University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 239 Topology from Physics Winter 2021
Assignment 5 – Solutions

Due 12:30pm Wednesday February 10, 2021

Thanks in advance for following the guidelines on hw01. Please ask me by email if you have any trouble.

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1. **Hodge star and adjoint of d.** Consider the inner product on (real) p-forms on a manifold without boundary $\mathcal{M}$

$$\langle B, A\rangle \equiv \int_{\mathcal{M}} (\star B) \wedge A.$$

Show (using integration by parts and $\star^2 = (-1)^k$) that the adjoint of the exterior derivative can be written as

$$d^\dagger = s \star d \star$$

where s is a sign depending on p and $\dim \mathcal{M}$.

Bonus problem: Find s.

$$\left\langle B, d^\dagger A\right\rangle = \langle dB, A\rangle \tag{1}$$

$$= \int_{\mathcal{M}} (\star dB) \wedge A = \int_{\mathcal{M}} dB \wedge \star A \tag{2}$$

$$\overset{\text{IBP}}{=} -(-1)^p \int_{\mathcal{M}} B \wedge d \star A \tag{3}$$

$$= -(-1)^p(-1)^k \int_{\mathcal{M}} \star B \wedge \star d \star A \qquad \star^2 = (-1)^k \tag{4}$$

$$= (-1)^{1+p+k} \langle B, \star d \star A\rangle . \tag{5}$$

Since this is true for all p-forms B, A and the inner product is linear in each argument, this says

$$d^\dagger = (-1)^{1+p+k} \star d \star .$$

Now what is k? To figure it out we have to be more careful about this signs. The Liebniz rule for forms is

$$d\left(B_p \wedge A\right) = dB_p \wedge A + (-1)^p B_p \wedge dA$$

which implies

$$(\star B) \wedge dA = -(-1)^{n-p} d(\star B) \wedge A + \text{total derivative} \,.$$

So we have

$$\langle B, dA\rangle = \int_{\mathcal{M}} \star B \wedge dA = -(-1)^{n-p}\int_{\mathcal{M}} d\star B \wedge A = (-1)^{n-p+1+x}\int \star(\star d \star B)\wedge A = \left\langle d^\dagger, A\right\rangle .$$

where $\star^2 = (-1)^x$ when acting on a q-form. Here $x = q(n-q)$ and $d \star B$ is a $n-p+1$ form if B is a p-form. Therefore

$$d^\dagger = (-1)^{p(n-p+1)} \star d \star .$$

2. **Supersymmetric harmonic oscillator.** Consider the quantum mechanical system with Hamiltonian

$$H = \frac{p^2}{2} + \frac{1}{2}\omega^2 x^2 + \frac{1}{2}\omega[\bar{\psi}, \psi].$$

(a) Using your knowledge of the ordinary harmonic oscillator, construct the spectrum. Consider both signs of ω.

The two parts of the system actually decouple, so the spectrum arises by taking their tensor product. The bosonic SHO has levels $\omega\frac{1}{2}, \omega(\frac{1}{2}+1), \omega(\frac{1}{2}+2)\cdots$. The fermionic oscilaltor has levels $\pm\omega\frac{1}{2}$. If $\omega > 0$, the empty state is the groundstate, while if $\omega < 0$, the filled state is the groundstate. Let's call the empty state bosonic. For $\omega > 0$ the spectrum is:

$$B : 0, \omega, 2\omega\cdots\,, \quad F : \omega, 2\omega, \cdots$$

For $\omega < 0$ the spectrum is:

$$F : \omega, 2\omega\cdots\,, \quad F : 0, \omega, 2\omega, \cdots\,.$$

Altogether, we can write this as

$$E_{N,k} = \frac{1}{2}|\omega|\left(2N + 1 - \text{sign}(\omega)(-1)^k\right), \quad N = 0, 1, 2..., k = 0, 1.$$

(b) Compute the thermal partition function at temperature β, $Z(\beta) = \text{tr}e^{-\beta H}$, and the Witten index $\text{tr}(-1)^F = \text{tr}(-1)^F e^{-\beta H}$.

Summing over the spectra above, we find

$$Z(\beta) = \sum_{N,k} e^{-\frac{\beta}{2}\left(2N+1-\text{sign}(\omega)(-1)^k\right)} = \frac{e^{-\beta|\omega|/2}}{1-e^{-\beta|\omega|}} 2\cosh\beta|\omega|/2 = \coth\beta|\omega|/2$$

and

$$\text{tr}(-1)^F = \sum_{N,k}(-1)^k e^{-\frac{\beta}{2}\left(2N+1-\text{sign}(\omega)(-1)^k\right)} = \frac{e^{-\beta|\omega|/2}}{1-e^{-\beta|\omega|}} \text{sign}(\omega) 2\sinh\beta|\omega|/2 = \text{sign}(\omega)\,.$$

Note that the overall sign depends on our convention for whether the empty state is bosonic or fermionic.

3. **Supersymmetry in $D = 0$ and localization.** I got this problem from §9.3 of the Clay Mirror Symmetry book.

   In this problem, we consider the 'action'

   $$S[x, \psi, \bar{\psi}] = \frac{1}{2}\left(\partial_x h(x)\right)^2 + a\bar{\psi}\psi\partial^2 h(x)$$

   for a field theory in $D = 0$ dimensions and the associated 'partition function'

   $$Z[h] \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx d\psi d\bar{\psi} e^{-S[x,\psi,\bar{\psi}]}.$$

   Here $\psi, \bar{\psi}$ are independent grassmann variables, but otherwise this is just an ordinary finite-dimensional integral.

   (a) Show that the action S is supersymmetric (for some choice of constant a), in the sense that it is invariant under the transformation

   $$\delta_\epsilon x = \epsilon\psi - \bar{\epsilon}\bar{\psi}, \quad \delta_\epsilon\psi = \bar{\epsilon}\partial h, \quad \delta_\epsilon\bar{\psi} = \epsilon\partial h.$$

   $\epsilon, \bar{\epsilon}$ are independent grassmann variables.

   The variation of the action is

   $$h'h''(\epsilon\psi - \bar{\epsilon}\bar{\psi}) + ah''(-\bar{\epsilon}h'\bar{\psi} + \psi\epsilon h') = 0$$

   if $a = -1$.

   (b) Prove the supersymmetry Ward identity

   $$0 = \left\langle \delta_\epsilon g(x, \psi, \bar{\psi}) \right\rangle \tag{6}$$

   where

   $$\langle \mathcal{O} \rangle \equiv \frac{1}{\sqrt{2\pi}} \int dx d\psi d\bar{\psi} e^{-S[x,\psi,\bar{\psi}]} \mathcal{O}$$

for any function $\mathcal{O}$ of the dynamical variables, and g is a function with good enough behavior at $|x| \to \infty$.

[Hint: there is no boundary of the integration region.]

[Second hint: Very generally, if the action and integration measure are invariant under some symmetry then

$$\langle \delta g \rangle \equiv \int \delta g e^{-S} = 0$$

where δg is the variation of g under the symmetry (as long as g behaves well at the boundaries of the integration region). This follows from changing variables in the integral. Consider, for example, the case $\int_{\mathbb{R}^2} dxdye^{-S[x,y]}$, with $S[x,y] = x^2 + y^2$ and $\delta x = y, \delta y = -x$ (rotations).]

In the case of an n-dimensional bosonic integral, we can make a change of coordinates (polar coords in the example) to new coordinates $r_1 \cdots r_{n-1}, \theta$ so that $\delta r_i = 0, \delta\theta = \epsilon$ is the action of the symmetry. In that case, $\delta f = \partial_\theta f$, and the integral is

$$\int d^n x \delta g e^{-S} = \int d^n x \delta \left(g e^{-S} \right) = \int d^{n-1} r f(r) d\theta \partial_\theta \left(g e^{-S} \right) = 0.$$

Note that the fact that the transformation is a symmetry of the measure means that the integrand has no dependence on θ. At the last step we used Stokes' theorem and the assumption that there are no boundaries of the range of integration that matter.

Here is a better way to think about it in general: with the assumption that the measure and the action are invariant,

$$\langle \delta g \rangle = \delta \left(\int g e^{-S} \right)$$

is just the change in the expectation value under a change of integration variables by the symmetry transformation, which is zero since the integral is coordinate invariant.

The supersymmetry Ward identity is the same idea but with grassmann variables.

As long as $\delta_\epsilon S = 0$, we have

$$\langle \delta g \rangle \equiv \int \delta g e^{-S} = \int \delta \left(g e^{-S} \right).$$

Assuming the integration measure is invariant under the symmetry as well, we can make a change of variables

(c) By choosing $g(x,\psi,\bar\psi) = \partial\rho(x)\psi$ (for some function $\rho(x)$) in the Ward identity (6), show that changing $h(x) \to h(x) + \rho(x)$ in the action does not change the partition function Z, *i.e.* $Z[h+\rho] = Z[h]$, for infinitesimal ρ.

With this choice of g, we have

$$\delta_\epsilon g = -\bar\epsilon\rho''\bar\psi\psi + \bar\epsilon\rho' h' = \bar\epsilon\delta_\rho S$$

where $\delta_\rho S$ is the change of S under the replacement $h \to h+\rho$ for infinitesimal ρ (*i.e.* to first order in ρ).

Now the hard part. In the next few parts, we wish to show that the integral $Z[h]$ is localized to loci where the supersymmetry variation of the fermions is zero. This is called the *supersymmetric localization principle*. In this case this means $0 = \delta\psi \propto \partial_x h(x) \equiv h'(x)$, critical points of h.

(d) Argue that if $h'(x) = 0$ has no real solutions, then $Z[h] = 0$. Hint: change integration variables to $\tilde x = x - \bar\psi\psi/h'(x)$.

Actually in the problem with just a single superfield integration variable, we can just do the integral over $\psi, \bar\psi$ to get

$$Z = \frac{1}{\sqrt{2\pi}}\int d\phi e^{-|h'(\phi)|^2/2} h''(\phi) = \frac{1}{\sqrt{2\pi}}\int_{h'(-\infty)}^{h'(\infty)} du e^{-u^2/2} = \frac{1}{\sqrt{2\pi}}\int d\left(\frac{\sqrt{\pi}}{2}\mathrm{erf}\left(h'(\phi)\right)\right) = 0$$

(where $u \equiv h'(x)$) since the error function has the same limits at $\pm\infty$ as long as $h'(\phi)$ has no zeros.

But for purposes of generalizing to many variables, its best to use the supersymmetry transformation. If h' has no zeros, we can do the suggested change of variables. Moreover, we can choose the parameter of the supersymmetry transformation to set $\bar\psi = 0$. This requires $\epsilon = \bar\epsilon = -\bar\psi/h'(\phi)$. (Notice that this requires $h' \neq 0$.) The action becomes $\frac{1}{2}(h')^2$. The measure becomes (from the jacobian of the bosonic variable) $d\phi d\psi d\bar\psi \to d\phi d\psi d\bar\psi\left(1 - \bar\psi\psi\frac{h''}{(h')^2}\right)$. Only the second term contributes, and

$$\int dx \frac{h''}{(h')^2} e^{-\frac{1}{2}(h')^2} = \int d\left(-\sqrt{\pi}\mathrm{erf}(h'(x)) - \frac{e^{-(h'(x))^2}}{h'(x)}\right) = 0$$

since the antiderivative has the same limits at $x \to \infty$ as long as $h'(x)$ has no zeros.

(e) Hence the integral gets contribution only from the neighborhood of critical points of h. Taylor expand to second order in h near such a critical point and evaluate its contribution.

Near a critical point x_0, the quadratic action is

$$S \simeq -\frac{1}{2}(h'')^2|_{x_0}(x-x_0)^2 + h''|_{x_0}\bar{\psi}\psi.$$

So the quadratic fluctuations integrate to (setting $x_0 = 0$ wlog)

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\ e^{-\frac{1}{2}(h'')^2|_{x_0}(x-x_0)^2}\int d\bar{\psi}d\psi e^{-h''|_{x_0}\bar{\psi}\psi} = \frac{h''(x_0)}{|h''(x_0)|} = \text{sign}(h''|_{x_0}).$$

(f) Add up the results and conclude that the partition function is an integer.

$$Z = \sum_{x_A|h'(x_A)=0}\frac{h''(x_A)}{|h''(x_A)|}$$

is a sum of terms, each of which is ± 1.

(g) Conclude using the result of part (3c) that $Z[h] = 0$ if h is polynomial of odd degree, while $Z[h] = \pm 1$ if h is a polynomial of even degree.

We can deform the potential any way we want. If the superpotential h is a polynomial of odd degree, its derivative h' has even degree, and we can deform it so that it never touches the real axis, just by adding/subtracting a large constant if $h'(x \to \infty) \to \pm\infty$. In contrast, if h' has odd degree, it must cross the origin an odd number of times and at least once. This one critical point contributes ± 1.

(h) Argue that the Witten index for the supersymmetric SHO of problem 2 reduces to $Z[h = \omega x^2]$ in the limit $\beta \to 0$.

The path integral expression for the Witten index

$$\text{tr}(-1)^F = \int_{\text{PBC}} DXD\bar{\psi}D\psi e^{-S}$$

with periodic boundary conditions for everyone around the thermal circle (of radius β, which does not matter). But we can choose $\beta \to 0$, so that only constant-in-time configurations contribute to the integral. The resulting integral is exactly $Z[h = \omega x^2]$. Note that the answer agrees, since there is a single critical point.

(i) [More optional] Generalize the results of this problem this problem to n variables $x_i, \psi_i, \bar{\psi}_i$, and a superpotential h which depends on all of them.

The only new ingredient is that the variation of x in the fermion mass term gives a term of the form

$$\partial_i\partial_j\partial_k h(\epsilon\psi^i - \bar{\epsilon}\psi^i)\psi^j\bar{\psi}^k$$

but symmetry of the mixed partials and antisymmetry $\{\psi^i, \psi^j\} = 0$ of the grassmann variables means this is zero.