University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 239 Topology from Physics Winter 2021
Assignment 8 – Solutions

Due 5pm Friday March 5, 2021

Thanks in advance for following the guidelines on hw01. Please ask me by email if you have any trouble.

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1. **Cech cohomology brainwarmer.** Check that the Cech coboundary operator δ is nilpotent: $\delta^2 = 0$.

$$(\delta^2\omega)_{\alpha_0\cdots\alpha_{p+2}} = \sum_i (-1)^i (\delta\omega)_{\alpha_0\cdots\widehat{\alpha_i}\cdots\alpha_{p+2}} \tag{1}$$

$$= \sum_{j<i} (-1)^{i+j} \omega_{\alpha_0\cdots\widehat{\alpha_j}\cdots\widehat{\alpha_i}\cdots\alpha_{p+2}} + \sum_{j>i} (-1)^{i+j-1} \omega_{\alpha_0\cdots\widehat{\alpha_i}\cdots\widehat{\alpha_j}\cdots\alpha_{p+2}} \tag{2}$$

$$= 0. \tag{3}$$

2. **Euler-Poincaré theorem for Cech cohomology.**

   Let X be a manifold with a finite good cover. Let β_p is the number of non-empty $(p+1)$-fold intersections $U_{\alpha_0\cdots\alpha_p}$. Show that the Euler character is

$$\chi(X) = \sum_p (-1)^p \beta_p.$$

   Since the cohomology groups over $\mathbb{R}$ can be computed by a complex with dimensions β_p, this follows by the same proof as the Euler-Poincaré theorem for homology that we proved earlier.

3. **Cech cohomology example.**

   Convince yourself that the computation of the Cech cohomology of the 2-sphere (with arbitrary coefficients) using the good cover described in lecture is the same as the computation of the homology of the tetrahedron cell complex.

4. **Homology of spheres.**

(a) Use the Mayer-Vietoris sequence to compute $H^q(S^n)$ using the open cover $S^n = U_N \cup U_S$, where $U_N(U_S)$ is the complement of the north (south) pole. Start with S^2 and work your way up.

$$
\begin{array}{lllll}
S^2 & & U_N \sqcup U_S & & U_N \cap U_S \simeq S^1 \\
\hline
H^2(S^2) & \to & \cancel{H^2(U_N)} \oplus \cancel{H^2(U_S)} & \to & \cancel{H^2(S^1)}\ 0 \\
H^1(S^2) & \to & \cancel{H^1(U_N)} \oplus \cancel{H^1(U_S)} & \to & \underbrace{H^1(S^1)}_{=\mathbb{R}} \\
0 \to H^0(S^2) & \to & \underbrace{H^0(U_N)}_{=\mathbb{R}} \oplus \underbrace{H^0(U_S)}_{=\mathbb{R}} & \to & \underbrace{H^0(S^1)}_{=\mathbb{R}}
\end{array}
$$

$$
\begin{array}{lllll}
S^2 & & U_N \sqcup U_S & & U_N \cap U_S \simeq S^1 \\
\hline
H^2(S^2) & \to & 0 & \to & 0 \\
 & & & & \cong \\
H^1(S^2) & \to & 0 & \to & \mathbb{R} \\
 & & & & 0 \\
0 \to H^0(S^2) & \to & \mathbb{R} \oplus \mathbb{R} & \to & \mathbb{R}
\end{array}
$$

$$
\begin{array}{ccccc}
S^3 & & U_N \sqcup U_S & & U_N \cap U_S \simeq S^2 \\
\hline
\to H^3(S^3) & \to & H^2\cancel{(U_N)} \oplus H^1\cancel{(U_S)} & \to & H^3\cancel{(S^2)}\ 0 \\
 & & & & \cong \\
\to H^2(S^3) & \to & H^2\cancel{(U_N)} \oplus H^2\cancel{(U_S)} & \to & H^2(S^2) \\
 & & & & = \mathbb{R} \\
\to H^1(S^3) & \to & H^1\cancel{(U_N)} \oplus H^1\cancel{(U_S)} & \to & \cancel{H^1(S^2)}\ 0 \\
0 \to H^0(S^3) & \to & \underset{=\mathbb{R}}{H^0(U_N)} \oplus \underset{=\mathbb{R}}{H^0(U_S)} & \to & \underset{=\mathbb{R}}{H^0(S^2)}
\end{array}
$$

(b) Consider the sphere $S^n(r) = \{\sum_{i=0}^n x_i^2 = r^2\} \subset \mathbb{R}^{n+1}$. Show that

$$\omega \equiv \frac{1}{r}\sum_{i=0}^{n}(-1)^i x_i dx_0 \wedge \cdots \widehat{dx_i} \wedge dx_n$$

(the one with the hat is omitted) is not exact by integrating it over S^n. It is a generator of $H^n(S^n)$. [Hint: $dr \wedge \omega = dx_0 \wedge \cdots dx_n$ is the volume form on $\mathbb{R}^{n+1}$.]

$S^n(r) = \partial B^{n+1}(r)$ so Stokes says

$$\int_{S^n(r)} \omega = \int_{B^{n+1}(r)} d\omega = \frac{n+1}{r}\int_{B^{n+1}(r)} dx_0 \wedge \cdots dx_n = \partial_r\left(\mathrm{vol}(B^{n+1}(r))\right) \neq 0.$$

Restricted to $S^n(r)$, however, $d\omega = 0$ since there are no $n+1$-forms on S^n. So on $S^n(r)$ ω is closed but not exact (if it were exact on S^n it couldn't integrate to something nonzero).