University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 239 Topology from Physics Winter 2021
Assignment 9

Due 5pm Friday March 12, 2021

Thanks in advance for following the guidelines on hw01. Please ask me by email if you have any trouble.

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1. **Intersection pairing and cohomology.** Because $2 + 2 = 4$, on a 4-manifold M_4, we can define a pairing on the integral 2-cycles, $[S_1], [S_2] \in H_2(M_4, \mathbb{Z})$, by $(S_1, S_2) \equiv$ the number of points in which S_1 and S_2 intersect, counted with orientation and multiplicity. The sign is plus if the volume form on S_1 wedge the volume form on S_2 agrees with the volume form on M_4.

   (a) Now consider the Poincaré dual perspective. Each 2-cycle S has a Poincaré dual 2-form η_S. Show that

   $$(S_1, S_2) = \int_{M_4} \eta_{S_1} \wedge \eta_{S_2}.$$

   Bonus: check the sign by considering representatives of $[\eta_{S_{1,2}}]$ supported in a small neighborhood of $S_{1,2}$, and looking at a coordinate system near an intersection point where S_1 is $x = y = 0$ and S_2 is $z = w = 0$.

   (b) Convince yourself of the following statement: There exists a basis of harmonic 2-forms $\alpha_I, I = 1..b_2(M_4)$ satisfying

   $$\int_{M_4} \alpha_I \wedge \alpha_J = K_{IJ}$$

   where K_{IJ} is the intersection matrix on some basis of the 2-cycles.

   (c) Show that K_{IJ} is symmetric.

   (d) What is the intersection form on $S^2 \times S^2$? On $\mathbb{CP}^2$? On S^4?

   (e) Bonus: define the connected sum $X_1 \# X_2$ of two n-manifolds $X_{1,2}$ to be the result of removing a small n-ball from each and gluing the resulting things together along the boundaries. What is the intersection form on $X_1 \# X_2$ in terms of those of X_1 and X_2?

   (f) Bonus: By thinking about the spectrum of the Hodge $\star$ operator on 2-forms, relate the signature of the matrix K (the number of positive eigenvalues minus the number of negative eigenvalues) to the Hirzebruch signature of M_4.

(g) Bonus: argue that K_{IJ} is unimodular, that is, it satisfies $\det K = \pm 1$.

2. **Dimensional reduction exercise.** Consider the following 3-form $\mathsf{U}(1)$ gauge theory in 6+1 dimensions. The degree of freedom is a 3-form potential C. Consider the action

$$S[C] = \frac{1}{4\pi} \int_{M_7} C \wedge dC$$

where M_7 is some smooth manifold. A field theory with this action is topological in the sense that no metric was required to write down the action.

(a) Show that S is gauge invariant if M_7 is closed, $\partial M_7 = 0$. The infinitesimal gauge transformation acts as $C \to C + d\lambda$ for some 2-form λ.

(b) Consider the case where $M_7 = M_4 \times \mathbb{R}^3$, where M_4 is some 4-manifold. Suppose that the intersection form on M_4 is $K_{IJ}, I = 1.. \dim H_2(M_4, \mathbb{Z})$ Plug in $C = \sum_I \alpha^I \wedge A^I(x)$, where α^I are the basis of harmonic 2-forms on M_4 from the previous part, and find the resulting 3d action for A^I.

3. **Fundamental group of an acyclic space.** In lecture we defined X by gluing two disks $B_{1,2}$ into a bouquet of two circles a and b by identifying their boundaries with the paths $a^5 b^{-3}$ and $b^3 (ab)^{-2}$. Use the van Kampen theorem twice to compute $\pi_1(X)$. That is, first use it compute $\pi_1(X \setminus B_1)$.

4. **Induced map on homotopy groups.** Like homology, π_q is a covariant functor from the category of topological spaces (and continuous maps) to the category of groups (and group homomorphisms). To see this, consider a map $\phi : (X, x_0) \to (Y, y_0)$. Given a representative of $\pi_q(X)$, $\alpha : (I^q, \partial I^q) \to (X, x_0)$, we can use ϕ to make a representative of $\pi_q(Y)$, namely $\phi \circ f : (I^q, \partial I^q) \to (Y, y_0)$. So we can define an induced map on the homotopy groups

$$\phi_\star [\alpha] \equiv [\phi \circ f].$$

Convince yourself that this is a group homomorphism in the sense that $\mathbb{1}_\star = \mathbb{1}$, $\phi \circ (\alpha \star \beta) = (\phi \circ \alpha) \star (\phi \circ \beta)$ and given also $\psi : (Y, y_0) \to (Z, z_0)$, we have $\psi_\star \circ \phi_\star = (\psi \circ \phi)_\star$.

Conclude that if $X \simeq Y$ then $\pi_1(X) \cong \pi_1(Y)$.

5. **$\mathbb{CP}^2$ is not anyone's boundary.**

In this problem we will show that the boundary of any compact manifold has even Euler character. Since $\chi(\mathbb{CP}^2)$ is odd, it cannot arise as the boundary of any compact 5-manifold.

(a) Here we will show that if $M = \partial V$ is a $2n$-dimensional manifold and V is compact, then $\dim_{\mathbb{Z}_2} H^n(M, \mathbb{Z}_2)$ is even. (If we assume V is oriented, then we can replace $\mathbb{Z}_2$ by any other field.)

Consider the following part of the long exact sequence for the homology of V relative to its boundary M:

$$\begin{array}{ccccc} H^n(V) & \xrightarrow{i^\star} & H^n(M) & \xrightarrow{\delta^\star} & H^{n+1}(V, M) \\ & & \downarrow f & & \downarrow g \\ & & H_n(M) & \xrightarrow{i_\star} & H_n(V) \end{array}$$

All coefficients are $\mathbb{Z}_2$. The vertical maps f and g are isomorphisms because of Poincaré duality (the one that relates homology and cohomology).

Use the fact that $\text{rank}(i_\star) = \text{rank}(i_\star)$ and the diagram to conclude that $\dim H^n(M) = 2\text{rank}(i^\star)$.

(b) Show that if $M = \partial V$, then $\chi(M)$ is even. Consider separately the cases where $\dim M$ is odd and even.

Hint: in the case where $\dim M = 2n$, relate $\chi(M)$ to $\dim_{\mathbb{Z}_2} H^n(M, \mathbb{Z}_2)$.

(c) What is $\chi(\mathbb{CP}^2)$? Conclude that $\mathbb{CP}^2$ represents a nontrivial cobordism class.

(d) What about $\mathbb{RP}^2$? Can an unoriented closed compact Riemann surface be a boundary? (Use the same argument.)

(e) What about $\mathbb{CP}^n$ for general n?