University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 239 Topology from Physics Winter 2021
Assignment 9 – Solutions

Due 5pm Friday March 12, 2021

Thanks in advance for following the guidelines on hw01. Please ask me by email if you have any trouble.

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1. **Intersection pairing and cohomology.** Because $2 + 2 = 4$, on a 4-manifold M_4, we can define a pairing on the integral 2-cycles, $[S_1], [S_2] \in H_2(M_4, \mathbb{Z})$, by $(S_1, S_2) \equiv$ the number of points in which S_1 and S_2 intersect, counted with orientation and multiplicity. The sign is plus if the volume form on S_1 wedge the volume form on S_2 agrees with the volume form on M_4.

   (a) Now consider the Poincaré dual perspective. Each 2-cycle S has a Poincaré dual 2-form η_S. Show that

   $$(S_1, S_2) = \int_{M_4} \eta_{S_1} \wedge \eta_{S_2}.$$

   Bonus: check the sign by considering representatives of $[\eta_{S_{1,2}}]$ supported in a small neighborhood of $S_{1,2}$, and looking at a coordinate system near an intersection point where S_1 is $x = y = 0$ and S_2 is $z = w = 0$.

   Everything I am saying here is explained on the first few pages of the book by Donaldson and Kronheimer, *Geometry of four-manifolds*.

   The definition of the Poincaré dual form η_S is $\int_S i^S_\star(\omega) = \int_{M_4} \eta_S \wedge \omega$ for all 2-forms ω, where i^S is the inclusion map $i : S \to M_4$. Since we can choose the Poincaré dual to have support in a small neighborhood about the cycle, we have

   $$(S_1, S_2) = \int_{S_1} i^{S_1}_\star \eta_{S_2} = \int_{M_4} \eta_{S_1} \wedge \eta_{S_2}.$$

   To verify the sign, choose coordinates near an intersection point and choose representatives where

   $$\eta_1 = \rho(x, y) dx \wedge dy, \quad \eta_2(z, w) dz \wedge dw$$

   with $\rho(x, y)$ a function supported in a small neighborhood about the origin. In this case

   $$\int \eta_1 \wedge \eta_2 = \int_{M_4} \rho(x, y)\rho(z, w) dx \wedge dy \wedge dz \wedge dw = \pm$$

gets contribution only from the neighborhood of the intersection and the sign is determined by comparing the orientations of S_1 and S_2 with that of M_4.

(b) Convince yourself of the following statement: There exists a basis of harmonic 2-forms $\alpha_I, I = 1..b_2(M_4)$ satisfying

$$\int_{M_4} \alpha_I \wedge \alpha_J = K_{IJ}$$

where K_{IJ} is the intersection matrix on some basis of the 2-cycles.

The intersection form is independent of cohomology representative. So we can appeal to the Hodge theorem to choose a harmonic representative of each class.

(c) Show that K_{IJ} is symmetric.

2-forms commute with each other.

(d) What is the intersection form on $S^2 \times S^2$? On $\mathbb{CP}^2$? On S^4?

$\sigma^x, 1$ and a 0-dimensional matrix.

The example of $\mathbb{CP}^2$ illustrates the following lesson: The self-intersection of a given 2-cycle can be nonzero. In terms of forms, this is clear because 2-forms are commuting objects. The definition is of the self-intersection is: take the two cycle S and deform it a little bit to another representative S? of the same homology class. Generically S and S? will intersect at a finite number of points, and this number counted with multiplicity depends only on the homology class.

In $\mathbb{CP}^2$ there is only one nontrivial generator of H_2. A representative is $S = \{\sum_{i=0}^{2} a_i z_i = 0\}$? the zero locus of arbitrary linear function of the homogeneous coordinates. So one representative is $S_2 = \{z_2 = 0\}$ and another is $S_1 = \{z_1 = 0\}$. $[S] = [S_1] = [S_2]$. The intersection of the latter two sets is the point $\{(z_0, 0, 0)\}$, so $K_{SS} = \#(S \cup S) = 1$.

(e) Bonus: define the connected sum $X_1 \# X_2$ of two n-manifolds $X_{1,2}$ to be the result of removing a small n-ball from each and gluing the resulting things together along the boundaries. What is the intersection form on $X_1 \# X_2$ in terms of those of X_1 and X_2?

Direct sum.

(f) Bonus: By thinking about the spectrum of the Hodge $\star$ operator on 2-forms, relate the signature of the matrix K (the number of positive eigenvalues minus the number of negative eigenvalues) to the Hirzebruch signature of M_4.

The key is that $\int \alpha \wedge \star\alpha \geq 0$ is a positive semi-definite norm on 2-forms. So if we choose a basis of forms $\alpha_\pm$ which are eigenvectors of $\star$, we have

$$0 \geq \int \alpha_\pm \wedge \star\alpha_\pm = \pm \int \alpha \wedge \alpha = \pm(\alpha, \alpha)$$

(where I denote the dual 2-cycle also as α because why not). We conclude that if $K_{II} < 0 (> 0)$ then η_{S_I} is in the anti-self-dual (self-dual) eigenspace of the Hodge $\star$. Therefore the number of negative (positive) diagonal entries of the intersection form is b_2^- (b_2^+) and the signature of the matrix K is equal to the Hirzebruch signature $b_2^+ - b_2^-$.

(g) Bonus: argue that K_{IJ} is unimodular, that is, it satisfies $\det K = \pm 1$.

This follows from the fact that the pairing between $H_2(M_4, \mathbb{Z})$ and $H^2(M_4, \mathbb{Z})$ is an isomorphism.

2. **Dimensional reduction exercise.** Consider the following 3-form $\mathsf{U}(1)$ gauge theory in 6+1 dimensions. The degree of freedom is a 3-form potential C. Consider the action

$$S[C] = \frac{1}{4\pi} \int_{M_7} C \wedge dC$$

where M_7 is some smooth manifold. A field theory with this action is topological in the sense that no metric was required to write down the action.

(a) Show that S is gauge invariant if M_7 is closed, $\partial M_7 = 0$. The infinitesimal gauge transformation acts as $C \to C + d\lambda$ for some 2-form λ.

$$\delta S = \frac{1}{4\pi} \int d\lambda \wedge dC \overset{\text{IBP}}{=} 0.$$

(b) Consider the case where $M_7 = M_4 \times \mathbb{R}^3$, where M_4 is some 4-manifold. Suppose that the intersection form on M_4 is $K_{IJ}, I = 1..\dim H_2(M_4, \mathbb{Z})$ Plug in $C = \sum_I \alpha^I \wedge A^I(x)$, where α^I are the basis of harmonic 2-forms on M_4 from the previous part, and find the resulting 3d action for A^I.

$$S[A] = \frac{K_{IJ}}{4\pi} \int_{\mathbb{R}^3} A^I \wedge dA^J.$$

3. **Fundamental group of an acyclic space.** In lecture we defined X by gluing two disks $B_{1,2}$ into a bouquet of two circles a and b by identifying their boundaries with the paths $a^5 b^{-3}$ and $b^3 (ab)^{-2}$. Use the van Kampen theorem twice to compute $\pi_1(X)$. That is, first use it compute $\pi_1(X \setminus B_1)$.

Decompose $Y \equiv X \backslash B_1$ into $U \cup V$ with $U = B_2$ and $V = Y\backslash$ a point in the middle of B_2. Then $\pi_1(U) = 0, \pi_1(V) = \langle a, b \rangle = \mathbb{F}_2$, the free group on two elements, and $\pi_1(U \cap V) = \langle g \rangle$. Therefore

$$\pi_1(Y) = \left\langle a, b | i_\star^V(g) = i_\star^U(g) \right\rangle = \left\langle a, b | b^3 (ab)^{-2} = e \right\rangle .$$

Now decompose X into $U' \cup V'$ with $U' = B_1$ and $V' = X\backslash$ a point in the middle of B_1. $\pi_1(U') = 0, \pi_1(V') = \pi_1(Y)$ from the previous step and $\pi_1(U' \cap V') = \langle h \rangle$ Therefore

$$\pi_1(X) = \left\langle a, b | b^3 (ab)^{-2} = e, i_\star^{V'}(h) = i_\star^{U'}(h) \right\rangle = \left\langle a, b | b^3 (ab)^{-2} = e, a^5 b^{-3} = e \right\rangle .$$

4. **Induced map on homotopy groups.** Like homology, π_q is a covariant functor from the category of topological spaces (and continuous maps) to the category of groups (and group homomorphisms). To see this, consider a map $\phi : (X, x_0) \to (Y, y_0)$. Given a representative of $\pi_q(X)$, $\alpha : (I^q, \partial I^q) \to (X, x_0)$, we can use ϕ to make a representative of $\pi_q(Y)$, namely $\phi \circ f : (I^q, \partial I^q) \to (Y, y_0)$. So we can define an induced map on the homotopy groups

$$\phi_\star[\alpha] \equiv [\phi \circ f].$$

Convince yourself that this is a group homomorphism in the sense that $\mathbb{1}_\star = \mathbb{1}$, $\phi \circ (\alpha \star \beta) = (\phi \circ \alpha) \star (\phi \circ \beta)$ and given also $\psi : (Y, y_0) \to (Z, z_0)$, we have $\psi_\star \circ \phi_\star = (\psi \circ \phi)_\star$.

Conclude that if $X \simeq Y$ then $\pi_1(X) \cong \pi_1(Y)$.

If $f : X \to Y$ and $g : Y \to X$ are the relevant maps then the induced map $f_\star : \pi_q(X, x_0) \to \pi_q(X, f(x_0))$ is an isomorphism with inverse $g_\star$.

5. $\mathbb{CP}^2$ **is not anyone's boundary.**

In this problem we will show that the boundary of any compact manifold has even Euler character. Since $\chi(\mathbb{CP}^2)$ is odd, it cannot arise as the boundary of any compact 5-manifold.

(a) Here we will show that if $M = \partial V$ is a $2n$-dimensional manifold and V is compact, then $\dim_{\mathbb{Z}_2} H^n(M, \mathbb{Z}_2)$ is even. (If we assume V is oriented, then we can replace $\mathbb{Z}_2$ by any other field.)

Consider the following part of the long exact sequence for the homology of V relative to its boundary M:

$$
\begin{array}{ccccc}
H^n(V) & \xrightarrow{i^\star} & H^n(M) & \xrightarrow{\delta^\star} & H^{n+1}(V, M) \\
 & & \big\downarrow f & & \big\downarrow g \\
 & & H_n(M) & \xrightarrow{i_\star} & H_n(V)
\end{array}
$$

All coefficients are $\mathbb{Z}_2$. The vertical maps f and g are isomorphisms because of Poincaré duality (the one that relates homology and cohomology).

Use the fact that $\text{rank}(i_\star) = \text{rank}(i^\star)$ and the diagram to conclude that $\dim H^n(M) = 2\text{rank}(i^\star)$.

This comes from statements 10.4 and 10.5 of chapter VI of Bredon's book.

The diagram implies that

$$\dim \text{im}(i^\star) = \dim \ker(\delta^\star) = \dim \ker(i_\star). \tag{1}$$

The first step is exactness of the sequence, and the second step is the fact that the vertical maps are isomorphisms. Then

$$\text{rank}(i^\star) = \dim \text{im}(i^\star) \overset{(1)}{=} \dim \ker(i_\star) = \dim H_n(M) - \text{rank}(i_\star) = \dim H^n(M) - \text{rank}(i^\star).$$

Therefore

$$\dim H_n(M) = 2\text{rank}(i^\star).$$

Since the rank of a linear map is an integer, $\dim H_n(M)$ is even.

(b) Show that if $M = \partial V$, then $\chi(M)$ is even. Consider separately the cases where $\dim M$ is odd and even.

Hint: in the case where $\dim M = 2n$, relate $\chi(M)$ to $\dim_{\mathbb{Z}_2} H^n(M, \mathbb{Z}_2)$.

If $\dim M = 2n+1$ is odd, its euler character is $\chi = (b_0 - b_{2n+1}) + (b_1 - b_{2n}) + \cdots + (b_n - b_{n+1}) = 0$ by Poincaré duality. If $\dim M = 2n$ is even, its euler character is $\chi = (b_0 - b_{2n}) + (b_1 - b_{2n-1}) + \cdots + b_n = b_n$ (also by Poincaré duality). Therefore in the latter case

$$\chi = b_n$$

is even.

(c) What is $\chi(\mathbb{CP}^2)$? Conclude that $\mathbb{CP}^2$ represents a nontrivial cobordism class.

Since $\mathbb{CP}^q$ has a single generator in each even dimension, the euler character of $\mathbb{CP}^2$ is three. A good way to describe the cohomology of $\mathbb{CP}^q$ is as the polynomial ring

$$H^\bullet(\mathbb{CP}^q) = \mathbb{R}[x]/x^{q+1},$$

where x is the generator of $H^2(\mathbb{CP}^2)$.

(d) What about $\mathbb{RP}^2$? Can an unoriented closed compact Riemann surface be a boundary? (Use the same argument.)

χ is odd for any unoriented closed compact Riemann surface, so the same argument applies.

(e) What about $\mathbb{CP}^n$ for general n?

$\mathbb{CP}^1 \simeq S^2 = \partial B^3$ is a boundary. More generally, since $\chi(\mathbb{CP}^q) = q + 1$, we conclude that for q even $\mathbb{CP}^q$ is not a boundary. For general odd q we can't say based on the results of this problem.