X,p)$ as usual. Noting that $\pi_k(X,p) = \pi_k(X,p,p)$, we also have a map

$$j_\star : \pi_k(X,p) = \pi_k(X,p,p) \to \pi_k(X,Y,p).$$

Finally, we can define

$$\partial_\star : \begin{array}{ccc} \pi_k(X,Y,p) & \to & \pi_{k-1}(Y,p) \\ \alpha & \mapsto & \alpha|_{\text{bottom face}} \end{array}.$$

This produces a long exact sequence on homotopy:

$$\cdots \to \pi_k(Y,p) \xrightarrow{i_\star} \pi_k(X,p) \xrightarrow{j_\star} \pi_k(X,Y,p) \xrightarrow{\partial_\star} \pi_{k-1}(Y,p) \xrightarrow{i_\star} \cdots .$$

Nash and Sen (page 117) use this sequence to show $\pi_q(S^n) = 0$ for $q < n$. I must admit, though, that I do not understand what they mean by $B \supset S^n$.

We can expect that there is a relation between $\pi_1(X, Y, p)$ and groundstates of the quantum double perturbed into a trivial phase in the region Y, or alternatively with gapped boundary conditions on Y. I have not verified it in detail. Here are some papers about gapped boundaries of the quantum double model.

4 Topological gauge theories and knot invariants

4.1 Topological field theory

One perspective on a D-dimensional topological field theory is the following: it is a machine which first of all associates to a closed D-manifold a number (the partition function). This was simple. Then, roughly, to a closed $(D-1)$-manifold Σ it associates a Hilbert space $\mathcal{H}_\Sigma$. I'll explain this next. For $k > 0$, to a closed $(D-1-k)$-manifold it associates a k-category. That's all I'm going to say about that[46].

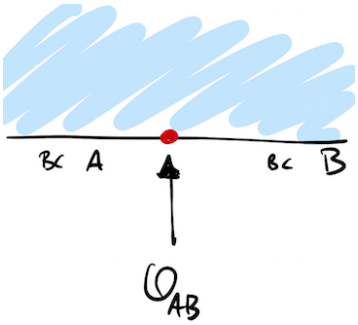
The way to think about the second statement is just through the path integral: The path integral on a D-manifold with boundary $\partial M = \Sigma$ produces a state $|\Psi_M\rangle$ in the Hilbert space $\mathcal{H}_\Sigma$. More precisely[47], the path integral with boundary conditions ϕ for the fields produces the wavefunction $\langle\phi|\Psi_M\rangle$.

Path integrals are wonderful tools but are difficult to define precisely. One way to avoid having to define them is to extract from them axioms that we'd like the resulting machine to satisfy. The axioms look ugly but each one is extremely simple and obvious:

- $\mathcal{H}_\emptyset = \mathbb{C}$.

[46] OK I can't resist. The association involving higher codimension objects is sometimes called an *extended* TQFT.

One way to think about it is that we associate codimension two loci with *boundary-condition-changing operators*. The objects of the category are the possible boundary conditions $\{A, B, \cdots\}$ and the morphisms between them are operators on the boundary $\mathcal{O}_{AB}$ across which the boundary conditions change, like disorder operators.



For example, in $D = 3$, these boundaries between regions of given boundary conditions are one-dimensional. There can be more than one morphism between two given objects. In that case we can imagine a codimension 3 defect across which we go from one BC-changing operator to another. And so on. (You can learn more here.)

[47] Actually you can already see that we are in trouble with the second statement above: not every $(D-1)$ manifold arises as the boundary of a smooth D-manifold, as you saw for $\mathbb{CP}^{2n}$ on the homework. So maybe the right statement is that the TFT associates a Hilbert space to $(D-1)$ manifolds which are bordant to nothing. So a better statement of the definition is that a D-dimensional TFT specifies a functor from a category of D-dimensional bordisms (perhaps with some structure like orientation or spin structure) to some other $\mathbb{C}$-linear category, such as the category of vector spaces. (In a category of D-dimensional bordisms, the objects are closed $(D-1)$-manifolds, and the morphisms are D-manifolds with boundary connecting them.) For more on this, see *e.g.* G. Moore, *Some Remarks on Topological Field Theory*.

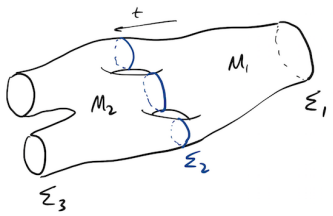
- $\mathcal{H}_{-\Sigma} = \mathcal{H}_\Sigma^\star$
- $\mathcal{H}_{\Sigma_1 \coprod \Sigma_2} = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2}$

From the last two it follows that if $\partial M = (-\Sigma_1) \coprod \Sigma_2$, a disjoint union of two components, then $Z(M) \in \mathcal{H}_{\Sigma_1}^\star \otimes \mathcal{H}_{\Sigma_2} = \mathrm{Hom}(\mathcal{H}_{\Sigma_1}, \mathcal{H}_{\Sigma_2})$, a linear operator from $\mathcal{H}_{\Sigma_1}$ to $\mathcal{H}_{\Sigma_2}$.

- If $\partial M_1 = (-\Sigma_1) \coprod \Sigma_2$ and $\partial M_2 = (-\Sigma_2) \coprod \Sigma_3$ and $N = M_1 \cup_{\Sigma_2} M_2$, that is, N is obtained by gluing M_2 to M_1 along the boundary Σ_2, then the resulting operators satisfy the composition law:

  $$Z(N) = Z(M_2) \circ Z(M_1) : \mathcal{H}_{\Sigma_1} \to \mathcal{H}_{\Sigma_3}.$$

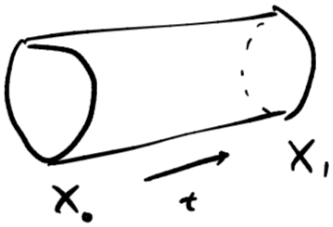
  This is just the analog of the rule for the time-evolution operator $U(t_1, t_3) = U(t_1, t_2)U(t_2, t_3)$.



So far we'd like these properties to hold for any quantum field theory. This next axiom is where the adjective 'topological' comes in:

- $Z(X \times I) = \mathbb{1}$ on $\mathcal{H}_X$. By the previous axiom, this object is an element of $\mathcal{H}_X^\star \otimes \mathcal{H}_X = \mathrm{Hom}(\mathcal{H}_X, \mathcal{H}_X)$, a linear operator on $\mathcal{H}_X$.

  By the usual logic of path integrals, it is the time-evolution operator. This says that the time-evolution operator is trivial, *i.e.* the Hamiltonian is zero.



Path integrals on $S^1 \times X$. Consider again $Z(I \times X) \in \mathcal{H}_X^\star \otimes \mathcal{H}_X$, the time evolution operator. Now glue X_1 to X_0 to make $S^1 \times X$. The last axiom above gives

$$Z(S^1 \times X) = \mathrm{tr}_{\mathcal{H}_X} \mathbb{1} = \dim \mathcal{H}_X. \tag{4.1}$$

A slight generalization is the following. Suppose we have a diffeomorphism $K : X \to X$. Then we can consider gluing X_1 to X_0 after doing the operation K. This space $S^1 \times_K X$ is called the 'mapping cylinder' of K and we have

$$Z(S^1 \times_K X) = \mathrm{tr}_{\mathcal{H}_X} \hat{K}$$

where $\hat{K}$ is the representation of K on $\mathcal{H}_X$.

[End of Lecture 19]

4.2 Chern-Simons theory

[Witten] I'm going to try to explain this in such a way that you can see that any (2+1)-dimensional gapped phase of matter with non-abelian anyons can produce 3-manifold invariants and knot invariants. In the interest of time, I'm going to avoid to the extent possible discussion of conformal field theory.

What do I mean by knot invariants? A knot is an embedding of the circle in some 3-manifold, say $\mathbb{R}^3$ or S^3. Given a knot, we'd like to know if we can untie it without cutting it or passing it through itself (this is called *isotopy*), and given two knots we'd like to know if they can be isotoped to each other. Much like phases of matter, it would be easiest to decide this if we could associate with each knot some topological quantity, invariant under isotopy. It will be fruitful to consider simultaneously *links*, embeddings of multiple, possibly linked circles in the 3-manifold, to which the same questions apply.

One nice such invariant is the Jones polynomial. Its original definition was actually already in terms of a statistical mechanics model associated with a projection into the plane of the knot in question. The Jones polynomial is shown to be an isotopy invariant by showing that it is preserved by a few basic (Riedemeister) moves on the projection which generate all isotopies. Here is a fully 3-dimensional description of the Jones polynomial as a physical observable of a topological gauge theory.

Suppose given a smooth, closed, oriented 3-manifold M and a knot K embedded in it. Let E be the trivial G-bundle with base M, where G is a simple compact Lie group (like $\mathsf{SU}(N)$). So $E = M \times \mathbb{C}^N$. And on E we'll consider a connection, *i.e.* a Lie-algebra-valued one-form $A_i = \sum_{A=1}^{\dim G} A_i^A T^A$ (an $N \times N$ matrix of 1-forms in the example), $i = 1..3$ a spatial index. Under an infinitesimal gauge transformation,

$$A_i \mapsto A_i - D_i\lambda, \qquad D_i\lambda \equiv \partial_i\lambda + [A_i, \lambda]. \tag{4.2}$$

The field strength is $F_{ij} \equiv [D_i, D_j] = \partial_i A_j - \partial_j A_i + [A_i, A_j]$.

A is going to be our dynamical variable. So far in this class, we've mostly been talking about Hamiltonian descriptions of physical systems, but here we're going to start with a path integral approach. The basic object required to do that is an action:

$$S_{\mathrm{CS}}[A] \equiv \frac{k}{4\pi}\int_M \mathrm{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) = \frac{k}{8\pi}\int_M d^3x\epsilon^{ijk}\mathrm{tr}\left(A_i\left(\partial_j A_k - \partial_k A_j + \frac{2}{3}[A_j, A_k]\right)\right).$$

A few comments about this choice of action:

- Unlike the Maxwell action, we can write it down without saying anything about a metric on M.

- It also has fewer derivatives than the Maxwell action, and so is more important for slowly-varying fields. If we add them both, the Chern-Simons term will win at low energies.

- The choice of orientation of M is required to define the integral of the 3-form. This breaks parity symmetry. A parity transformation takes the level k to $-k$.

- The main point in life of this functional is that the equations of motion are

$$0 = \frac{\delta S_{\rm CS}[A]}{\delta A} \propto F$$

  – they say that A is a flat connection. This equation is very different from the Maxwell equations. Here there are no propagating degrees of freedom; this is not a surprise since in order to propagate they need to know how far they are propagating, a metric. If we added the Maxwell action, we would find that the excitations are massive and become more and more massive as the coefficient in front of the Maxwell term decreases. Chern-Simons theory is a theory of groundstates, a topological field theory.

- In particular, the theory is scale-invariant; k is a dimensionless coupling.

- The Chern-Simons term is invariant under the infinitesimal gauge transformation (4.2) if $\partial M = \emptyset$.

  But as we saw earlier, a better expression for the gauge transformation, which incorporates global information, is $A \mapsto A^g \equiv g^{-1}Ag - g^{-1}dg$, where $g : M \to G$. Consider for a moment putting our theory on $M = S^3$. Such maps $g : S^3 \to G$ are classified up to homotopy by $\pi_3(G) = \mathbb{Z}$. The variation of $S_{\rm CS}$ under a gauge transformation in the homotopy class labelled by the integer ν is determined by

$$S_{\rm CS}[A^g] = S_{\rm CS}[A] + 2\pi k\nu$$

  where

$$\nu = \frac{1}{12\pi}\int_M \mathrm{tr}\left(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg\right) \in \mathbb{Z}$$

  is the winding number we discussed above in (3.32). The sum over connections is weighted by $e^{\mathbf{i}S_{\rm CS}[A]}$; in order that $e^{\mathbf{i}S_{\rm CS}[A^g]} \stackrel{!}{=} e^{\mathbf{i}S_{\rm CS}[A]}$ we must have $k \in \mathbb{Z}$, the level is an integer.

  This means that, although this is an interacting field theory, the coupling constant k cannot change a little bit under renormalization and so cannot change at all. Note that $k \to \infty$ is weak coupling, where the action becomes quadratic (after a rescaling of A).

What are the observables of this theory? There are no (gauge-invariant) local operators. There are only Wilson loops

$$W_R(C) \equiv \mathrm{tr}_R P e^{\mathbf{i} \oint_C A}.$$

Here R is a choice of representation of G whose generators appear in $A = \sum_A A^A T^A$. And C is a knot. Since there's no way to measure distances, isotopic knots give the same operator. But trying to move a segment of C through some other one will change the operator, since, as usual in quantum field theory, putting things on top of each other is dangerous. So we are going to think about the complex numbers

$$Z(M; \{C_r, R_r\}_r) \equiv \int [dA]\ e^{\mathbf{i} S_{\mathrm{CS}}[A]} \prod_{r=1}^{n} W_{R_r}(C_r) \tag{4.3}$$

where $C_r \cap C_{r'} = \emptyset$.

The abelian case. If G is abelian, say $\mathsf{U}(1)$, the action is gaussian $S[a] = \frac{k}{8\pi} \int \epsilon^{ijk} a_i \partial_j a_k$ and we can just do the integral (4.3).

$$\begin{aligned} Z(S^3, \{C_r, n_r\}) &= \int [da] e^{\mathbf{i}S} \prod_r e^{\mathbf{i} n_r \oint_{C_r} a} = \int [da] e^{\mathbf{i}S + \mathbf{i}\int ja} \qquad \text{with } j^\mu(x) \equiv \sum_r n_r \oint_{C_r} dx^\mu \delta^3(x - x_r(s)) \\ &= \mathcal{N} \exp \mathbf{i} \int_x \int_y j_x D^{-1}_{xy} j_y \\ &= \mathcal{N} \exp \frac{\mathbf{i}}{2k} \sum_{rs} n_r n_s \oint_{C_r} dx^i \oint_{C_s} dy^j \epsilon_{ijk} \frac{(x-y)^k}{|x-y|^3}. \end{aligned} \tag{4.4}$$

For $r \neq s$, and assuming $C_r \cap C_s = \emptyset$, the integral is

$$\ell(C_r, C_s) = \frac{1}{4\pi} \oint_{C_r} dx^i \oint_{C_s} dy^j \epsilon_{ijk} \frac{(x-y)^k}{|x-y|^3} \in \mathbb{Z}, \tag{4.5}$$

Gauss' formula for the linking number of the two curves. It is the same quantity appearing in the Hopf invariant[48]

[48] This linking number comes in many guises. What's being computed directly in (4.5) is the *degree* (multiplicity of the preimage of a point) of the map

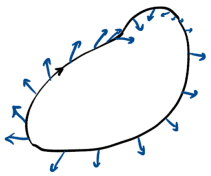
$$L : \begin{array}{ccc} C_r \times C_s \simeq T^2 & \to & S^2 \\ (x, y) & \mapsto & \frac{x-y}{|x-y|} \end{array}.$$

It is also the same as

$$\int \omega_r \wedge \eta_s = \#\left(D_r \cup C_s\right)$$

where $\eta_{r,s} = d\omega_{r,s}$ are compactly-supported Poincaré duals (2-forms) of $C_{r,s}$, which are exact on S^3; $\omega_{r,s}$ are then the Poincaré duals of disks $D_{r,s}$ bounding $C_{r,s}$. As in the discussion of the Hopf invariant, these expressions are independent of choices of representatives.

To define the terms with $r = s$, the self-linking requires a little more input, called a *framing* of the knot. This is a tiny vector at each point on the curve C, not tangent to C. The idea is that we displace C slightly along this direction to make a new curve C' and define



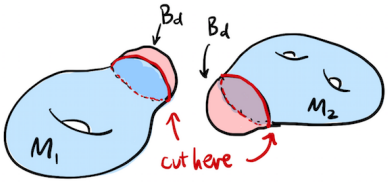
$$\ell(C_r, C_r) \equiv \ell(C_r, C_r'). \tag{4.6}$$

This is point-splitting regularization of the integral (4.5).

The regularized linking number (4.6) depends on the number of times the framing vector winds around as it goes around C. Twisting the framing t times changes ℓ_{rr} by t, which changes the answer for (4.4) by $Z \to Ze^{2\pi i t \frac{n_r^2}{k}}$. On S^3 there is a sort of canonical framing: just choose t for each curve so that $\ell(C, C) = 0$. But this is a physical ambiguity: the Wilson line describes the propagation of a particle with fractional spin, an anyon. (In this abelian Chern-Simons theory, its spin is $1/k$). In addition to its path through spacetime, to fully determine the amplitude it accrues, we must also specify the number of rotations t it undergoes during its trajectory.

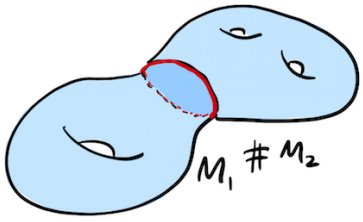
The idea for how we are going to compute (4.3) in the non-abelian case is by chopping up M into manageable parts.

First let's define the (purely topological) notion of connected sum: take two d-manifolds M_1, M_2 and cut out a d-ball from each one. Each one then has a component of its boundary homeomorphic to S^2. Glue these two boundaries together so that there is no boundary any more to get their connected sum $M_1 \# M_2$. The picture at right is the d-dimensional case.



First claim: suppressing the dependence on the knots, which I just assume don't go near the cut-out regions,

$$Z(M)Z(S^3) = Z(M_1)Z(M_2), \quad \text{or} \quad \frac{Z(M)}{Z(S^3)} = \frac{Z(M_1)}{Z(S^3)}\frac{Z(M_2)}{Z(S^3)}. \tag{4.7}$$



Notice that this is consistent with the fact that (from the definition) $M_1 \# S^3 = M_1$, S^3 is the identity for this product operation.

To see why (4.7) is true, let's think about the partition function associated with M_1 with the ball cut out. According to the axioms of TFT, this produces a state in the Hilbert space associated with S_2, $|M_1\rangle \in \mathcal{H}_{S^2}$. And similarly, the path integral on M_2 with the ball cut out (with the opposite orientation) produces $\langle M_2| \in \mathcal{H}^\star_{S^2}$. Therefore $Z(M) = \langle M_2|M_1\rangle$. Similarly we can regard $M = S^3$ as the connected sum of two balls

B_L and B_R, and $Z(S^3) = \langle B_L | B_R \rangle$.

But now here comes the point: this system has a unique groundstate on S^2 – $\mathcal{H}_{S^2}$ is one-dimensional. I'll explain this unsurprising[49] statement more in a bit but for now let's exploit the consequences.

In a one-dimensional vector space, any two vectors are linearly dependent and we have the identity

$$\langle M_2 | M_1 \rangle \langle B_L | B_R \rangle = \langle M_2 | B_R \rangle \langle B_L | M_1 \rangle \tag{4.8}$$

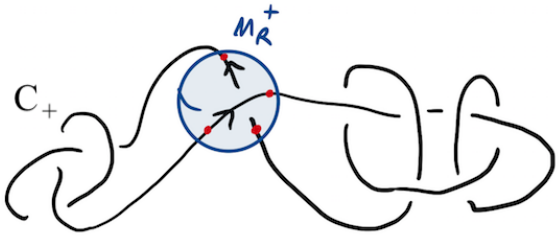
which is (4.7). Since there could have been Wilson loops all over away from the cut, we have the factorization formula for unlinked (so that we avoid cutting them) knots (in any reps)

$$\frac{Z(S^3; C_1 \cdots C_s)}{Z(S^3)} = \prod_{r=1}^{s} \frac{Z(S^3; C_r)}{Z(S^3)}. \tag{4.9}$$

A useful shorthand for this formula is

$$\langle C_1 \cdots C_s \rangle = \prod_{r=1}^{s} \langle C_r \rangle . \tag{4.10}$$

Now let's focus on a knot C_+ in S^3. Zoom in on a region around a single crossing (*i.e.* what would be a crossing in a particular plane projection of the knot). Cut out a small 3-ball M_R containing the two strands involved. The boundary of this ball is a $S^2 \backslash$ four punctures where the two Wilson lines enter and exit, labelled by representations $R, R, R^\star, R^\star$ of G.



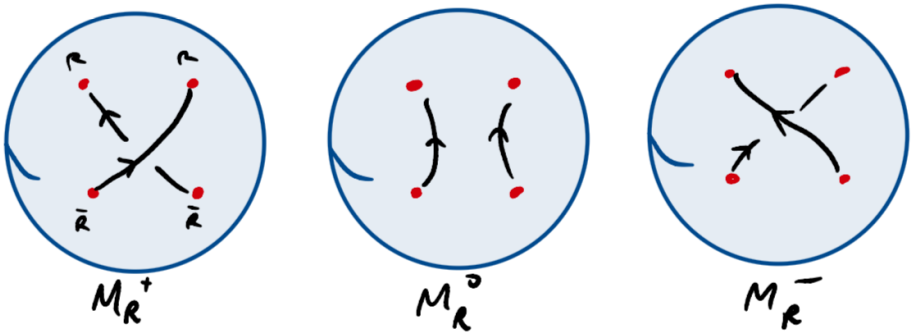
The path integral on the ball M_R produces a state $\left| M_R^+ \right\rangle \in \mathcal{H}_{S^2 \backslash RRR^\star R^\star}$. The path integral on its complement M_L in S^3 produces a state $\langle M_L | \in \mathcal{H}^\star_{S^2 \backslash RRR^\star R^\star}$ and $Z(S^3; C_+, R) = \left\langle M_L | M_R^+ \right\rangle$.

Now let me pull out one more fact from our future canonical analysis, which is that for $R =$ the fundamental (N-dimensional) representation of $\mathsf{SU}(N)$, this Hilbert space is two-dimensional.

This fact has a very dramatic consequence. Consider the following three ways of

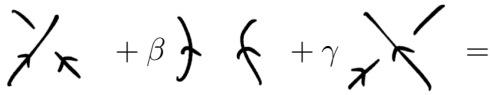
[49] The toric code and quantum double models all had a unique groundstate on S^2.

filling in the connections between the punctures:



Each of these produces a vector in $\mathcal{H}_{S^2\setminus RRR^*R^*}$. But in a two-dimensional vector space, any three vectors enjoy a linear relation: $\alpha\left|M_R^+\right\rangle+\beta\left|M_R^0\right\rangle+\gamma\left|M_R^-\right\rangle=0$. Now consider gluing back in the rest of the path integral, which means taking the overlap with $\langle\chi|$:

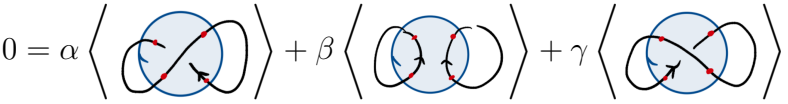
$$\begin{aligned}\alpha\left\langle\chi|\Psi^+\right\rangle+\beta\left\langle\chi|\Psi^0\right\rangle+\gamma\left\langle\chi|\Psi^-\right\rangle&=0\\ \text{or}\quad \alpha Z(S^3;C_+,R)+\beta Z(S^3;C_0,R)+\gamma Z(S^3;C_-,R)&=0\end{aligned}\tag{4.11}$$

or α  $=0.$

This is called a *Skein relation*. With a particular choice of α,β,γ, such a relation can be used to determine the Jones polynomial (and its many generalizations) for any knot or link. It determines the invariant for all knots (and links) in S^3 by induction on the number of crossing p in a plane projection. Suppose we know Z for all knots with $\le p-1$ crossings. If β were zero, we could untie any knot by repeatedly using the relation $Z_+\overset{?}{=}-\frac{\gamma}{\alpha}Z_-$. $\beta\neq 0$, but the coefficient of β involves a knot with one fewer crossing.

The simplest way to fill in the rest of the knot is:

$$0=\alpha\left\langle\ \right\rangle+\beta\left\langle\ \right\rangle+\gamma\left\langle\ \right\rangle\tag{4.12}$$



$$\alpha Z(C)+\beta Z(C^2)+\gamma Z(C)\tag{4.13}$$

from which we learn $Z(C^2)=-\frac{\alpha+\gamma}{\beta}Z(C)$. Using also (4.9) in the form $\langle C^2\rangle=\langle C\rangle^2$ we have

$$\langle C\rangle=-\frac{\alpha+\gamma}{\beta}.$$

Now it behooves us to determine α,β,γ in terms of the defining parameters N,k of Chern-Simons theory. Before we do that, let me explain about the Hilbert spaces $\mathcal{H}_\Sigma$.

Canonical picture. Consider the case $M = \mathbb{R} \times \Sigma_g$, a closed Riemann surface of genus g, perhaps punctured by a collection of Wilson loops stretched along the 'time' direction. On this space, it's a good idea to consider $A_0 = 0$ gauge, since the 0 direction is special anyway. In this gauge, the action takes the quadratic (!) form

$$S = \frac{k}{8\pi} \int_{\Sigma_g} \int dt \epsilon^{ij} \mathrm{tr} A_i \frac{d}{dt} A_j + \text{sources}$$

(the extra terms come from the Wilson lines). This just says that A_x and A_y are canonically conjugate variables, and $H = 0$. The non-linearities are not completely forgotten, though, because we must still impose the Gauss law constraint

$$0 = \frac{\delta S}{\delta A_0^A} = \frac{k}{4\pi} \epsilon_{ij} F_{ij}^A - \sum_{r=1}^{s} \delta^2(x - p_r) T_r^A, \tag{4.14}$$

that is, away from the Wilson loop insertions, the connection on Σ_g is flat. Here T_r^A is Lie algebra generator in the representation R_r made from some degrees of freedom, localized at the puncture r, whose associated Hilbert space is the carrier space of R_r.

We conclude that the phase space is the moduli space of flat G connections on $\Sigma_g \backslash$ the punctures (with some prescribed boundary conditions at the punctures) modulo gauge transformations. Without punctures, this has dimension $(2g-2) \dim G < \infty$.

Note the contrast with the quantum double model, where the groundstates themselves were labelled by flat G connections. This is a sense in which Chern-Simons theory is like a square root of the quantum double model.

We saw previously that a G-bundle on S^2 is specified by its transition functions on the equator in $\pi_1(G)$. So as long as G is simply-connected ($\pi_1(SU(N)) = 0$) there is a unique state in $\mathcal{H}_{S^2}$.

To understand what happens with punctures, consider the semiclassical limit $k \to \infty$. Then a naive guess is that the Hilbert space on $S^2 \setminus r$ points is $\mathcal{H} \stackrel{?}{=} \otimes_r R_r \equiv \mathcal{H}_0$, just the representation space. This is almost right, except consider the holonomy of the gauge field around a tiny contractible circle: $0 = \oint_{C_0} A$. But on the sphere, this tiny contour surrounds everything else. Comparing to the Gauss law (4.14) we see that the total charge on S^2 must vanish. That is, the R_r must fuse, somehow, to the singlet representation of G. So in fact:

$$\mathcal{H} = \text{the } G\text{-invariant subspace of } \mathcal{H}_0.$$

For $k < \infty$ the only difference is that the allowed irreps R_r are restricted to a subset of all the irreps of G (they are called the "integrable reps of the affine Lie algebra at level k").

With this information, we see

$$\mathcal{H}_{S^2} = \mathbb{C}. \quad \mathcal{H}_{S^2\setminus R_a} = \begin{cases} \mathbb{C}, & R_a = R_1 \\ 0 & \text{else} \end{cases}. \quad \mathcal{H}_{S^2\setminus\{R_a,R_b\}} = \mathbb{C}\delta_{R_a,R_b^\star} \equiv \mathbb{C}\delta_{a\bar{b}}. \quad \mathcal{H}_{S^2\setminus\{R_a,R_b,R_c\}} = V_{abc}^1 \tag{4.15}$$

the fusion space where the three reps fuse to the singlet, $R_a \otimes R_b \otimes R_c = V_{abc}^1 \otimes R_1 \oplus \cdots$ of dimension $\dim V_{abc}^1 \equiv N_{abc}$. Finally, since $\square \otimes \square = \square\!\square \oplus \square\!\square$, we have

$$(\square \otimes \square) \otimes (\overline{\square} \otimes \overline{\square}) = (\square\!\square \oplus \square\!\square) \otimes \left(\overline{\square\!\square} \oplus \overline{\square\!\square}\right) = \left(\square\!\square \otimes \overline{\square\!\square}\right) \oplus (\square\!\square \otimes \overline{\square\!\square}) \oplus \text{non-singlets}$$

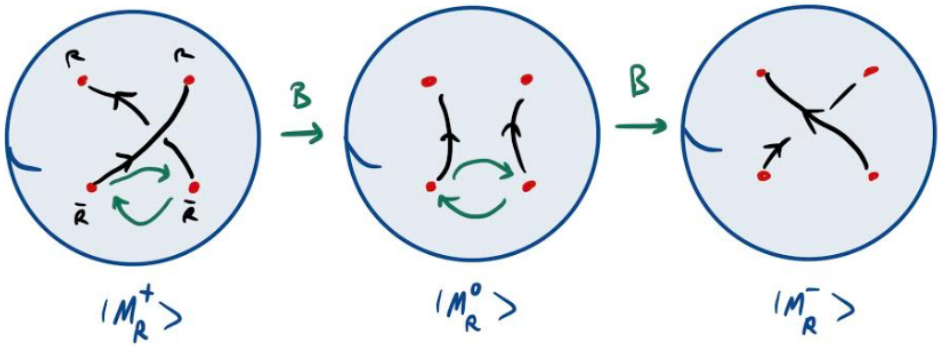
has a two-dimensional singlet subspace as claimed above.

Notice, by the way, that any 2d topological state with an anyon type a satisfying $a \otimes a \otimes \bar{a} \otimes \bar{a}$ is two-dimensional would lead to the same conclusions as above, *i.e.* a Skein relation (with different values of α, β, γ). For example, the quantum double model with gauge group $\mathsf{SL}(2, \mathbb{F}_3)$ (two-by-two matrices with elements in the field of three elements of determinant one) has three 2-dimensional reps with this property.

Braiding. The three vectors $\left|M_R^{\pm,0}\right\rangle$ are related in a very simple way: To get M_R^0 from M_R^+, just interchange the two outgoing punctures by a clockwise rotation on the surface of the sphere. Crucially, during this process, the strands of the Wilson loops inside the ball never touch. To get from M_R^- from M_R^0 just do the same thing again. If we call B the action of this operation on $\mathcal{H}_{S^2\setminus R\bar{R}\bar{R}}$, we have

$$\left|M_R^0\right\rangle = B\left|M_R^+\right\rangle, \quad \left|M_R^-\right\rangle = B^2\left|M_R^+\right\rangle.$$

That is,



This *braiding operator* B is a unitary operator on a two-dimensional Hilbert space, so its characteristic equation takes the form

$$B^2 - yB + z = 0, \quad \text{with } y = \mathrm{tr}B, z = \det B. \tag{4.16}$$

Acting (4.16) on $|\Psi_+\rangle$, we can rewrite the Skein relation (4.11) as

$$z\left|M_R^+\right\rangle - y\left|M_R^0\right\rangle + \left|M_R^-\right\rangle = 0. \tag{4.17}$$

The eigenvalues of the braiding matrix B (determining z, y and hence $\alpha, \beta\gamma$) are part of the anyon data. For the $\mathsf{SU}(N)_k$ model (which I will not explain), and accounting for the framing ambiguity in defining the Wilson lines, this leads to

$$-q^{N/2}Z_+ + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})Z_0 + q^{-N/2}Z_- = 0, \quad q \equiv e^{\frac{2\pi\mathbf{i}}{N+k}}$$

and we conclude from (4.10) that for the unknot in S^3,

$$\langle C \rangle = \frac{q^{N/2} - q^{-N/2}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}.$$

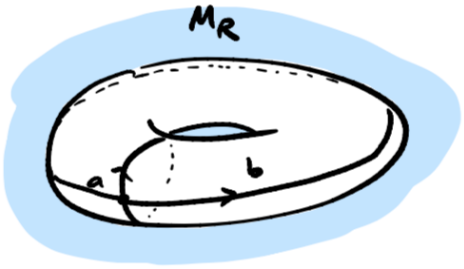
A few physics checks: reflection positivity of the Chern-Simons theory requires $\langle C \rangle \geq 0$. In the weak-coupling $k \to \infty$ limit, we can ignore quantum fluctuations and the classical solution is just $A = 0$, and we get $\langle C \rangle = \mathrm{tr}\mathbb{1} = N$. A less apparent check is that $\langle C \rangle = 0$ for $q^N = 1$, *i.e.* $k = 0$. In this case there are no integrable representations.

Dehn surgery on 3-manifolds. Consider a curve $C \subset M$ (which may or may not host a Wilson line). Thicken C to a tubular neighborhood M_R of C, a solid torus, $B^2 \times S^1$. Cut out M_R from M. Its boundary is $\partial M_R = T^2$. So we're going to think about $\mathcal{H}_{T^2}$. Before we do, let's introduce a purely mathematical operation. If we glue back M_R to M_L (the rest of M) we just get M back. But consider for a moment a basis of the first homology of $\partial M_R = T^2$. It is generated by two 1-cycles a, b. Let's say that in M_R, a is the contractible one. The torus enjoys an $\mathsf{SL}(2,\mathbb{Z})$ worth of large diffeomorphisms which act upon the first homology as

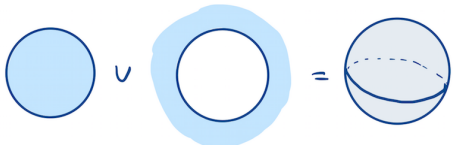
$$\begin{pmatrix} n \\ m \end{pmatrix} \mapsto K \begin{pmatrix} n \\ m \end{pmatrix} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} n \\ m \end{pmatrix}, \quad ad - bc = 1.$$

If we act with such a transformation before we glue back M_L, we get a different 3-manifold $\tilde{M}$.

For example, $S^3 = M_L \cup_{T^2} M_R$. Here both M_L and M_R are solid tori, $B^2 \times S^1$. I tried to draw them at right. In M_L a is contractible. In M_R, however, it is the *other* cycle b which is contractible. By the van Kampen theorem (thicken the overlap between M_L and M_R to a torus shell), therefore, all the homotopy classes are destroyed by this gluing. If, however, before gluing M_L to M_R we act with the $\mathsf{SL}(2,\mathbb{Z})$ transformation $K = S \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ which exchanges a and b, then it is a which is contractible in both patches, and we get



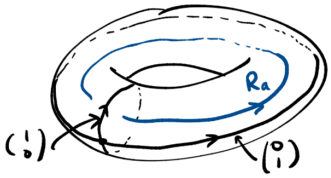
$$\tilde{M} = \left(\underbrace{B_2 \text{ glued to } B_2}_{=S^2} \right) \times S^1 = S^2 \times S^1.$$



In fact, every 3-manifold can be obtained from S^3 (or from $S^2 \times S^1$) by a sequence of such operations.

Therefore, if $Z(M) = \langle M_L | M_R \rangle$ then $Z(\tilde{M}) = \langle M_L | \hat{K} | M_R \rangle$ where $\hat{K}$ is the representation of the modular transformation K on $\mathcal{H}_{T^2}$.

Now back to Chern-Simons theory. Here is a basis for $\mathcal{H}_{T^2}$ due to Erik Verlinde: Let $|v_a\rangle$ be the state given by the path integral on the solid torus with a Wilson loop in the representation R_a running around the center of the non-contractible cycle. So the state $|v_0\rangle$ with the singlet rep is the state $|M_R\rangle$ defined above if there is no Wilson line.



Here $a = 1..t$ runs over the allowed 'integrable' irreps of G (t depends on the level k). In this basis,

$$\hat{K} |v_a\rangle = K_a{}^b |v_b\rangle .$$

Now, given $M \supset C$, a knot and an irrep R_a of G, cut out a tubular neighborhood M_R of C, a solid torus. We have $Z(M, C) = \langle M_L | v_{R_a} \rangle$ and

$$Z(\tilde{M}, C) = \langle M_L | \hat{K} | v_{R_a} \rangle = \sum_b K_a{}^b \langle M_L | v_{R_b} \rangle .$$

For example, if $R = R_0$ is the trivial rep, then

$$Z(\tilde{M}, C) = \sum_b K_0{}^b \langle M_L | v_{R_b} \rangle = \sum_b K_0{}^b Z(M, R_b).$$

Now apply (4.1) to $S^1 \times S^2$ with marked points in irreps $\{R_a\}$:

$$Z(S^2 \times S^1, \{R_a\}) = \dim \mathcal{H}_{S^2 \setminus \{R_a\}}.$$

In particular, from (4.15) we have

$$Z(S^2 \times S^1) = 1. \quad Z(S^2 \times S^1, R_a) = \delta_{a,0}. \quad Z(S^2 \times S^1, \{R_a, R_b\}) = \delta_{a\bar{b}}. \tag{4.18}$$

$$Z(S^2 \times S^1, \{R_a, R_b, R_c\}) = N_{abc}. \quad Z(S^2 \times S^1, \square\square\overline{\square\square}) = 2\ . \tag{4.19}$$

[50]

Now take $\tilde{M} = S^3$ and $M = S^2 \times S^1$ (so $K = S$ is the element of $\mathsf{SL}(2,\mathbb{Z})$ involved in the surgery). Then

$$Z(S^3) = \sum_b S_0{}^b \underbrace{Z(S^2 \times S^1, R_b)}_{=\delta_{b,0}} = S_0{}^0$$

where I wrote $S_a{}^b$ for the matrix elements of the 'S-matrix' – the matrix representation of the S transformation on $\mathcal{H}_{T^2}$. Similarly,

$$Z(S^3, R_a) = \sum_b S_a{}^b \underbrace{Z(S^2 \times S^1, R_b)}_{=\delta_{b,0}} = S_a{}^0 = \langle C_a \rangle\, Z(S^3)$$

from which we conclude

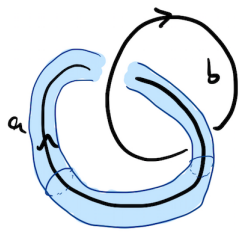
$$\langle C_a \rangle = \frac{S_a{}^0}{S_0{}^0}.$$

For the case of $G = \mathsf{SU}(2)_k$, the allowed irreps are spin $a/2$ for $a = 0...k$ and the S-matrix is

$$S_{ab} \equiv S_a{}^c \delta_{c\bar{b}} = \sqrt{\frac{2}{k+2}} \sin \frac{\pi(a+1)(b+1)}{k+2}, a, b = 0...k.$$

Finally, without introducing any more ideas (I think that's enough for now), we can derive the *Verlinde formula* determining the fusion coefficients N_{abc} in terms of the S-matrix.

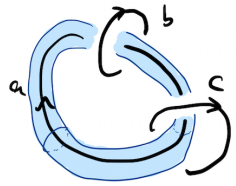
[End of Lecture 20]



Consider two Wilson loops with irreps R_a and R_b which are linked unknots. First notice that by doing surgery on a solid torus thickening R_a,

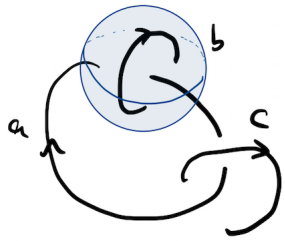
$$Z(S^3, R_a \text{ linking } R_b) = \sum_c S_a{}^c Z(S^2 \times S^1, \{R_c, R_b\}) = S_a{}^c \delta_{b,\bar{c}} \equiv S_{ab}. \tag{4.20}$$

[50] Side remark: Note that any knot S_C that lies in M_R is a *satellite* knot of C. The state created by a Wilson loop along S_C has an expansion in the above basis of $\mathcal{H}_{T^2}$, $|S_C\rangle = \sum_a \alpha_a |v_a\rangle$ with some coefficients α_a that are universal numbers independent of M_L.



And do the same thing for R_a linking R_b and R_c which are not linked with each other:

$$Z(S^3, R_a \text{ linking } R_b, R_c) = \sum_d S_a{}^d Z(S^2 \times S^1, \{R_b, R_c, R_d\}) = \sum_d S_a{}^d N_{bcd}. \tag{4.21}$$



We can derive another relation by cutting out a ball containing R_b but not R_c. $\mathcal{H}_{S^2 \setminus \{R_b, R_b^\star\}}$ is one-dimensional, so we can apply the same logic as we used in deriving (4.8) to conclude that the LHS of (4.21) satisfies

$$Z(S^3, R_a \text{ linking } R_b, R_c) = \frac{1}{Z(S^3, R_a)} Z(S^3, R_a \text{ linking } R_b) Z(S^3, R_a \text{ linking } R_c) \overset{(4.20)}{=} \frac{S_{ab} S_{ac}}{S_{a0}}. \tag{4.22}$$

Comparing (4.22) and (4.21), we find

$$\sum_d S_a{}^d N_{bcd} = \frac{S_{ab} S_{ac}}{S_{a0}} \tag{4.23}$$

Multiplying the BHS of (4.23) by $(S^{-1})_d{}^a$ and summing over a we conclude

$$N_{bcd} = \sum_a \frac{S_{ab} S_{ac} (S^{-1})_d{}^a}{S_{a0}}$$

which is the Verlinde formula. (4.23) can be interpreted as the statement that the S-matrix diagonalizes the fusion rules: Regard $(N_a)_b{}^c \equiv N_{ab}^c \equiv N_{abd}\delta^{c\bar{d}}$ as a collection of matrices. These matrices all commute (because the fusion product in irreps is commutative: $a \otimes b = b \otimes a$; this also means N_{abc} is completely symmetric) and so can be simultaneously diagonalized. (4.23) is the eigenvalue equation:

$$(N_b)_c{}^d S_{ad} = \left(\frac{S_{ab}}{S_{a0}}\right) S_{ac} \quad \text{is } (N_b)_c{}^d v_d = \lambda v_c \quad \text{with} \quad v_d \equiv S_{ad}, \lambda \equiv \left(\frac{S_{ab}}{S_{a0}}\right).$$

4.3 Links to the future

Given more time, the next topic would have been topological gauge theories defined on cell decompositions of spacetime in a subdivision-invariant way.

A warmup example for this is 2d Yang-Mills theory, as described in §2.3 of this paper. This is not quite a topological field theory, in that physics depends on *areas* (a small subset of the metric data), but all the rest of the structure of the axioms is

realized. This discussion also goes through for a finite gauge group, where it really is a TQFT.

Then in any dimension, Dijkgraaf and Witten define a generalization of lattice gauge theory which includes a twisting by a group cocycle. One reason this is an important example is that models related to them by a process called 'ungauging' produce solvable representatives of a large class of Symmetry Protected Topological phases, as described by Chen, Gu and Wen (especially §6, 7).