University of California at San Diego – Department of Physics – Prof. John McGreevy

# Physics 239 Topology from Physics Winter 2021
# Assignment 7

**Due 5pm Friday February 26, 2021**

Thanks in advance for following the guidelines on hw01. Please ask me by email if you have any trouble.

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1. **Cohomology ring.**

   The wedge product $A \wedge B$ introduces a product structure on the cohomology of a manifold $\mathcal{M}$, *i.e.* $H^\bullet_{\text{de Rham}}(\mathcal{M})$ is actually a ring, not just an abelian group. Show that this product is well-defined in the sense that $[A] \wedge [B] = [A \wedge B]$.

2. **Pullback on forms.** Check that the pullback operation on forms commutes with the exterior derivative, and so is a chain map on de Rham complexes.

3. **Kunneth formula.**

   Consider a manifold $\mathcal{M} = X \times Y$ defined as a Cartesian product of two others. That is, a point in $\mathcal{M}$ can be labelled as $(x, y)$, with $x \in X$ and $y \in Y$.

   (a) Show that the de Rham complex on $\mathcal{M}$ is

   $$\Omega^p(\mathcal{M}) = \bigoplus_{k=0}^{n} \Omega^k(X) \otimes \Omega^{p-k}(Y)$$

   where $n = \dim \mathcal{M} = \dim X + \dim Y$, with the coboundary operator $d = d_X \otimes 1\!\!1_Y \pm 1\!\!1_X \otimes d_Y$. Fix the sign. (Note that this operation defines the product of complexes $\Omega^\bullet(X \times Y) = \Omega^\bullet(X) \otimes \Omega^\bullet(Y)$.)

   (b) Relate the Betti numbers of $\mathcal{M}$ to those of $X$ and $Y$.