University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 239 Topology from Physics Winter 2021 Assignment 7 – Solutions

Due 5pm Friday February 26, 2021

Thanks in advance for following the guidelines on hw01. Please ask me by email if you have any trouble.

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1. **Cohomology ring.**

   The wedge product $A \wedge B$ introduces a product structure on the cohomology of a manifold $\mathcal{M}$, *i.e.* $H^\bullet_{\text{de Rham}}(\mathcal{M})$ is actually a ring, not just an abelian group. Show that this product is well-defined in the sense that $[A] \wedge [B] = [A \wedge B]$.

   We just need the Liebniz rule on forms:

   $$d(A \wedge B) = dA \wedge B + (-1)^p A \wedge dB$$

   if A is a p-form. Therefore, changing A and/or B by a coboundary changes $A \wedge B$ by a coboundary.

2. **Pullback on forms.** Check that the pullback operation on forms commutes with the exterior derivative, and so is a chain map on de Rham complexes.

   Just use $f_i \equiv f^\star(y_i) = y_i \circ f$ as local coordinates on M. If they are not good coordinates, the form will vanish so it is OK. Given a p-form on N $g_{i_1 \cdots i_p} dy^{i_1} \wedge \cdots \wedge dy^{i_p}$, we want to show

   $$d(f^\star(g)) = f^\star(dg).$$

   The LHS is

   $$d\left(g_{i_2 \cdots i_{p+1}}(f_i) df_{i_2} \wedge \cdots df_{i_{p+1}}\right) = \partial_{[i_1} g_{i_2 \cdots i_{p+1}]}(f_i) df_{i_1} \wedge df_{i_2} \wedge \cdots df_{i_{p+1}}.$$

   The RHS is the same expression, by definition of $f^\star$.

3. **Kunneth formula.**

   Consider a manifold $\mathcal{M} = X \times Y$ defined as a Cartesian product of two others. That is, a point in $\mathcal{M}$ can be labelled as (x, y), with $x \in X$ and $y \in Y$.

   (a) Show that the de Rham complex on $\mathcal{M}$ is

   $$\Omega^p(\mathcal{M}) = \bigoplus_{k=0}^{n} \Omega^k(X) \otimes \Omega^{p-k}(Y)$$

where $n = \dim \mathcal{M} = \dim X + \dim Y$, with the coboundary operator $d = d_X \otimes 1\!\!1_Y \pm 1\!\!1_X \otimes d_Y$. Fix the sign. (Note that this operation defines the product of complexes $\Omega^\bullet(X \times Y) = \Omega^\bullet(X) \otimes \Omega^\bullet(Y)$.)

The idea is just that any form on $X \times Y$ is of the form

$$\sum_{q=0}^{\dim X} \sum_{p=0}^{\dim Y} \omega_{i_1 \cdots i_q j_1 \cdots j_p}(x, y) dx^{i_1} \wedge \cdots \wedge dx^{i_q} \wedge dy^{j_1} \wedge \cdots \wedge dy^{j_p}.$$

The terms with fixed $p + q$ are the elements of $\Omega^{p+q}(\mathcal{M})$. We can choose a basis of such forms where $\omega_{i_1 \cdots i_q j_1 \cdots j_p}(x, y) = \omega^X_{i_1 \cdots i_q}(x) \omega^Y_{j_1 \cdots j_p}(y)$. On the basis elements,

$$d\left(\omega^X_q \wedge \omega^Y_p\right) = d\omega^X_q \wedge \omega^Y_p + (-1)^q \omega^X_q \wedge d\omega^Y_p.$$

(b) Relate the Betti numbers of $\mathcal{M}$ to those of X and Y.

$$b^p(X \times Y) = \sum_{k=0}^{n} b^k(X) b^{p-k}(Y).$$