University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 239 Topology from Physics Winter 2021
Assignment 4

Due 12:30pm Wednesday February 3, 2021

Thanks in advance for following the guidelines on hw01. Please ask me by email if you have any trouble.

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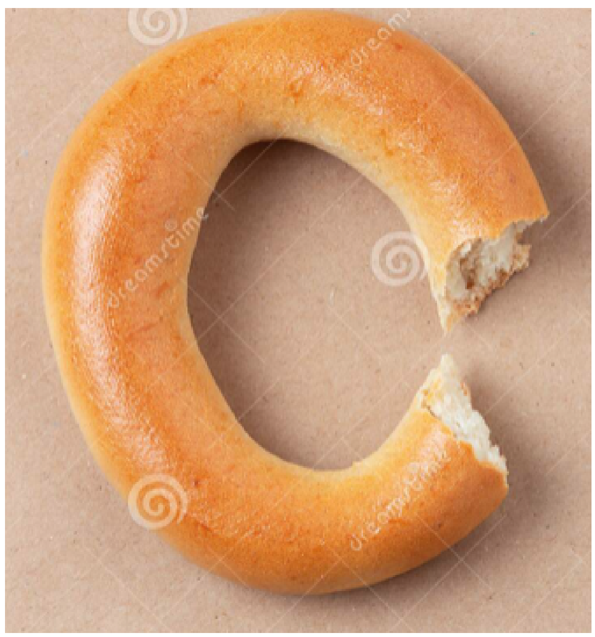
1. **Subdivision invariance.** Subdivide a complex C made of a single 3-simplex (and its sub-simplices) into a complex $\hat{C}$ with four 3-simplices. Show that the complex $\hat{C}/C$ has no homology.

2. **Relative homology.** Take a torus X (like the surface of a bagel) and take a bite Y out of it. Choose the bite so that both Y and $X \setminus Y$ are annuli.



   Choose a cell decomposition of X so that Y is closed (meaning that the boundaries of all cells in Y are also in Y). (This means that $X \setminus Y$ has rough boundary conditions and Y has smooth boundary conditions.) Compute $H_\bullet(X, \mathbb{Z})$, $H_\bullet(Y, \mathbb{Z})$, $H_\bullet(X/Y, \mathbb{Z})$. Show that your answers are consistent with the long exact sequence.

   [Note that a more common situation is where one uses the long exact sequence to learn $H_\bullet(X, \mathbb{Z})$ from $H_\bullet(Y, \mathbb{Z})$ and $H_\bullet(X/Y, \mathbb{Z})$.]

3. **Subdivision invariance and entanglement renormalization.**

(a) Verify that conjugation by the control-X gate

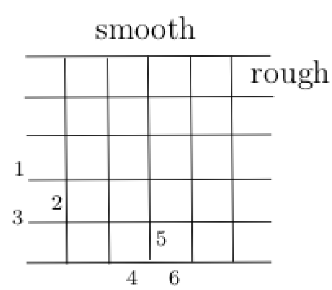
$$\mathsf{CX} \equiv P_C(0) \otimes \mathbb{1}_T + P_C(1) \otimes X_T$$

(with $P_C(0) = |0\rangle\langle 0|_C = \frac{1}{2}(1 + Z_C)\,, P_C(1) = |1\rangle\langle 1|_C = \frac{1}{2}(1 - Z_C)$), accomplishes the operations ($\mathcal{O} \leftrightarrow \mathsf{CX}\mathcal{O}\mathsf{CX}$)

$$\begin{aligned} 1_C Z_T &\leftrightarrow Z_C Z_T \\ 1_C X_T &\leftrightarrow 1_C X_T \\ Z_C 1_T &\leftrightarrow Z_C 1_T \\ X_C 1_T &\leftrightarrow X_C X_T \end{aligned}$$

(b) Find the Hamiltonians resulting from the operations described in the lecture notes which add a new plaquette or add a new vertex to the cell complex. (Note that some arrows were reversed in the vertex-addition-circuit in a previous version of the lecture notes.) Show that each one has the same topological groundstate degeneracy as the toric code on the new cell complex.

4. **A topological qubit.** Consider the toric code on this cell complex:

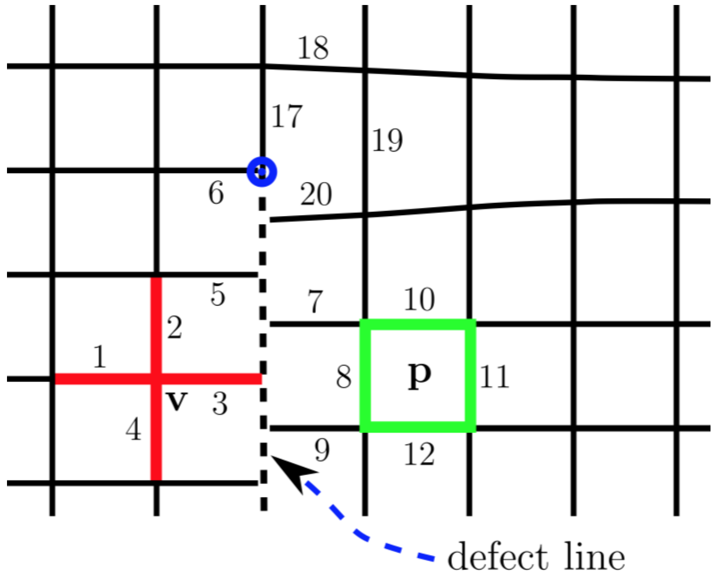


Recall that rough boundary conditions means that plaquette terms get truncated, such as the term $-X_1X_2X_3$, while smooth boundary conditions mean that star terms get truncated, such as the term $-Z_4Z_5Z_6$.

Show that there is a two-dimensional space of groundstates. A good way to do this is using the algebra of string operators which terminate at the various components of the boundary without creating excitations.

5. **Duality wall.** [Bonus problem] Show that the following hamiltonian realizes a

duality wall in the toric code.



What this means is that when crossing the wall, an e particle turns into an m particle and vice versa. (More precisely, there is a string operator which transports an e particle to the wall, and can be completed by a string operator transporting an m particle away from the wall, without creating any excitations.)

To be more precise about the figure: the dotted line carries no degrees of freedom. The terms in the hamiltonian along the wall are of the form

$$H = ... - X_2 X_5 X_3 Z_7 - Z_3 X_7 X_8 X_9$$

and there is a term at the end of the wall (the little blue circle) of the form $-Z_6 Y_{17} X_{18} X_{19} X_{20}$. Show that these terms commute with each other and all the usual star and plaquette terms, such as $A = Z_1 Z_2 Z_3 Z_4$ and $B = X_8 X_{10} X_{11} X_{12}$.

What can you say about the end of the duality wall (the little blue circle)?