University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 239 Topology from Physics Winter 2021
Assignment 6 – Solutions

Due 5pm Friday February 19, 2021

Thanks in advance for following the guidelines on hw01. Please ask me by email if you have any trouble.

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1. **Euler characteristics of spheres.** Consider the N-dimensional unit sphere

$$S^N \equiv \{(x_0, \cdots, x_N) | x_0^2 + x_1^2 + \cdots x_N^2 = 1\}.$$

Let k be the isometry taking $x_N \to -x_N$. Show that the supersymmetric non-linear sigma model on S^N has

$$\mathrm{tr}(-1)^F = 1 + (-1)^N, \quad \mathrm{tr}(-1)^F K = 1 - (-1)^N$$

where K is the unitary operation implementing k and commuting with supersymmetry. Hint: use the Morse function $h = x_0$.

We use coordinates $x_1 \cdots x_N$. The height function is $\pm\sqrt{1 - \sum_{i=1}^N x_i^2}$ in the upper and lower hemispheres. In terms of $x^2 \equiv \sum_{i=1}^N x_i^2$, the derivatives are $\partial_i h = \pm \frac{-x_i}{\sqrt{1-x^2}}$ and

$$\partial_i \partial_j h = \mp \frac{\delta_{ij}(1 - x^2) - x_i x_j}{(1 - x^2)^2}.$$

The critical points are then the north and south poles where $x_1 = \cdots = x_N = 0$ and $h = \pm 1$. The eigenvalues of the hessian are $\pm(-1, -1 \cdots - 1)$ at these points. Therefore $\mathrm{tr}(-1)^F = 1 + (-1)^N$. To compute the Lefschetz number of k, we must consider how k acts on the classical groundstates associated with the two critical points. In order to be supersymmetric, it must take $\psi^N \to -\psi^N$ as well as $x^N \to -x^N$. Therefore, if the Nth fermion mode is occupied in the groundstate then that groundstate has negative K eigenvalue. At the north pole, all the eigenvalues of the mass matrix are negative and therefore all the modes are occupied. Therefore this vacuum has $K = -1$. This is the one that contributed the $(-1)^N$. Therefore $\mathrm{tr}(-1)^F K = 1 - (-1)^N$.

2. **Cohomology of $\mathbb{CP}^N$.** [Bonus problem] Complex projective space can be defined as

$$\mathbb{CP}^N \equiv \{(z_0, z_1 \cdots z_N)\}/(z \sim \lambda z), \lambda \in \mathbb{C}^\star \equiv \mathbb{C} \setminus 0.$$

Compute the Witten index for the supersymmetric NLSM with this target space.

Construct the groundstates of the supersymmetric non-linear sigma model on this space.

An alternative definition is

$$\mathbb{CP}^N = \{(z_0, z_1 \cdots z_N), \sum_{i=0}^{N} |z_i|^2 = 1\}/(z \sim e^{\mathbf{i}\theta} z)$$

where we can regard the equivalence relation as a gauge redundancy.

One good set of Morse functions is $h = \sum_j c_j |z_j|^2$ for generic numbers c_j. There are $N + 1$ critical points: there is one in each coordinate patch, $z_i \neq 0$. (On the coordinate patch where $z_i \neq 0$, we can set $z_i = 1$ by a gauge transformation, and use the other $z_{j\neq i}$ as local coordanites; $h = c_i + \sum_{j\neq i} c_j |z_j|^2$ and $\partial_{z_j} h = c_j \bar{z}_j$ (no sum on j).) The Morse indices are even because changing the sign of c_j flips the eigenvalues in both the $\mathrm{Re}\, z_j$ and $\mathrm{Im}\, z_j$ directions.

3. **Integral formula for the Euler character from the path integral**

In this problem, we use the independence of Witten index $\mathrm{tr}(-1)^F e^{-\beta H}$ from β to derive an expression for the Euler number of a manifold in terms of an integral involving the Riemann curvature tensor over the manifold.

Consider the limit $\beta \to 0$ of the path-integral representation of the Witten index, and argue that the finite-action field configurations contributing to the path-integral localize to constant modes independent of time. This reduces the path-integral to a zero-dimensional QFT, involving an integration over the manifold and some grassmann variables. (This could also be derived using the localization principle and the supersymmetry transformation of the fermionic fields). Show that the fermionic integration brings down Riemann curvature terms from the quartic fermionic term in the action leading to the desired integral over the manifold.

Show in this way that the Euler character vanishes for odd-dimensional manifolds.

The key point is that the Witten index is independent of β, and when we take $\beta \to 0$ only constant configurations contribute. Then

$$\mathrm{tr}(-1)^F = \mathcal{N} \int d^n\phi \sqrt{\gamma} \int \prod_i d\bar{\psi}_i d\psi_i \; e^{R_{ijkl}(\phi)\bar{\psi}^i \psi^k \bar{\psi}^j \psi^l}.$$

Notice the order of the indices – this is very important since the antisymmetry properties of the ψ matches the antisymmetry properties of the Riemann tensor $R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij}$ (see pp. 75-78 here).

Since we must pull down four ψs at a time, and we get two ψs for each dimension of $\mathcal{M}$, in odd dimensions we cannot saturate the grassmann integrals and we get zero.

For n even, the integrand is

$$\int \prod_i d\bar{\psi}_i d\psi_i \; e^{R_{ijkl}(\phi)\bar{\psi}^i\psi^k\bar{\psi}^j\psi^l} = \frac{1}{(n/2)!}\sum_{\sigma,\sigma'\in S_n}\prod_{k=1}^{n/2} R_{\sigma_{2k-1}\sigma_{2k}\sigma'_{2k-1}\sigma_{2k}}(-1)^{\sigma+\sigma'}$$

There is an overall prefactor which is more difficult to fix.

This exercise is from the clay book on mirror symmetry.

Bonus problem: Find an integral formula for the Hirzebruch index. This requires first finding a path integral representation.